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An Integral Equation Method for Conformal Mapping of Doubly Connected Regions

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Abstract Based on a boundary relationship satisfied by a function which is analytic in a doubly connected region bounded by two closed Jordan curves, an integral equation is constructed. Some applications considered are the conformal mappings from a doubly connected region bounded by two closed smooth Jordan curves onto: (a) an annulus, and (b) a unit disk. Among the kernels involved are the Kerzman-Stein and the Neumann kernels.

Keywords Conformal mapping, Integral equation, Doubly connected region, Kerzman-Stein kernel, Neumann kernel.

Abstrak *Berdasarkan kepada hubungan sempadan yang ditepati oleh satu fungsi yang analisis dalam rantau terkait ganda dua yang dibatasi antara dua lengkung Jordan, sebuah persamaan kamiran dibina. Beberapa penggunaannya ialah pemetaan konformal dari rantau terkait ganda dua antara dua lengkung Jordan ke: (a) anulus (b) bulatan unit. Antara inti kamiran yang terlibat ialah inti Kerzman-Stein dan inti Neumann.*

Katakunci *Pemetaan konformal, Persamaan kamiran, Rantau terkait ganda dua, Inti Kerzman-Stein, Inti Neumann.*

1 Introduction

Integral equation methods for conformal mapping of interior, exterior and doubly connected regions are presently still a subject of interest. The classical integral equation methods of Symm [17, 18, 19] are well known for computing these maps by means of the singular Fredholm integral equations of the first kind. Various reformulations and modifications of Symm's method have been considered by Hough and Papamichael [10], Henrici [7], Berrut [4] and Reichel [16].

Conformal mappings of interior, exterior and doubly connected regions based on integral equations of the second kind are also encountered in the literature [1, 5, 8, 20]. Some notable ones are the integral equations of Warschawski, Banin, Lichtenstein, Gerschgorin, and Kerzman-Stein-Trummer. Among these, the Kerzman-Stein-Tummer integral equation (briefly, KST integral equation) is more recent. The KST integral equation, whose unique solution is the Szegö kernel, was derived by Kerzman and Trummer [11] using operator-theoretic approach. Henrici [8, pp. 560-563] gave a markedly different derivation of the KST integral equation based on a function-theoretic approach. Henrici's approach was recently extended in [15] to derive an integral equation of the second kind whose unique solution is the Bergman kernel. Both the Szegö and Bergman kernels are related to the Riemann map, which is a conformal map of interior regions onto a unit disk.

Crucial to Henrici's approach in deriving the KST integral equation for the Szegö kernel [8] and the adaptation of the approach in deriving the integral equation for the Bergman kernel is the used of certain boundary relationships satisfied by the Szegö and the Bergman kernels. Thus recently in [12], based on a certain boundary relationship satisfied by a function which is analytic in a region interior to a closed Jordan curve, an integral equation is constructed. Special realizations of this integral equation are the integral equations for the Szegö kernel and the Bergman kernel. Similar construction of an integral equation based on a certain boundary relationship satisfied by a function which is analytic in a region exterior to a closed Jordan curve has been given in [13]. This result gives rise to an integral equation that proves to be useful for the purpose of numerical exterior mappings.

In this paper we extend the above construction to a doubly connected region. Using a boundary relationship satisfied by a function which is analytic in a doubly connected region bounded by two closed Jordan curves, we construct an integral equation related to the boundary relationship. Some applications considered are the conformal mappings from a doubly connected region bounded by two closed smooth Jordan curves onto: (a) an annulus, and (b) a unit disk. The second mapping (b) is also known as the Ahlfors map (see e.g. Bell [2, 3]).

2 An Integral Equation Related to a Boundary Relationship

Let Γ_0 and Γ_1 be two smooth Jordan curves in the complex plane such that Γ_1 lies in the interior of Γ_0. Denote by Ω the finite doubly connected region bounded by Γ_0 and Γ_1. The positive direction of the contour $\Gamma = \Gamma_0 \cup \Gamma_1$ is usually that for which Ω is on the left as one traces the boundary (see Fig. 1).

It is well known that if h is analytic and single-valued in Ω and continuous on $\Omega \cup \Gamma$, we have [9, p. 176]

$$\frac{1}{2\pi i}\int_\Gamma \frac{h(w)}{w-z}\,dw = \frac{1}{2}h(z), \quad z \in \Gamma. \tag{1}$$

Suppose $D(z)$ is analytic and single-valued with respect to $z \in \Omega$ and is continuous on $\Omega \cup \Gamma$. Suppose further that D satisfies the boundary relationship

$$D(z) = c(z)\left[\frac{T(z)Q(z)D(z)}{P(z)}\right]^{-}, \qquad z \in \Gamma, \tag{2}$$

where the minus sign in the superscript denotes complex conjugation, $T(z) = z'(t)/|z'(t)|$ is the complex unit tangent function at $z \in \Gamma$, while c, P, and Q are complex-valued functions defined on Γ with the following properties:

