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Multiscale Boundary Element Method for Acoustic Wave Model

Nor Afifah Hanim Binti Zulkefli, Yeak Su Hoe* and Munira Binti Ismail
Department of Mathematical Sciences, Universiti Teknologi Malaysia
81310 UTM Johor Bahru, Malaysia
*Corresponding author: s.h.yeak@utm.my,

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Abstract In numerical methods, boundary element method has been widely used to solve acoustic problems. However, it suffers from certain drawbacks in terms of computational efficiency. This prevents the boundary element method from being applied to large-scale problems. This paper presents proposal of a new multiscale technique, coupled with boundary element method to speed up numerical calculations. Numerical example is given to illustrate the efficiency of the proposed method. The solution of the proposed method has been validated with conventional boundary element method and the proposed method is indeed faster in computation.

Keywords Acoustic wave model; Boundary element method; Multiscale technique.

Mathematics Subject Classification 65K05.

1 Introduction

Your acoustic waves often exist in vibration or impinged on by incident wave, which is also known as an infinite medium outside of structure. Solving acoustic wave problems can be used to predict sound field for noise control. The acoustic wave equation for one-dimension space can be written as:

$$\frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} + Q\delta\left(x, x_Q\right) = 0, \tag{1}$$

where $\varphi = \varphi\left(x, t\right)$ is the perturbation acoustic pressure at point x and time t, c is the speed of sound, and $Q\delta(x, x_Q)$ represents a possible point source located at x_Q [1].

Boundary Element Method (BEM) is an important numerical technique. Efficient in modelling and the ability to reduce the dimension of the problem are some principal advantages of the BEM over other numerical methods. The BEM mesh is much easier to generate for three dimensional problems or infinite domain problems by using the dimension reduction in the Boundary Integral Equation (BIE) formulations [2]. Integral equations and boundary value problem for systems of partial differential equations are frequently solved by using BEM [3], and as it is one of the most effective methods to solve numerical computation of the acoustic wave problems. BEM is very suitable for modeling symmetrical problems, since discretization is done only at the boundary, while computation is easy and efficient when in use. However, BEM has disadvantages. Matrices system of equations will frequently be populated in BEM in the case of non-symmetric or non-linear problems. These are not accessible by conventional BEM. This means that the storage requirements and computational time will tend to grow according to the square of the problem size [4]. BEM often suffers from drawbacks usually in terms of the computational efficiency, which leads to a linear system of equations with dense coefficient matrix. This prevents the boundary element method from being applied to large-scale problems [1].

Solving the problem of acoustic wave by just using BEM is much slower since the method is heavily loaded with numerical integration. To overcome this problem, this paper presents proposal of multiscale technique, coupled with boundary element method to speed up the large scale acoustic problems, with the help of Fortran, which is a numeric and scientific programming language. Numerical example is given to illustrate the efficiency of the proposed method. The solution of the proposed method has been validated with conventional boundary element method and the former method is indeed faster in computation.

2 Multiscale Boundary Element Method

This proposed method is aimed as improvement of BEM to solve numerical computation of the acoustic problems, by couplingmultiscale technique with BEM to speed up the acoustic problems in large-scale with the help of Fortran.

2.1 Multiscale Technique

Multiscale is also known as multi-resolution, multilevel, multigrid, etc. Past studies have demonstrated that all scale-born complexities can be effectively overcome or drastically reduced by multiscale algorithms [5]. Multiscale modeling or multiscale mathematics is the field of solving problems which have important features at multiple scales of time and space. For example, multiscale modelling and computation are required in studying natural porous media that have extreme heterogeneity [6].

For regions involving large-scale problems, a new multiscale method can be developed by breaking down the regionsfrom bigger mesh to produce smaller-scale meshes. In this study, piecewise Newton interpolation has been used, since this interpolation technique can predict better initial guess solution of higher resolution.This interpolation is used to get values at positions in between the data points. The points are simply joined by straight line segments. Each segment is bounded by two data points and can be interpolated independently.

In this study, conjugate gradient and interpolation have been implemented as a multiscale technique coupled with BEM. The positive definition of quadratic function takes the form of

$$f(x) = \frac{1}{2}x^T Qx + b^T x + c, \tag{2}$$

where x is the unknown vector, b is the known right-hand-side vector and c is a real number. $x, b \in \Re^n$, $Q = Q^T \rangle 0$, the gradient vectors $\left\{g^k\right\}$ are mutually orthogonal, as

$$\left(g^k\right)^T g^i = 0, \text{ for } i \neq k. \tag{3}$$

Moreover, the search direction vectors are mutually Q-conjugate [7]. In other words,

$$\left(d^k\right)^T Qd^i = 0, \text{ for } i \neq k. \tag{4}$$

The basic conjugate gradient method which is designed for quadratic function is applied as below,

Algorithm 1

Step 1: Set $k = 0$ to select the initial point x^0.

