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A Unified Presentation of Certain Subclass of Analytic Functions with Negative Coefficients

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Abstract The main objective of this paper is to introduce and investigate various properties and characteristics of a unified class of analytic functions with negative coefficient in the unit disc. The results presented here involve distortion inequalities and growth and distortion theorem involving fractional integrals and fractional derivatives.

Keywords Analytic functions, Univalent functions, Riemann - Liouville operators , Growth and distortion theorem, Fractional derivatives

1 Introduction

Let S denotes the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open unit disk

$$U = \{z : |z| < 1\}. \tag{1.2}$$

Let T denote the subclass of S consisting functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \ (a_n \geq 0). \tag{1.3}$$

Following work of Joshi & Srivastava [3] , we say that a function $f \in T$ is in the class $T^*_\lambda(A, B, \alpha, \beta, \gamma)$ if

$$\left| \frac{zF'_\lambda(z)/F_\lambda(z) - 1}{(B-A)\gamma\left(\frac{zF'_\lambda(z)}{F_\lambda(z)} - \alpha\right) - B\left(\frac{zF'_\lambda(z)}{F\lambda(z)} - 1\right)} \right| < \beta \tag{1.4}$$

where $\quad z \in U, 0 \leq \alpha < 1, 0 < \beta \leq 1, -1 \leq A \leq B \leq 1, 0 < B \leq 1$

$$\frac{B}{(B-A)} < \gamma \leq \left\{ \begin{array}{ll} B/(B-A)\alpha, & \alpha \neq 0 \\ 1, & \alpha = 0 \end{array} \right\} \tag{1.5}$$

and

$$\begin{aligned} F_\lambda(z) &= (1-\lambda)f(z) + \lambda z f'(z), \lambda \geq 0, \lambda \in T \\ &= z - \sum_{n=2}^{\infty} [1+(n-1)\lambda] a_n z^n. \end{aligned}$$

We also define a subclass $C_\lambda^*(A,B,\alpha,\beta,\gamma)$ of T by,

$$C_\lambda^*(A,B,\alpha,\beta,\gamma) = \{f \in T, zf'(z) \in T_\lambda^*(A,B,\alpha,\beta,\gamma)\}.$$

By specializing suitably, the parameters α,β,γ,A and B the aforementioned classes $T_\lambda^*(A,B,\alpha,\beta,\gamma)$ and $C_\lambda^*(A,B,\alpha,\beta,\gamma)$ will reduce to several known interesting subclasses earlier studied by Owa & Aouf [5] , Silverman [9] , Srivastava & Owa [10] and several others ([1], [2], [7]).

The following known results for the class $T_\lambda^*(A,B,\alpha,\beta,\gamma)$ and $C_\lambda^*(A,B,\alpha,\beta,\gamma)$ will be required in our present investigation.

Lemma 1 ([3]). *Let $f \in T$ be defined by (1.3). Then $f(z)$ is in the class $T_\lambda^*(A,B,\alpha,\beta,\gamma)$ if and only if*

$$\sum_{n=2}^{\infty} a_n D(n,A,B,\alpha,\beta,\gamma,\lambda) \leq (B-A)\gamma(1-\alpha) \tag{1.6}$$

where the parameters are constrained as in (1.5), and

$$D(n,A,B,\alpha,\beta,\gamma,\lambda) = [1+(n-1)\lambda][(n-1)+\beta(B-A)\gamma(n-\alpha)-B(n-1)].$$

The result (1.6) is sharp.

Lemma 2 ([4]). *Let $f \in T$ be defined by (1.3). Then $f(z)$ is in the class $C_\lambda^*(A,B,\alpha,\beta,\gamma)$ if and only if*

$$\sum_{n=2}^{\infty} n a_n D(n,A,B,\alpha,\beta,\gamma,\lambda) \leq (B-A)\gamma(1-\alpha) \tag{1.7}$$

The result (1.7) is sharp.