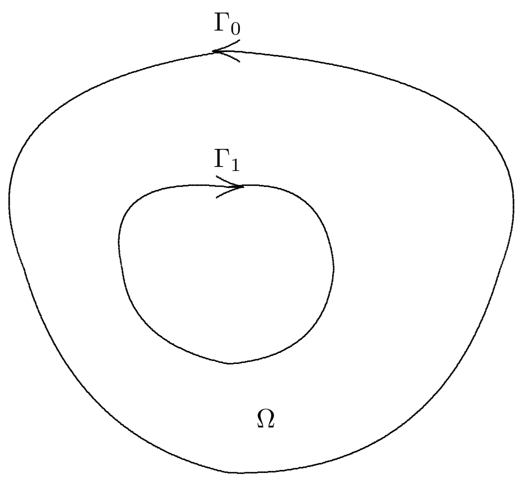

Figure 1: A Doubly Connected Region.

(P1) $c(z) = \begin{cases} c_0, & \text{if } z \in \Gamma_0 \\ c_1, & \text{if } z \in \Gamma_1 \end{cases}$, where c_0 and c_1 are complex constants,

(P2) $P(z)$ is analytic and single-valued with respect to $z \in \Omega$,

(P3) $P(z)$ is continuous on $\Omega \cup \Gamma$,

(P4) P has a finite number of zeroes at $a_1, a_2, ..., a_n$ in Ω,

(P5) $P(z) \neq 0, Q(z) \neq 0, z \in \Gamma$.

Note that the boundary relationship (2) also has the following equivalent form:

$$P(z) = \overline{c(z)} \, \frac{T(z)Q(z)D(z)^2}{|D(z)|^2}, \qquad z \in \Gamma. \tag{3}$$

By means of (1), an integral equation for D may be constructed that is related to the boundary relationship (2) as shown below.

Theorem 2.1 *Let u and v be any complex-valued functions that are defined on Γ. Then*

$$\frac{1}{2}\left[v(z) + \frac{u(z)}{\overline{T(z)Q(z)}}\right] D(z) + PV\frac{1}{2\pi i}\int_{\Gamma}\left[\frac{u(z)}{(\overline{w}-\overline{z})\overline{Q(w)}} - \frac{v(z)T(w)}{w-z}\right] D(w)\,|dw|$$

$$= -c(z)\, u(z) \left[\sum_{a_j \; inside \; \Gamma} \operatorname*{Res}_{w=a_j} \frac{D(w)}{(w-z)P(w)}\right]^{-}$$

$$-u(z)(c_0 - c_1)\left[\frac{1}{2\pi i}\int_{\Gamma_2} \frac{D(w)}{(w-z)P(w)}\,dw\right]^{-}, \qquad z \in \Gamma, \tag{4}$$

where the minus sign in the superscript denotes complex conjugation and where

$$\Gamma_2 = \begin{cases} -\Gamma_1, & \text{if } z \in \Gamma_0, \\ \Gamma_0, & \text{if } z \in \Gamma_1. \end{cases}$$

Proof. Consider the integral

$$I_1(z) = \mathrm{PV}\frac{1}{2\pi i}\int_\Gamma \frac{v(z)T(w)D(w)}{w-z}\,|dw|, \qquad z \in \Gamma.$$

Using $T(w)|dw| = dw$ and (1), since D is analytic on Ω, we obtain

$$I_1(z) = \frac{1}{2}v(z)D(z), \qquad z \in \Gamma.$$

Next we consider the integral

$$I_2(z) = \mathrm{PV}\frac{1}{2\pi i}\int_\Gamma \frac{u(z)D(w)}{(\overline{w}-\overline{z})\overline{Q(w)}}\,|dw|, \qquad z \in \Gamma.$$

Using the boundary relationship (2) and $\overline{T(w)}|dw| = \overline{dw}$, we get

$$I_2(z) = -u(z)\left[\mathrm{PV}\frac{1}{2\pi i}\int_\Gamma \frac{\overline{c(w)}D(w)}{(w-z)P(w)}\,dw\right]^-.$$

Taking property (P1) into account, we obtain

$$I_2(z) = -u(z)\left[\overline{c_0}\,\mathrm{PV}\frac{1}{2\pi i}\int_{\Gamma_0} \frac{D(w)}{(w-z)P(w)}\,dw + \overline{c_1}\,\mathrm{PV}\frac{1}{2\pi i}\int_{\Gamma_1} \frac{D(w)}{(w-z)P(w)}\,dw\right]^-.$$