Step 2: $g^0 = \nabla f(x^0)$.. If $g^0 = 0$, stop; go to step 9: else, set $d^0 = -g^0$.

Step 3: $a^k = \dfrac{d^{k^T} g^k}{Qd^{k^T} d^k}$.

Step 4: $x^{k+1} = x^k + \alpha^k d^k$.

Step 5: $g^{k+1} = \nabla f\left(x^{k+1}\right)$. If $g^{k+1} = 0$, stop; go to step 9.

Step 6: $\beta^k = \dfrac{g^{(k+1)^T} Qd^k}{d^{k^T} Qd^k}$.

Step 7: $d^{k+1} = -g^{k+1} + \beta^k d^k$.

Step 8: Set $k = k + 1$; go to step 3.

Step 9: End

This method was first proposed for quadratic function; later it was futher developed into a method for general functions.

2.2 Boundary Element Method

BEM is a general numerical method for solving boundary of initial value problem by formulating it into boundary integral equations. Solving the integral equations given solution at the boundary can give an approximate solution to the problem. Conceptually, it works by constructing "a mesh" over the modelled surface. By using the dimension reduction in Boundary Integral Equation formulations, BEM mesh is much easier to generate for three dimensional problems or infinite domain problems.

The governing equation for acoustic wave problems can be written as

$$\nabla^2\varphi + k^2\varphi + Q\delta\left(x, x_Q\right) = 0, \ \ \forall x \in \mathrm{E}, \tag{5}$$

where $Q\delta(x, x_Q)$ is a typical point source located at x_Q in the acoustic domain E. $\nabla^2$is the Laplace operator. Firstly, an integral equationmust be formed from acoustic wave equation. The fundamental solution, denoted as $G\left(x, y, \omega\right)$, satisfies:

$$\nabla^2 G\left(x, y, \omega\right) + k^2 G\left(x, y, \omega\right) + \delta\left(x, y\right) = 0, \ \ \forall x, y \in \Re^2/\Re^3. \tag{6}$$

The derivative is taken at field point y and the dirac delta function, which represents the unit derivative, is taken at field x. The diracdelta function, $\delta\left(x, y\right)$ in two and three dimensions, has the following sifting properties:

$$\int_V f(y)\,\delta(x,y)\,dV(y) = \begin{cases} f(x), & \text{if } x \in V, \\ 0, & \text{if } x \notin V \cup S', \end{cases} \tag{7}$$

$$\int_V f(y)\,\frac{\partial}{\partial x_i}\delta(x,y)\,dV(y) = \begin{cases} -\dfrac{\partial}{\partial x_i} f(x), & \text{if } x \in V, \\ 0, & \text{if}\,, x \notin V \cup S'. \end{cases} \tag{8}$$

The fundamental solution is given by [8,1,9] as shown in equation (9)

$$G(x,y,\omega) = \begin{cases} \dfrac{i}{4} H_0^{(1)}(kr)\,, & \text{for 2D}, \\ \dfrac{1}{4\pi r} e^{ikr}, & \text{for 3D} \end{cases} \tag{9}$$

where r is the distance between the source point x and field point y, and its normal derivative is expressed in [1] and is given by equation (10)

$$F(x,y,\omega) \equiv \frac{\partial G(x,y,\omega)}{\partial n(y)} = \begin{cases} \dfrac{i}{4} H_1^{(1)}(kr)\, r_{,l}\, n_{/}(y)\,, & \text{for 2D}, \\ \dfrac{1}{4\pi r^2}(ikr-1)\, r_{,j}\, n_j(y)\, e^{ikr}, & \text{for 3D} \end{cases} \tag{10}$$

where $H_n^{(1)}()$ denotes the Hankel function of the first kind, j is subscripts for coordinate components, and n is the component of the outward normal.

In this method, the Green-Gauss theorem is used next, which is multi-dimensionally equivalent in terms of integration by parts, where E is a domain bounded by boundary S of the structure, expressed as:

$$\int_E \left[u\nabla^2 v - v\nabla^2 u\right] dE = \int_S \left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right] dS, \tag{11}$$

for any two continuous functions u and v. Let $v(y) = \varphi(y,)$ which satisfies Equation (5), and $u(y) = G(x,y,\omega)$ which satisfies Equation (6). From Equation (11), the following equation is formed [1,10,11,12]:

$$\int_E \left[G\nabla^2 \varphi - \varphi\nabla^2 G\right] dE = \int_S \left[G\frac{\partial \varphi}{\partial n} - \varphi\frac{\partial G}{\partial n}\right] dS. \tag{12}$$

Applying Equations (5), (6) and (7) yields

$$\varphi(x) = \int_S \left[G(x,y,\omega)\,q(y) - F(x,y,\omega)\,\varphi(y)\right] dS(y) + QG(x,x_Q,\omega),\ \forall x \in S, \tag{13}$$

where $q = \partial\varphi/\partial n$ and $QG(x,x_Q,\omega)$ are due to the point source at x_Q. Equation (13) is the integral equation of the solution φ that represents the acoustic problem (5) which is inside domain S.