In view of Lemma 1 and Lemma 2 , we find it to be worthwhile to present here a unified study of the classes $T_\lambda^*(A,B,\alpha,\beta,\gamma)$ and $C_\lambda^*(A,B,\alpha,\beta,\gamma)$ by introducing new subclass $P_\lambda^*(A,B,\alpha,\beta,\gamma,\sigma)$. Indeed, we say that a function $f(z)$ defined by(1.3) belong to the class $P_\lambda^*(A,B,\alpha,\beta,\gamma,\sigma)$, if and only if

$$\sum_{n=2}^{\infty} \frac{[1-\sigma+\sigma n]D(n,A,B,\alpha,\beta,\gamma,\lambda)}{(B-A)\gamma((1-\alpha)} a_n \leq 1 \tag{1.8}$$

where $0 \leq \sigma \leq 1$, and other parameters are constrained as in (1.5).

The main aim of the present paper is to investigate various properties and characteristics of the general class $P_\lambda^*(A,B,\alpha,\beta,\gamma,\sigma)$.

2 A Set of Distortion Inequalities

Distortion inequalities are inequalities that give upper and lower bounds for functions in class of analytic functions. We establish the following distortion inequalities for the functions belonging to the class $P_\lambda^*(A,B,\alpha,\beta,\gamma,\sigma)$.

Theorem 1 *If a function, $f(z)$, defined by (1.3) is in the class $P_\lambda^*(A,B,\alpha,\beta,\gamma,\sigma)$ then*

$$\begin{aligned}&|z|-\frac{(B-A)\gamma\beta(1-\alpha)}{(1+\sigma)D(2,A,B,\alpha,\beta,\gamma,\lambda)}|z|^2\leq|f(z)|\\&\leq|z|+\frac{(B-A)\gamma\beta(1-\alpha)}{(1+\sigma)D(2,A,B,\alpha,\beta,\gamma,\lambda)}|z|^2\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}&1-\frac{(B-A)\gamma\beta(1-\alpha)}{(1+\sigma)D(2,A,B,\alpha,\beta,\gamma,\lambda)}|z|\leq|f'(z)|\\&\leq1+\frac{(B-A)\gamma\beta(1-\alpha)}{(1+\sigma)D(2,A,B,\alpha,\beta,\gamma,\lambda)}|z|.\end{aligned}\tag{2.2}$$

Proof We find from (1.8) that

$$\sum_{n=2}^{\infty}a_n\leq\frac{(B-A)\gamma\beta(1-\alpha)}{(1+\sigma)D(2,A,B,\alpha,\beta,\gamma,\lambda)}.\tag{2.3}$$

Using (1.3) and (2.3), we readily have (for $z\in U$),

$$|f(z)|\geq|z|-|z|^2\sum_{n=2}^{\infty}a_n\geq|z|-\frac{|z|^2(B-A)\gamma\beta(1-\alpha)}{(1+\sigma)D(2,A,B,\alpha,\beta,\gamma,\lambda)}$$

and

$$|f(z)|\leq|z|+|z|^2\sum_{n=2}^{\infty}a_n\leq|z|+\frac{|z|^2(B-A)\gamma\beta(1-\alpha)}{(1+\sigma)D(2,A,B,\alpha,\beta,\gamma,\lambda)}\tag{2.4}$$

which prove assertion (2.1) of Theorem 1.

Also, from (1.3), we find for $z\in U$, that

$$|f'(z)|\geq1-|z|\sum_{n=1}^{\infty}na_n\geq1-\frac{2(1-\alpha)\gamma\beta(B-A)}{(1+\sigma)D(2,A,B,\alpha,\beta,\gamma,\lambda)}|z|\tag{2.5}$$

and

$$|f'(z)|\leq1+|z|\sum_{n=1}^{\infty}na_n\leq1+\frac{2(1-\alpha)\gamma\beta(B-A)}{(1+\sigma)D(2,A,B,\alpha,\beta,\gamma,\lambda)}|z|\tag{2.6}$$

which prove the assertion (2.2) of Theorem 1.

Finally, each of the results (2.1) and (2.2) is sharp for the functions $f(z)$ given by

$$f(z)=z-\frac{(B-A)\gamma\beta(1-\alpha)z^2}{(1+\sigma)D(2,A,B,\alpha,\beta,\gamma,\lambda)}\tag{2.7}$$

we complete the proof of Theorem 1.