This can be written in two different ways, i.e. either

$$I_2(z) = -u(z)\left[\overline{c_0}\,\mathrm{PV}\frac{1}{2\pi i}\int_{\Gamma} \frac{D(w)}{(w-z)P(w)}\,dw + (\overline{c_1}-\overline{c_0})\,\mathrm{PV}\frac{1}{2\pi i}\int_{\Gamma_1} \frac{D(w)}{(w-z)P(w)}\,dw\right]^-. \tag{5}$$

or

$$I_2(z) = -u(z)\left[(\overline{c_0}-\overline{c_1})\,\mathrm{PV}\frac{1}{2\pi i}\int_{\Gamma_0} \frac{D(w)}{(w-z)P(w)}\,dw + \overline{c_1}\,\mathrm{PV}\frac{1}{2\pi i}\int_{\Gamma} \frac{D(w)}{(w-z)P(w)}\,dw\right]^-. \tag{6}$$

By substituting $z = z_0 \in \Gamma_0$ in (5) and applying the residue theory and formula (1) to the first integral, (5) becomes

$$\begin{aligned} I_2(z_0) &= -u(z_0)\left[\frac{\overline{c_0}}{2}\frac{D(z_0)}{P(z_0)} + \overline{c_0}\sum_{j=1}^{n}\operatorname*{Res}_{w=a_j}\frac{D(w)}{(w-z_0)P(w)} + \frac{\overline{c_1}-\overline{c_0}}{2\pi i}\int_{\Gamma_1}\frac{D(w)}{(w-z_0)P(w)}\,dw\right]^- \\ &= -u(z_0)\frac{c_0}{2}\frac{\overline{D(z_0)}}{\overline{P(z_0)}} - c_0\,u(z_0)\left[\sum_{j=1}^{n}\operatorname*{Res}_{w=a_j}\frac{D(w)}{(w-z_0)P(w)}\right]^- \\ &\qquad -u(z_0)(c_1-c_0)\left[\frac{1}{2\pi i}\int_{\Gamma_1}\frac{D(w)}{(w-z_0)P(w)}\,dw\right]^-. \end{aligned}$$

Applying the boundary relationship (2) to the first term on the right hand side yields

$$I_2(z_0) = -\frac{u(z_0)D(z_0)}{2\overline{T(z_0)Q(z_0)}} - c_0\, u(z_0)\left[\sum_{j=1}^{n} \operatorname*{Res}_{w=a_j} \frac{D(w)}{(w-z_0)P(w)}\right]^{-} \\ -u(z_0)(c_1-c_0)\left[\frac{1}{2\pi i}\int_{\Gamma_1} \frac{D(w)}{(w-z_0)P(w)}\,dw\right]^{-}. \qquad (7)$$

Next we substitute $z = z_1 \in \Gamma_1$ in (6) and apply the residue theory and formula (1) to the second integral, in which (6) becomes

$$I_2(z_1) = -u(z_1)\left[\frac{\overline{c_1}}{2}\frac{D(z_1)}{P(z_1)} + \overline{c_1}\sum_{j=1}^{n} \operatorname*{Res}_{w=a_j} \frac{D(w)}{(w-z_1)P(w)} + \frac{\overline{c_0}-\overline{c_1}}{2\pi i}\int_{\Gamma_0} \frac{D(w)}{(w-z_1)P(w)}\,dw\right]^{-}$$

$$= -u(z_1)\frac{c_1}{2}\frac{\overline{D(z_1)}}{\overline{P(z_1)}} - c_1\, u(z_1)\left[\sum_{j=1}^{n} \operatorname*{Res}_{w=a_j} \frac{D(w)}{(w-z_1)P(w)}\right]^{-} \\ -u(z_1)(c_0-c_1)\left[\frac{1}{2\pi i}\int_{\Gamma_0} \frac{D(w)}{(w-z_1)P(w)}\,dw\right]^{-}.$$

Applying the boundary relationship (2) to the first term on the right hand side yields

$$I_2(z_1) = -\frac{u(z_1)D(z_1)}{2\overline{T(z_1)Q(z_1)}} - c_1\, u(z_1)\left[\sum_{j=1}^{n} \operatorname*{Res}_{w=a_j} \frac{D(w)}{(w-z_1)P(w)}\right]^{-} \\ -u(z_1)(c_0-c_1)\left[\frac{1}{2\pi i}\int_{\Gamma_0} \frac{D(w)}{(w-z_1)P(w)}\,dw\right]^{-}. \qquad (8)$$

In short (7) and (8) can be combined as

$$I_2(z) = -\frac{u(z)D(z)}{2\overline{T(z)Q(z)}} - c(z)\, u(z)\left[\sum_{j=1}^{n} \operatorname*{Res}_{w=a_j} \frac{D(w)}{(w-z)P(w)}\right]^{-} \\ -u(z)(c_0-c_1)\left[\frac{1}{2\pi i}\int_{\Gamma_2} \frac{D(w)}{(w-z)P(w)}\,dw\right]^{-},$$

where Γ_2 is defined as in Theorem 2.1. Finally looking at $I_2(z) - I_1(z)$ yields

$$\mathrm{PV}\frac{1}{2\pi i}\int_{\Gamma} \frac{u(z)D(w)}{(\overline{w}-\overline{z})\overline{Q(w)}}\,|dw| - \mathrm{PV}\frac{1}{2\pi i}\int_{\Gamma} \frac{v(z)T(w)D(w)}{w-z}\,|dw|$$

