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The Effect of a Uniform Vertical Magnetic Field on the Onset of Steady Marangoni Convection in a Semi-infinitely Deep Layer of Conducting Fluid

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Abstract In this paper we use classical linear stability theory to analyse the effect of a uniform vertical magnetic field on the onset of steady thermocapillary-driven Marangoni convection in a semi-infinitely deep layer of quiescent electrically-conducting fluid. We obtain an exact expression for the marginal stability curve for the onset of steady convection in the presence of a magnetic field. We show that increasing the magnetic field strength stabilises the layer.

Keywords Convection, Marangoni

Abstrak *Di dalam makalah ini kami menggunakan teori kestabilan linear klasik untuk menganalisa kesan medan magnet terhadap olakan mantap Marangoni dalam lapisan mengufuk bendalir yang tak terhingga dalamnya. Kami perolehi penyelesaian tepat bagi lengkung kestabilan sut apabila tercetusnya olakan mantap dengan mengambilkira kewujudan medan magnet. Kami dapati bahawa medan magnet menstabilkan lapisan mengufuk bendalir tersebut.*

Katakunci *Olakan, Marangoni*

1 Introduction

The onset of thermocapillary-driven (Marangoni) convection in a layer of fluid which is heated from above or below is a fundamental model problem for several material processing technologies, such as semiconductor crystal growth from melt in microgravity conditions where, as Schwabe [11] describes, typically thermocapillary rather than buoyancy forces are the dominant mechanism driving the flow.

In his pioneering work Pearson [7] showed that thermocapillary effects will drive steady Marangoni convection in a fluid layer of finite depth provided that the layer is heated sufficiently strongly from below. Pearson's [7] work was restricted to the limit of strong surface tension in which the free upper surface is non-deformable, but subsequently Scriven and Sternling [12], Smith [14] and Takashima [15] showed that the presence of free-surface deformation has a dramatic destabilising effect on the long-wave modes. The temporal growth rates of this instability have recently been investigated by Regnier and Lebon [9] and Wilson and Thess [21]. Very recently, Hashim and Wilson [4] extended Regnier and Lebon's [9] work to include the effect of a magnetic field. Takashima [16] showed numerically that oscillatory Marangoni convection can also occur, but only if the layer is heated sufficiently strongly from above and the free surface is deformable.

In practice, uncontrolled convection often results in unsatisfactory end-products, such as poor crystal quality and poor weld penetration. Thus there has been considerable practical, experimental and theoretical interest in understanding various additional physical mechanisms for suppressing (or possibly eliminating altogether) the onset of convection. The effects of a body force due to an externally-imposed magnetic field on the onset of convection have been studied theoretically (extending the pioneering theoretical analyses of Rayleigh [8] and Pearson [7]) by several authors, for example, Wilson [18, 19, 20] and Hashim and Wilson [5]. When a magnetic field is imposed on an electrically-conducting liquid, the liquid motion is reduced because of the interaction between the imposed magnetic field and the induced electric current. A review of the use of magnetic fields in semiconductor crystal growth was presented by Series and Hurle [13].

All of the studies mentioned above dealt with layers of *finite* depth. The analysis of the onset of Marangoni convection in a *semi-infinitely deep* layer is simpler than that of finite depth layer. While this case is sufficiently simple for us to make significant analytical progress it still retains many of the qualitative features of the finite-depth problem. Studying this problem also allows us to isolate the influence of surface effects from those due to the presence of the lower boundary. The first analysis of Marangoni convection in a semi-infinitely deep layer of fluid was performed by Scanlon and Segel [10]. They studied the linear and weakly non-linear regimes of Marangoni convection in the case of non-deformable free upper surface and infinite Prandtl number (defined in Section 2). More recently Velarde *et al.* [17] conducted a linear stability analysis of oscillatory Marangoni convection in a semi-infinitely deep layer of fluid with free-surface deformation. In particular, they presented some numerically-calculated marginal stability curves and critical values of the Marangoni number for the onset of convection and the corresponding analytical results in the asymptotic limit of high frequency of oscillation. Subsequently Garcia-Ybarra and Velarde [2] generalised this work to investigate the onset of oscillatory Marangoni convection in semi-infinitely deep single- or two-component liquid layers both with or without Soret thermal diffusion. Very recently, Hashim and Wilson [6] obtained for the first time a detailed description of the marginal stability curves for the onset of oscillatory Marangoni convection in a semi-infinitely deep layer of fluid.