3 Growth and Distortion Theorems Involving Fractional Calculus Operators

We begin by recalling the Fractional integrals operators $\mathbb{I}_{o,z}^{\delta,\mu,\eta}$ as follows (defined by Srivastava et. al. [6]):

Definition 1 ([6]) Let $\mathbb{R}_+$ and $\mu, \eta \in \mathbb{R}$. Then in terms of familiar (Gauss hypergeometric function ${}_2F_1$ the fractional integral operators $\mathbb{I}_{o,z}^{\delta,\mu,\eta}$ is defined by

$$\mathbb{I}_{o,z}^{\delta,\mu,\eta} = \frac{z^{-\delta-\mu}}{\Gamma(\delta)} \int_0^z (z-t)^{\lambda-1} \, {}_2F_1(\delta+\mu, -\eta; \, \delta, 1-t/z) f(t) dt \tag{3.1}$$

where the function $f(z)$ is analytic in a simply connected region of the z-plane containing the origin, with the order

$$f(z) = 0(|z|^k), \; (z \to 0) \tag{3.2}$$

for

$$k > \max\{0, \mu-\eta\} - 1 \tag{3.3}$$

and the multiplicity of $(z-t)^{\delta-1}$ is removed by requiring $\log(z-t)$ to be real when $z-t \in \mathbb{R}_+$.

Next, under the same constraints as in Definition1, above, an extended definition of the fractional derivative operator $\mathbb{J}_{o,z}^{\delta,\mu,\eta}$ is given by,

$$\begin{gathered}\mathbb{J}_{o,z}^{\delta,\mu,\eta} = \frac{d^n}{dz^n}\Big(\frac{z^{\delta-\mu}}{\Gamma(n-\delta)} \int_0^z (z-t)^{n-\delta-1} \, {}_2F_1(\mu-\delta, n-\eta; \, n-\delta, 1-t/z) f(t) dt\Big) \\ (n-1 \leq \delta < n; \, n \in \mathbb{N}, \, \mu, \eta \in \mathbb{R})\end{gathered} \tag{3.4}$$

The fractional derivative operator $\mathbb{J}_{o,z}^{\delta,\mu,\eta}(0 \leq \delta < 1)$, studied recently in Raina [7] and Raina & Srivastava [8], follows from (3.4) when $n = 1$. The fractional calculus operators defined by (3.1) and (3.4) include the Riemann-Liouville operators as their particular cases (c.f . [10]).

$$\mathbb{J}_{o,z}^{\delta,-\delta,\eta} f(z) = D_z^{-\delta} f(z), \;\; \delta > 0 \tag{3.5}$$

and

$$\mathbb{J}_{o,z}^{\delta,\delta,\eta} f(z) = D_z^{-\delta} f(z), \;\; \delta \geq 0. \tag{3.6}$$

We now prove the following growth and distortion theorem involving the fractional calculus operators define by (3.1) and (3.4).

Theorem 2 *Let $\delta \in \mathbb{R}_+$ and $\mu, \eta \in \mathbb{R}$ such that*

$$\mu < 2, \eta > max\{-\delta, \mu\} - 2 \quad and \quad \mu(\delta+\mu) \leq 3\delta.$$

If $f(z) \in P_\lambda^*(A, B, \alpha, \beta, \gamma, \sigma)$, then

$$|\mathbb{I}_{o,z}^{\delta,\mu,\eta} f(z)| \geq \frac{\Gamma(2-\mu+\eta)}{\Gamma(2-\mu)\Gamma(2-\delta+\eta)} |z|^{1-\mu} \Big\{1 - \frac{2(B-A)\gamma\beta(1-\alpha)(2-\mu+\eta)}{(1+\sigma)D(2, A, B, \alpha, \beta, \gamma, \lambda)} |z|\Big\} \tag{3.7}$$

and

$$|\mathbb{I}_{o,z}^{\delta,\mu,\eta} f(z)| \leq \frac{\Gamma(2-\mu+\eta)}{\Gamma(2-\mu)\Gamma(2-\delta+\eta)} |z|^{1-\mu} \Big\{1 + \frac{2(B-A)\gamma\beta(1-\alpha)(2-\mu+\eta)}{(1+\sigma)D(2, A, B, \alpha, \beta, \gamma, \lambda)} |z|\Big\} \tag{3.8}$$

with $z \in U$, *if* $\mu \leq 1$ *and* $z \in U \setminus \{0\}$ *if* $\mu > 1$.