$$= -\frac{u(z)D(z)}{2\overline{T(z)Q(z)}} - c(z)\, u(z)\left[\sum_{j=1}^{n} \operatorname*{Res}_{w=a_j} \frac{D(w)}{(w-z)P(w)}\right]^{-} \\ -u(z)(c_0-c_1)\left[\frac{1}{2\pi i}\int_{\Gamma_2} \frac{D(w)}{(w-z)P(w)}\,dw\right]^{-} - \frac{1}{2}v(z)D(z),$$

which is equivalent to (4). This completes the proof.
Remark 1. If $P(z)$ does not have any zeros in Ω, then the first term of the right hand side of (4) will not appear.

3 Application to Conformal Mapping

Let $w = f(z)$ be the analytic function which maps the doubly connected region Ω bounded by the two smooth Jordan curves Γ_0 and Γ_1 onto the annulus $A = \{w : \mu < |w| < 1\}$ so that Γ_0 and Γ_1 correspond respectively to $|w| = 1$ and $|w| = \mu$ (see Figure 2 below).

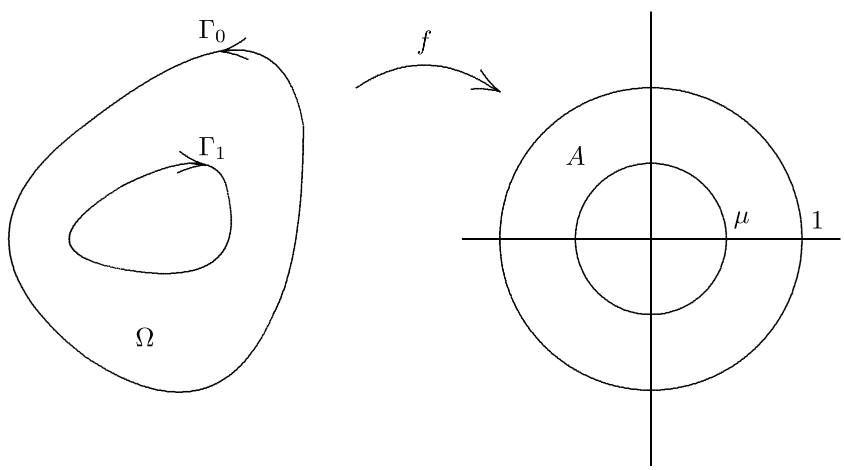

Figure 2: Conformal Mapping of Ω Onto A.

As is well known such a mapping function f exists up to a rotation of the annulus, and the function f can be made unique by prescribing that

$$f(a) > 0$$

for some predetermined point $a \in \Omega$ (see Henrici [8, Chap. 17] and Nehari [14, Chap. 7]). Due to Γ being analytic, the mapping function f can be extended to a function that is analytic on $\Omega \cup \Gamma$. Moreover, f' is different from zero and analytic in $\Omega \cup \Gamma$. Thus a single-valued analytic square root of $f'(z)$, denoted by $\sqrt{f'(z)}$, may be defined on $\Omega \cup \Gamma$ (for details, see Bell [2, pp. 38-44]).

Suppose the boundary curves Γ_0 and Γ_1 have the following parametric representations:

$$\Gamma_0 : z = z_0(t), \quad 0 \le t \le \beta_0,$$
$$\Gamma_1 : z = z_1(t), \quad 0 \le t \le \beta_1.$$

Hence

$$f(z_0(t)) = e^{i\theta_0(t)}, \quad 0 \le t \le \beta_0, \tag{9}$$
$$f(z_1(t)) = \mu e^{i\theta_1(t)}, \quad 0 \le t \le \beta_1, . \tag{10}$$

Then it can be shown that

$$f(z_0(t)) = \frac{1}{i} T(z_0(t)) \frac{f'(z_0(t))}{|f'(z_0(t))|}, \tag{11}$$

$$f(z_1(t)) = \frac{\mu}{i} T(z_1(t)) \frac{f'(z_1(t))}{|f'(z_1(t))|}. \tag{12}$$

The boundary relationships (11) and (12) can be combined as

$$f(z) = \frac{|f(z)|}{i} T(z) \frac{f'(z)}{|f'(z)|}, \qquad z \in \Gamma. \tag{13}$$

Note that the values of $|f(z)|$ is either 1 or μ for $z \in \Gamma$.

By comparing (13) with (3) yields three possible assignments:

(A1) $c(z) = i|f(z)|$, $P(z) = f(z)$, $D(z) = \sqrt{f'(z)}$, $Q(z) = 1$.

(A2) $c(z) = i$, $P(z) = f(z)$, $D(z) = \sqrt{f'(z)}$, $Q(z) = |f(z)|$.

(A3) $c(z) = i$, $P(z) = 1$, $D(z) = \sqrt{f'(z)/f(z)}$, $Q(z) = 1$.