3 Numerical Example

Numerical example in 2D is presented in this section to demonstrate the efficiency and speed of the multiscale Boundary Element Method for the numerical computation of acoustic wave problems, as comparisonto Boundary Element Method. All computations have been done using Fortran compiler. Average error $\bar{E}$ and average speed up rate between mesh have been compared as well. The formula of average error is defined as

$$\bar{E} = \frac{\int E dx}{\int 1 dx} \tag{14}$$

which the formula of average speed up rate is defined as

$$\bar{S} = \frac{\text{Total speed up BEM}}{\text{Total speed up MBEM}} \times 100\%. \tag{15}$$

Accordingly, the following governing equation for acoustic wave equations has been considered.

$$\nabla^2 \varphi + k^2 \varphi + Q\delta\left(x, x_Q\right) = 0, \;\; \forall x \in S. \tag{16}$$

The boundary conditions considered are Neumann boundary condition $q \equiv \partial\varphi/\partial n = \bar{q}$ on S, in which the over bar indicates the prescribed value for the function. Here, φ is the perturbation acoustic pressure, S is the boundary of the domain, and n is the outward normal of the boundary S. Figure 1 shows the boundary conditions, and Figure 2 shows the mesh of the graph of the region that has been discretized by using multiscale technique to produce smaller-sized mesh.

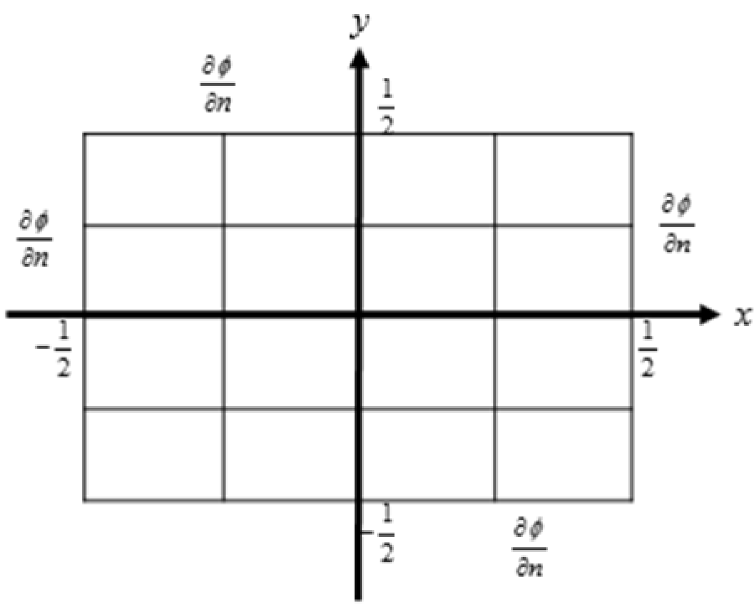

Figure 1: Neumann Boundary Conditions

Example

$$\varphi\left(x\right) = \int_S \left[G\left(x, y, \omega\right) q\left(y\right) - F\left(x, y, \omega\right) \varphi\left(y\right)\right] dS\left(y\right) + QG\left(x, x_Q, \omega\right), \;\; \forall x \in S.$$

Figure 3 shows the boundary conditions of the problem.

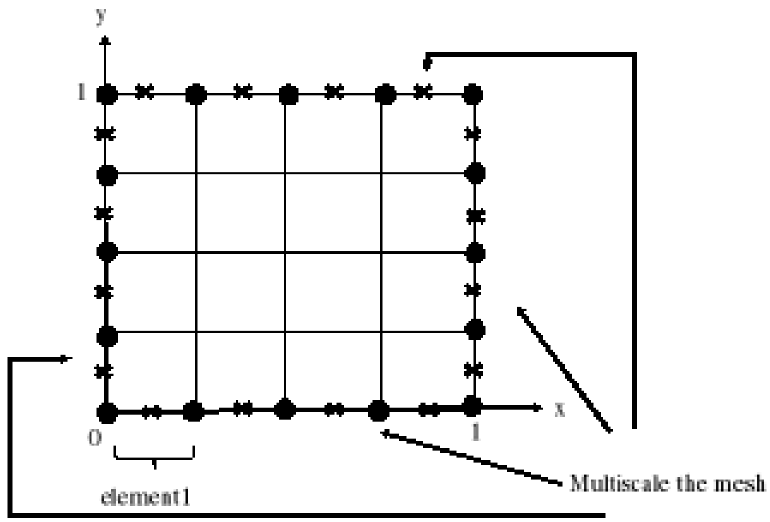

Figure 2: Mesh of the Problem

4 Results and Discussion

Table 1 shows the total CPU time used to solve the problem between 32, 64, 128 and 256 sizes by the use of BEM and MBEM. Table 2 shows the average error between mesh, while Table 3 shows the average speed up rate between meshes.