The equalities in (3.7) and (3.8) are attained by the function f(z) given by (2.7).

Proof. Since function $f(z) \in P^*_\lambda(A, B, \alpha, \beta, \gamma, \lambda, \sigma)$ it follows that,

$$\frac{(1+\sigma)D(2, A, B, \alpha, \beta, \gamma, \lambda)}{(B-A)\gamma\beta(1-\alpha)} \sum_{n=2}^{\infty} a_n \leq \frac{(1-\sigma+\sigma n)D(2, A, B, \alpha, \beta, \gamma, \lambda))}{(B-A)\gamma\beta(1-\alpha)} \tag{3.9}$$

which gives

$$\sum_{n=1}^{\infty} a_n \leq \frac{(B-A)\gamma\beta(1-\alpha)}{(1+\sigma)D(2, A, B, \alpha, \beta, \gamma, \lambda))}. \tag{3.10}$$

¿From (1.8), (3.1) and known formula [6 , P.415 Lemma 3], we obtain

$$|\mathbb{I}^{\delta,\mu,\eta}_{o,z} f(z)| \geq \frac{\Gamma(2-\mu+\eta)}{\Gamma(2-\mu)\Gamma(2-\delta+\eta)} \{1 - |z| \sum_{n=2}^{\infty} \psi(n) a_n\} \tag{3.11}$$

where

$$\psi(n) := \frac{(2)_{n-1}(2-\mu+\eta)_{n-1}}{(2-\mu)_{n-1}(2+\delta+\eta)_{n-1}}, \ n \geq 2 \tag{3.12}$$

and $(\delta)_n : \Gamma(\delta+n)/\Gamma(\delta)$. We observe that the function $\psi(n)$ is a non increasing function, under the hypotheses of Theorem 2 and the desired inequality (3.7) is easily obtained on using (3.10) to (3.12). Assertion (3.8) can be proved in a similar manner .

Theorem 3 *Let* $\delta \in \mathbb{R}_+$ *and* $\mu, \eta \in \mathbb{R}$ *such that*

$$\mu < 2, \eta > max\{\delta, \mu\} - 2 \quad and \quad \mu(\delta - \eta) \geq 3\delta.$$

If $f(z) \in P^*_\lambda(A, B, \alpha, \beta, \gamma, \lambda, \sigma)$, *then*

$$|\mathbb{J}^{\delta,\mu,\eta}_{o,z}| \geq \frac{\Gamma(2-\mu=\eta)}{\Gamma(2-\mu)\Gamma(2-\delta+\eta)} |z|^{1-\mu} \Big\{1 - \frac{2(B-A)\gamma\beta(1-\alpha)(2-\mu-\eta)}{(1+\sigma)D(2, A, B, \alpha, \beta, \gamma, \lambda)}\Big\} \tag{3.13}$$

and

$$|\mathbb{J}^{\delta,\mu,\eta}_{o,z}| \leq \frac{\Gamma(2-\mu=\eta)}{\Gamma(z-\mu)\Gamma(2-\delta+\eta)} |z|^{1-\mu} \Big\{1 - \frac{2(B-A)\gamma\beta(1-\alpha)(2-\mu-\eta)}{(1+\sigma)D(2, A, B, \alpha, \beta, \gamma, \lambda)}\Big\}. \tag{3.14}$$

The proof of Theorem 3 is straightforward and hence omitted.

4 Conclusion

The work presented in this paper gives unified approach for earlier subclasses studied by many researches. (Further research can be done by finding extreme points and radius results for this class).

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