By means of Theorem 2.1 and Remark 1, the integral equation related to the boundary relationship (13) with the (A1) assignments, along with the choice of $u(z) = \overline{T(z)Q(z)}$ and $v(z) = 1$, is

$$\sqrt{f'(z)} + \int_\Gamma A(z,w)\sqrt{f'(w)}\,|dw| = -i(1-\mu)\overline{T(z)}\left[\frac{1}{2\pi i}\int_{\Gamma_2} \frac{\sqrt{f'(w)}}{(w-z)f(w)}\,dw\right]^-, \quad z \in \Gamma, \tag{14}$$

where

$$A(z,w) = \begin{cases} \overline{H(w,z)} - H(z,w), & \text{if } w, z \in \Gamma, w \neq z \\ 0, & \text{if } w = z \in \Gamma, \end{cases} \tag{15}$$

and

$$H(w,z) = \frac{1}{2\pi i}\frac{T(z)}{z-w}, \qquad w \in \Omega \cup \Gamma, z \in \Gamma, w \neq z. \tag{16}$$

The kernel A is known as the Kerzman-Stein kernel [11] and is smooth and skew-Hermitian. The kernel H is usually referred to as the Cauchy kernel. It remains a disadvantage of the integral equation (14) that μ is unknown and the evaluation of the integral on the right hand side is yet undetermined.

As for the assignment (A2), since $Q(z) = |f(z)|$, the application of Theorem 2.1 will yield an integral equation with its kernel involving μ which is unknown. Theorem 2.1 may not be applied to assignment (A3) since $D(z)$ is not analytic and single-valued in Ω as required.

If we now square both sides of the boundary relationship (13), we get

$$f(z)^2 = -|f(z)|^2 T(z)^2 \frac{f'(z)^2}{|f'(z)|^2}, \qquad z \in \Gamma. \tag{17}$$

By comparing (17) with (3) yields still another three possible assignments:

(B1) $c(z) = -|f(z)|^2$, $P(z) = f(z)^2$, $D(z) = f'(z)$, $Q(z) = T(z)$.

(B2) $c(z) = -1$, $P(z) = f(z)^2$, $D(z) = f'(z)$, $Q(z) = |f(z)|^2 T(z)$.

(B3) $c(z) = -1$, $P(z) = 1$, $D(z) = f'(z)/f(z)$, $Q(z) = T(z)$.

Application of Theorem 2.1 with Remark 1 to the case (B1), along with the choice of $u(z) = \overline{T(z)Q(z)}$ and $v(z) = 1$, gives

$$f'(z) + \int_\Gamma M(z,w) f'(w)\, |dw| = (1-\mu^2)\overline{T(z)}^2 \left[\frac{1}{2\pi i}\int_{\Gamma_2} \frac{f'(w)}{(w-z)f(w)^2}\, dw\right]^-, \quad z \in \Gamma, \tag{18}$$

where M is the kernel defined by [15]

$$M(z,w) = \begin{cases} \dfrac{T(w)}{2\pi i}\left[\dfrac{\overline{T(z)^2}}{\overline{w}-\overline{z}} - \dfrac{1}{w-z}\right], & \text{if } w, z \in \Gamma, w \neq z \\ \dfrac{1}{2\pi}\dfrac{\mathrm{Im}\left[z''(t)\overline{z'(t)}\right]}{|z'(t)|^3}, & \text{if } w = z \in \Gamma. \end{cases} \tag{19}$$

If we multiply both sides of (18) by $T(z)$ and use the fact that $T(z)\overline{T(z)} = |T(z)|^2 = 1$, we obtain

$$T(z)f'(z) + \int_\Gamma N(z,w)T(w) f'(w)\, |dw| = (1-\mu^2)\overline{T(z)} \left[\frac{1}{2\pi i}\int_{\Gamma_2} \frac{f'(w)}{(w-z)f(w)^2}\, dw\right]^-, \tag{20}$$

for $z \in \Gamma$ and where N is the Neumann kernel defined by [8, p. 282]

$$N(z,w) = \begin{cases} \dfrac{1}{\pi}\mathrm{Im}\left[\dfrac{T(z)}{z-w}\right], & \text{if } w, z \in \Gamma, w \neq z \\ \dfrac{1}{2\pi}\dfrac{\mathrm{Im}\left[z''(t)\overline{z'(t)}\right]}{|z'(t)|^3}, & \text{if } w = z \in \Gamma. \end{cases} \tag{21}$$

Besides the presence of the unknown μ, the integral equations (18) and (20) have the disadvantage that the integrals on their right hand side are also undetermined.

As for the assignment (B2), since $Q(z) = |f(z)|^2 T(z)$, the application of Theorem 2.1 will yield an integral equation with its kernel involving μ which is unknown.