In the present paper we use a classical linear stability theory to investigate the effect of a uniform vertical magnetic field on the onset of steady thermocapillary-driven Marangoni convection in a semi-infinitely deep layer of quiescent electrically-conducting fluid. We obtain an explicit expression for the marginal stability curve for the onset of steady convection. In particular, this work extends parts of Scanlon and Segel's [10], Velarde *et al.*'s [17] and

Garcia-Ybarra and Velarde's [2] work to include the effect of a magnetic field, and is an essential first step before embarking upon further non-linear studies.

2 Problem Formulation

The basic equations which express the interactions between the fluid motions and the magnetic fields consist of Maxwell's equations together with a suitably modified form of the linear momentum equation. Neglecting the displacement currents (since we are not concerned with the effects due to the propagation of electromagnetic waves), Maxwell's equations are given by

$$\nabla \cdot \mathbf{H} = 0, \tag{1}$$

$$\nabla \times \mathbf{H} = 4\pi \mathbf{J}, \tag{2}$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \tag{3}$$

where $\mathbf{H}$, $\mathbf{J}$ and $\mathbf{E}$ are respectively the magnetic field, the current density and the electric field, and μ is the magnetic permeability. Furthermore, for a moving medium, $\mathbf{H}$, $\mathbf{J}$ and $\mathbf{E}$ must satisfy Ohm's law given by

$$\mathbf{J} = \sigma(\mathbf{E} + \mu\, \mathbf{U} \times \mathbf{H}), \tag{4}$$

where σ is the electrical conductivity and $\mathbf{U}$ is the fluid velocity vector. The homogeneous incompressible electrically-conducting Newtonian fluid in the presence of a magnetic field can be shown as

$$\left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla\right) \mathbf{H} = (\mathbf{H} \cdot \nabla)\mathbf{U} + \eta \nabla^2 \mathbf{H}, \tag{5}$$

by using equations (1)–(4), where $\eta = 1/4\pi\mu\sigma$.

The motion of an electrically-conducting fluid in the presence of a magnetic field will give rise to a Lorentz force which acts on the fluid so that an extra body force term $\mathbf{L}$ appears in the Navier-Stokes equation which, with buoyancy forces in the bulk of the fluid neglected, can be written as

$$\left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla\right) \mathbf{U} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{U} + \frac{1}{\rho}\mathbf{L}, \tag{6}$$

where p is the fluid pressure, ρ is the fluid density, ν is the kinematic viscosity and $\mathbf{L}$ is the Lorentz force. In general, $\mathbf{L}$ is given by

$$\mathbf{L} = q\mathbf{E} + \mu\, \mathbf{J} \times \mathbf{H}, \tag{7}$$

where q is the electric charge density. Assuming that the magnitude of the speed of a fluid element is much smaller than the speed of light (i.e. we neglect terms of order $|\mathbf{U}|^2/c^2$), $\mathbf{L}$ takes the simpler form

$$\mathbf{L} = \mu\, \mathbf{J} \times \mathbf{H}, \tag{8}$$

or, according to equation (2),

$$\mathbf{L} = \frac{\mu}{4\pi}(\nabla \times \mathbf{H}) \times \mathbf{H}. \tag{9}$$

Using an alternative form for $\mathbf{L}$,

$$\mathbf{L} = -\nabla\left(\frac{\mu|\mathbf{H}|^2}{8\pi}\right) + \frac{\mu}{4\pi}(\mathbf{H}\cdot\nabla)\mathbf{H}, \tag{10}$$

allows us to write the equation of motion (6) in the form

$$\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)\mathbf{U} = -\frac{1}{\rho}\nabla\Pi + \nu\nabla^2\mathbf{U} + \frac{\mu}{4\pi\rho}(\mathbf{H}\cdot\nabla)\mathbf{H}, \tag{11}$$

where $\Pi = p + \mu|\mathbf{H}|^2/8\pi$ is the magnetic pressure. The equation of heat conduction is

$$\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)T = \kappa\nabla^2 T, \tag{12}$$

where T is the temperature and κ is the thermal diffusivity.