The efficiency of Multiscale Boundary Element Method for solving acoustic problem has been proven by the results.Themultiscale technique approach, by combination of conjugate gradient and interpolation, can significantly improve the conditioning of the Boundary Element Method systems of equations, thus can facilitate faster convergence when the multiscale Boundary Element Method is applied.

Table 1 shows the comparison of total CPU time in seconds. The total CPU times for 32, 64, 128 and 256 sizes by using the Multiscale Boundary Element Method prove that it is clearly faster than Boundary Element Method. Table 2 displays comparison of average error between meshes. The average error indicates that the solution is closer to the exact solution when the mesh is larger. Evidently, sizes 32, 64, 128 and 256 are more efficient than initial size 16.

Table 1: Total CPU Time Used by Boundary Element Method and Multiscale Boundary Element Method

n size	BEM	MBEM
	Total CPU time used (second)	Total CPU time used (second)
32	0.0156	0.0145
64	0.0625	0.0468
128	0.3437	0.1094
256	0.5209	0.1273

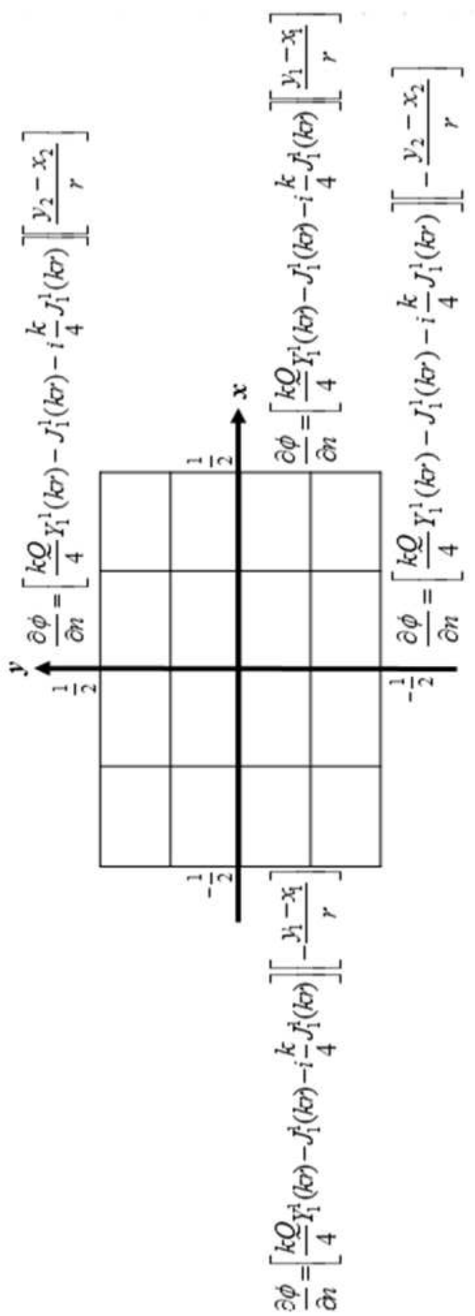

Figure 3: Boundary Conditions

Table 2: Average Error between Meshes by Boundary Element Method and Multiscale Boundary Element Method

n size	BEM	MBEM
	Average error, $\overline{E}$	Average error, $\overline{E}$
16	0.065107	0.004138
32	0.041841	0.002496
64	0.004045	0.002002
128	0.002502	0.001786
256	0.001071	0.000802

Table 3: Average Speed up Rate Between Meshes

n size	Average speed up rate (%)
32	0.01075862
64	0.0133547
128	0.03141682
256	0.04091909

The comparison of the average error between the two methods proves that the Multiscale Boundary Element Method is more efficient than Boundary Element Method. Meanwhile, the comparison of the average speed up rate between meshes in Table 3 shows that speed is faster when the mesh is larger. The numerical example presented here clearly demonstrates the efficiency of the proposed Multiscale Boundary Element Method for solving acoustic problem.

5 Conclusion

Based on numerical results, it can be concluded that the Multiscale Boundary Element Method is faster compared with Boundary Element Method. This paper is expected to establish a numerical library for the solution of numerical computation of acoustic equation. The proposed method can be used as a reference for future studies in many fields of science and engineering. For validation purposes against other (future) experimental and numerical results, the numerical results obtained will serve as reference and can be used. More researches need to be done to improve the Boundary Element Method.

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