Application of Theorem 2.1 with Remark 1 to the case with (B3) assignment, along with the choice of $u(z) = \overline{T(z)Q(z)}$ and $v(z) = 1$, gives

$$\frac{f'(z)}{f(z)} + \int_\Gamma M(z,w)\frac{f'(w)}{f(w)}\, |dw| = 0, \quad z \in \Gamma, \tag{22}$$

where M is the kernel as defined in (19). Note that we have also used the fact that $c(z) = i$ implies $c_0 = c_1 = i$. If we multiply both sides of (22) by $T(z)$, we obtain

$$T(z)\frac{f'(z)}{f(z)} + \int_\Gamma N(z,w)T(w)\frac{f'(w)}{f(w)}\, |dw| = 0, \quad z \in \Gamma, \tag{23}$$

where N is again the Neumann kernel.

Equation (23) has an interesting consequence. From (9) and (10), we see that

$$f'(z_0(t))z_0'(t) = i\,\theta_0'(t)e^{i\theta_0(t)} = i\,f(z_0(t))\theta_0'(t), \quad 0 \le t \le \beta_0,$$
$$f'(z_1(t))z_1'(t) = i\,\mu\theta_1'(t)e^{i\theta_1(t)} = i\,f(z_1(t))\theta_1'(t), \quad 0 \le t \le \beta_1.$$

In short,

$$f'(z(\tau))z'(\tau) = i\,f(z(\tau))\theta'(\tau), \quad z(\tau) \in \Gamma, 0 \le \tau \le \beta,$$

which implies

$$z'(\tau)\frac{f'(z(\tau))}{f(z(\tau))} = i\,\theta'(\tau), \quad z(\tau) \in \Gamma, 0 \le \tau \le \beta.$$

Substituting into (23) and using the definition that $T(z(\tau)) = z'(\tau)/|z'(\tau)|$, we get

$$\theta'(\tau) + \int_0^\beta k(\tau,\sigma)\theta'(\sigma)\,d\sigma = 0, \quad 0 \le \tau \le \beta, \tag{24}$$

where

$$k(\tau,\sigma) = |z'(\tau)|N(z(\tau),z(\sigma)) = \begin{cases} \dfrac{1}{\pi}\mathrm{Im}\left[\dfrac{z'(\tau)}{z(\tau)-z(\sigma)}\right], & \text{if } \tau \ne \sigma \\ \dfrac{1}{2\pi}\mathrm{Im}\left[\dfrac{z''(\tau)}{z'(\tau)}\right], & \text{if } \tau = \sigma. \end{cases}$$

The integral equation (24) is also known as the Warschawski's integral equation for the doubly connected region [8, p. 466].

4 Application to the Ahlfors Map

Besides the annulus, another canonical region that could serve for the conformal mapping of a doubly connected region is the unit disk. Such a conformal map is known as the Ahlfors map. The Ahlfors map shares many of the geometric features one would expect of a "Riemann mapping function" of a multiply connected region.

As before, let Ω be a doubly connected region bounded by two smooth Jordan curves Γ_0 and Γ_1. Then it is well known that the Ahlfors map $w = F(z)$ is the unique single-valued analytic function that maps Ω onto a unit disk $|w| < 1$ that is covered precisely two times and such that (see Nehari [14, p. 378] and Bell [3, pp. 47-52])

$$F(a) = 0, \quad F'(a) > 0,$$

for a fixed $a \in \Omega$. Thus the map F is a two-to-one map. Also F extends to be continuous in $\Omega \cup \Gamma$, where $\Gamma = \Gamma_0 \cup \Gamma_1$, F' is nonvanishing on the boundary, and F maps each boundary curves Γ_0 and Γ_1 one-to-one manner onto the boundary of the unit disk. The point $a \in \Omega$ is a simple zero for F. Since F is a two-to-one map, it must has another simple zero besides the one at a. Denote this other zero by b.

Suppose the boundary curves Γ_0 and Γ_1 have the parametric representations given by

$$\Gamma_0 : z = z_0(t), \quad 0 \le t \le \beta_0,$$
$$\Gamma_1 : z = z_1(t), \quad 0 \le t \le \beta_1.$$

Hence

$$F(z_0(t)) = e^{i\theta_0(t)}, \quad 0 \le t \le \beta_0, \tag{25}$$
$$F(z_1(t)) = e^{i\theta_1(t)}, \quad 0 \le t \le \beta_1, . \tag{26}$$

In short we have

$$F(z(\tau)) = e^{i\theta(\tau)}, \quad z(\tau) \in \Gamma, 0 \le \tau \le \beta. \tag{27}$$

Then it can be shown that

$$F(z) = \frac{T(z)}{i} \frac{F'(z)}{|F'(z)|}, \qquad z \in \Gamma. \tag{28}$$

Hence to solve the Ahlfors map it is sufficient to compute the boundary values of $F'(z)$.