We wish to examine the stability of a semi-infinitely deep layer of incompressible, Newtonian fluid which is unbounded in the horizontal x- and y-directions and extends to minus infinity in the vertical z-direction. The layer is subject to an externally-imposed uniform vertical magnetic field of strength H, a uniform vertical temperature gradient β and bounded above by a free surface which is initially at $z = 0$ and at constant temperature T_2 and is in contact with a passive gas which remains at constant pressure and constant temperature T_∞. When motion occurs the free surface will be deformed and then we denote its position by $z = f(x, y, t)$. The fluid motion is driven entirely by the thermocapillary effect at the free surface, where the surface tension, τ, is dependent on temperature T according to the simple linear law, $\tau = \tau_0 - \gamma(T - T_2)$, where τ_0 is the value of τ at $T = T_2$ and $-\gamma > 0$ is the coefficient of thermal surface tension variation. We neglect buoyancy forces in the bulk of the fluid (equivalent to taking the coefficient of the thermal expansion of the fluid to be zero) but include the effect of gravity to allow for the presence of gravity-driven surface waves. At the free surface we have the usual kinematic condition and conditions of continuity of the normal and tangential stresses, and the temperature obeys Newton's law of cooling, $-k\partial T/\partial\mathbf{n} = h(T - T_\infty)$, where h is the heat transfer coefficient between the free surface and the passive medium above, k is the thermal conductivity of the fluid and $\mathbf{n}$ is the outward unit normal to the free surface. The boundary condition on the magnetic field depend on the electrical properties of the medium adjoining the fluid (Chandrasekhar [1]). For simplicity, we assume that the media above and below the fluid are both perfect electrical conductors. All perturbations decay to zero as $z \to -\infty$.

We shall investigate the linear stability of a basic state in which the fluid is at rest, the free surface is flat, the temperature gradient across the layer is constant, the magnetic field is uniform, and the pressure is constant. To simplify the analysis we non-dimensionalise the governing equations and boundary conditions using d, $d^2/\nu, \nu/d, \beta d\nu/\kappa, \nu H/\eta$ as appropriate scales for length, time, velocity, temperature and magnetic field respectively. As a result the following non-dimensional groups arise: the Marangoni number, $M = \gamma\beta d^2/\rho\nu\kappa$, the Prandtl number, $P_1 = \nu/\kappa$, the crispation (capillary) number, $C_r = \rho\nu\kappa/\tau_0 d$, the Biot number, $B_i = hd/k$, and the Bond number, $B_o = \rho g d^2/\tau_0$, where g denotes acceleration due to gravity. In the absence of a natural geometrical lengthscale in the problem we choose $d = \rho\nu\kappa/\tau_0$ corresponding to setting $C_r = 1$ without loss of generality. However, we shall retain C_r explicitly in what follows for clarity. In addition to the dimensionless groups mentioned above we have the Chandrasekhar number (the square of the Hartmann number) $Q = \mu H^2 d^2/4\pi\rho\nu\eta$ and the magnetic Prandtl number $P_2 = \nu/\eta$.

3 Linearised Problem

We analyse the linear stability of the basic state in the usual manner by seeking perturbed solutions for any quantity $\Phi(x, y, z, t)$ in terms of normal modes in the form

$$\Phi(x, y, z, t) = \Phi_0(x, y, z) + \phi(z) \exp\left[i(a_x + a_y) + st\right],$$

where Φ_0 is the value of Φ in the basic state and $a = (a_x^2 + a_y^2)^{1/2}$ is the total horizontal wave number of the perturbation. The unknown temporal exponent s will, in general, be complex.