4.1 Connection with the Kerzman-Stein Kernel

Comparison of (28) with (3) yields the assignment

$$P(z) = F(z), c(z) = i, Q(z) = 1, D(z) = \sqrt{F'(z)}.$$

By means of Theorem 2.1, the integral equation related to the boundary relationship (28), along with the choice of $u(z) = \overline{T(z)Q(z)}$ and $v(z) = 1$, is

$$\sqrt{F'(z)} + \int_\Gamma A(z,w)\sqrt{F'(w)}\,|dw| = -i\overline{T(z)} \left[\sum_{j=1}^{2} \operatorname*{Res}_{w=a_j} \frac{\sqrt{F'(w)}}{(w-z)F(w)} \right]^{-}, \quad z \in \Gamma, \tag{29}$$

where $a_1 = a$, $a_2 = b$ and A being the Kerzman-Stein kernel as defined in (15). Since a and b are simple zeros of F, by means of residue theory, we get

$$\sqrt{F'(z)} + \int_\Gamma A(z,w)\sqrt{F'(w)}\,|dw| = -i\overline{T(z)} \left[\frac{\sqrt{F'(a)}}{(a-z)F'(a)} + \frac{\sqrt{F'(b)}}{(b-z)F'(b)} \right]^{-}.$$

Multiplying both sides by $\sqrt{F'(a)}$ and using the fact that $F'(a) > 0$, gives

$$\sqrt{F'(a)F'(z)} + \int_\Gamma A(z,w)\sqrt{F'(a)F'(w)}\,|dw| = -i\overline{T(z)} \left[\frac{1}{(a-z)} + \frac{\sqrt{F'(a)}}{(b-z)\sqrt{F'(b)}} \right]^{-}, \tag{30}$$

which resembles very much like the Kerzman-Stein integral equation for the Szegö kernel in the case of a simply connected region [11].

4.2 Connection with the Neumann Kernel

If we now square both sides of the boundary relationship (28), we get

$$F(z)^2 = -T(z)^2 \frac{F'(z)^2}{|F'(z)|^2}, \qquad z \in \Gamma. \tag{31}$$

This time comparison of (31) with (3) yields the assignments:

$$c(z) = -1, P(z) = F(z)^2, D(z) = F'(z), Q(z) = T(z).$$

Application of Theorem 2.1 again, along with the choice of $u(z) = \overline{T(z)Q(z)}$ and $v(z) = 1$, gives the integral equation

$$F'(z) + \int_\Gamma M(z,w)F'(w)\,|dw| = \overline{T(z)}^2 \left[\sum_{j=1}^{2} \operatorname*{Res}_{w=a_j} \frac{F'(w)}{(w-z)F(w)^2}\right]^-, \quad z \in \Gamma, \tag{32}$$

where M is the kernel defined by (19). If we multiply both sides of (32) by $T(z)$ and apply the fact that $T(z)\overline{T(z)} = |T(z)|^2 = 1$ and the residue theory, we obtain

$$T(z)F'(z) + \int_\Gamma N(z,w)T(w)F'(w)\,|dw| = -\frac{\overline{T(z)}}{F'(a)(\overline{a}-\overline{z})^2} + \overline{T(z)}\left[\operatorname*{Res}_{w=b} \frac{F'(w)}{(w-z)F(w)^2}\right]^-, \quad z \in \Gamma, \tag{33}$$

where N is the Neumann kernel defined in (21). The integral equations (32) and (33) closely resemble the integral equations for the Bergman kernel in the case of a simply connected region [15].

4.3 Connection with the Szegö Kernel

If D is a n-connected region, Bell [2] has shown that the Ahlfors map F and the Szegö kernel function S are related on the boundary Γ of Ω by

$$F(z) = \frac{S(z,a)T(z)}{i\,S(a,z)}, \qquad a \in \Omega, z \in \Gamma. \tag{34}$$

Since S is Hermitian, i.e. $S(a,z) = \overline{S(z,a)}$, (34) is equivalent to

$$F(z) = \frac{S(z,a)T(z)}{i\,\overline{S(z,a)}}.$$

Multiply and divide the right hand side by $S(z,a)$ yields

$$F(z) = \frac{T(z)}{i}\frac{S(z,a)^2}{|S(z,a)^2|}, \qquad z \in \Gamma. \tag{35}$$

Thus to compute the Ahlfors map F, it is sufficient to compute the boundary values of S. We now show how to compute the boundary values of S by using an integral equation method.

Restricting our discussion to a doubly connected region, comparison of (35) with (3) gives the assignment

$$P(z) = F(z), c(z) = i, Q(z) = 1, D(z) = S(z,a).$$

By means of Theorem 2.1, the integral equation related to the boundary relationship (35), along with the choice of $u(z) = \overline{T(z)Q(z)}$ and $v(z) = 1$, is

$$S(z,a) + \int_\Gamma A(z,w)S(w,a)\,|dw| = -i\overline{T(z)}\left[\operatorname*{Res}_{w=a} \frac{S(w,a)}{(w-z)F(w)}\right]^{-}, \quad z \in \Gamma, \tag{36}$$

where A being the Kerzman-Stein kernel again as defined in (15). Note that no evaluation of residue is required at $z = b$, which is a simple zero of F besides a. This is because S and F have a common zero at b. Furthermore $S(z,a)/F(z) = L(z,a)$, where L is the Garabedian kernel, and L has a simple pole at $z = a$ [3, p. 106]. By means of the residue theory, we get

$$S(z,a) + \int_\Gamma A(z,w)S(w,a)\,|dw| = -i\overline{T(z)}\frac{S(a,a)}{(\overline{a}-\overline{z})F'(a)}, \quad z \in \Gamma, \tag{37}$$

But it is known that [3, pp. 49-51]

$$F'(a) = 2\pi S(a,a).$$

Hence (37) becomes

$$S(z,a) + \int_\Gamma A(z,w)S(w,a)\,|dw| = \frac{1}{2\pi i}\frac{\overline{T(z)}}{\overline{a}-\overline{z}}, \quad z \in \Gamma, \tag{38}$$

a result as obtained by Bell [2]. Notice the exact resemblance of this integral equation with the Kerzman-Stein integral equation for the Szegö kernel of the simply connected region [11].

If we square both sides of (35), the result is

$$F(z)^2 = -T(z)^2 \frac{S(z,a)^4}{|S(z,a)^4|}, \qquad z \in \Gamma. \tag{39}$$

Comparison of (39) with (3) now gives the assignment

$$P(z) = F(z)^2, c(z) = -1, Q(z) = T(z), D(z) = S(z,a)^2.$$

By means of Theorem 2.1, the integral equation related to the boundary relationship (39), along with the choice of $u(z) = \overline{T(z)Q(z)}$ and $v(z) = 1$, is

$$S(z,a)^2 + \int_\Gamma M(z,w)S(w,a)^2\,|dw| = \overline{T(z)^2}\left[\operatorname*{Res}_{w=a} \frac{S(w,a)^2}{(w-z)F(w)^2}\right]^{-}, \quad z \in \Gamma, \tag{40}$$

where M is the kernel defined in (19).

To evaluate the residue above, we use the fact that if $f(w) = g(w)/h(w)$ where g and h are analytic at a, and $g(a) \neq 0$, $h(a) = h'(a) = 0$, $h''(a) \neq 0$, which means a is a double pole of $f(w)$, then [6, p. 671]

$$\operatorname*{Res}_{w=a} f(w) = 2\frac{g'(a)}{h''(a)} - \frac{2}{3}\frac{h'''(a)g(a)}{h''(a)^2}. \tag{41}$$

Applying (41) with $g(w) = S(w,a)^2/(w-z)$ and $h(w) = F(w)^2$, and after several algebraic manipulations, we obtain

$$\operatorname*{Res}_{w=a} \frac{S(w,a)^2}{(w-z)F(w)^2} = \frac{S'(a,a)}{2\pi^2(a-z)S(a,a)} - \frac{F''(a)}{8\pi^3(a-z)S(a,a)} - \frac{1}{4\pi^2(z-a)^2}. \tag{42}$$

To further simplify this expression, we use the fact that

$$F''(a) = 4\pi S'(a,a), \qquad a \in \Omega.$$

This is proven by applying (41) with $g(w) = S(w,a)^2$ and $h(w) = F(w)^2$, and the result that [3, p. 51]

$$\operatorname*{Res}_{w=a} \frac{S(w,a)^2}{F(w)^2} = \operatorname*{Res}_{w=a} L(w,a)^2 = 0.$$

In summary, (40) becomes

$$S(z,a)^2 + \int_\Gamma M(z,w)S(w,a)^2\,|dw| = -\frac{\overline{T(z)^2}}{4\pi^2(\overline{a}-\overline{z})^2}, \quad z \in \Gamma. \tag{43}$$

Multiplying both sides by $T(z)$ gives

$$T(z)S(z,a)^2 + \int_\Gamma N(z,w)T(w)S(w,a)^2\,|dw| = -\frac{\overline{T(z)}}{4\pi^2(\overline{a}-\overline{z})^2}, \quad z \in \Gamma, \tag{44}$$

where N is the Neumann kernel again.

5 Conclusion

In this paper we have shown how a boundary relationship satisfied by a function which is analytic in a doubly connected region bounded by two closed Jordan curves can be transformed to an integral equation. Special cases obtained are integral equations related to the conformal mapping of doubly connected regions onto an annulus or a unit disk. These integral equations closely resemble the boundary integral equations for the Szegö kernel and the Bergman kernel of the simply connected regions. One special case gives rise to the classical Warschawski's integral equation for the doubly connected regions, thus rendering a new method for its derivation. We also manage to derive an integral equation obtained by Bell related to the Ahlfors map. Unlike Bell who used the Kerzman-Stein method, our method is basically Henrici's method. But we also obtain more, i.e., an integral equation related to the Ahlfors map whose kernel is the Neumann kernel.

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