Substituting into the governing equations and neglecting terms of the second and higher orders in the perturbations we obtain the corresponding linearised equations (see, for example, Hashim [3])

$$\begin{aligned}(D^2 - a^2 - sP_1)T + w &= 0, & (13)\\ (D^2 - a^2 - sP_2)h_z + Dw &= 0, & (14)\\ (D^2 - a^2)[(D^2 - a^2 - s)w + QDh_z] &= 0. & (15)\end{aligned}$$

The corresponding linearised boundary equations are

$$\begin{aligned}sf - w &= 0, & (16)\\ P_1C_r[(D^2 - 3a^2 - Q - s)Dw + sP_2Qh_z] - a^2(a^2 + B_o)f &= 0, & (17)\\ P_1(D^2 + a^2)w + a^2M(P_1T - f) &= 0, & (18)\\ h_z &= 0, & (19)\\ P_1DT + B_i(P_1T - f) &= 0, & (20)\end{aligned}$$

evaluated at $z = 0$ and

$$\begin{aligned}w &\to 0, & (21)\\ Dw &\to 0, & (22)\\ h_z &\to 0, & (23)\\ T &\to 0, & (24)\end{aligned}$$

as $z \to -\infty$. Here $w(z)$, $T(z)$, h_z and f denote the vertical variation of the z-component of velocity, temperature and magnetic field, and the magnitude of the free surface deflection of the perturbation respectively and the operator $D = \mathrm{d}/\mathrm{d}z$ denotes differentiation with respect to z. Note that the boundary conditions (19) and (23) correspond to the case when the medium adjoining the fluid layer is a perfect electrical conductor, i.e. no magnetic field can cross the boundary (see, for example, Chandrasekhar [1]).

4 Solution of the Linearised Problem

The complete solution of the linear stability problem is determined once we have solved equations (13)–(15) subject to the boundary conditions (16)–(24). In the next subsection we shall concentrate on the special case when $s = 0$. The general case $s \neq 0$ will be dealt with in the forthcoming paper.

4.1 Onset of Steady Convection

In the special case $s = 0$ (corresponding to the onset of steady convection) the magnetic field h_z can be eliminated entirely from the problem. In this case equation (15) can be written, using equations (13) and (14), as

$$(D^2 - a^2)[(D^2 - a^2)^2 - QD^2]T = 0. \tag{25}$$

The general solution for $T(z)$ obtained from equation (25) with appropriate decay as $z \to -\infty$ is simply

$$T(z) = A_1 e^{az} + A_2 e^{\xi_1 z} + A_3 e^{\xi_2 z}, \tag{26}$$

where

$$\xi_1 = \frac{1}{2}\left[(4a^2 + Q)^{\frac{1}{2}} + Q^{\frac{1}{2}}\right], \qquad \xi_2 = \frac{1}{2}\left[(4a^2 + Q)^{\frac{1}{2}} - Q^{\frac{1}{2}}\right],$$

and A_i for $i = 1, 2, 3$ are arbitrary constants. The corresponding general expressions for $w(z)$ and f calculated from equations (13) and (17) are

$$w(z) = -A_2 Q^{\frac{1}{2}} \xi_1 e^{\xi_1 z} + A_3 Q^{\frac{1}{2}} \xi_2 e^{\xi_2 z}, \tag{27}$$

$$f = \frac{P_1 C_r Q^{\frac{1}{2}} (4a^2 + Q)^{\frac{1}{2}}}{a^2 + B_o} [\xi_1 A_2 - \xi_2 A_3]. \tag{28}$$

The boundary conditions (16) and (20) yield

$$A_2 = \frac{\xi_2}{\xi_1} A_3 \quad \text{and} \quad A_1 = -\frac{4a^2 + 2(4a^2 + Q)^{\frac{1}{2}} B_i}{\xi_1 (a + B_i)} A_3,$$

and so the solutions for $T(z)$, $w(z)$ and f are

$$T(z) = \left[-\frac{4a^2 + 2(4a^2 + Q)^{\frac{1}{2}} B_i}{\xi_1 (a + B_i)} e^{az} + \frac{\xi_2}{\xi_1} e^{\xi_1 z} + e^{\xi_2 z}\right] A_3, \tag{29}$$

$$w(z) = -\frac{2a^2 Q^{\frac{1}{2}} [e^{\xi_1 z} - e^{\xi_2 z}] A_3}{\xi_1}, \tag{30}$$

$$f = 0, \tag{31}$$

where A_3 is arbitrary. The remaining boundary condition (18) yields

$$\left[Q(4a^2 + Q)^{\frac{1}{2}} [a + B_i] + aM\{2a - (4a^2 + Q)^{\frac{1}{2}}\}\right] A_3 = 0.$$

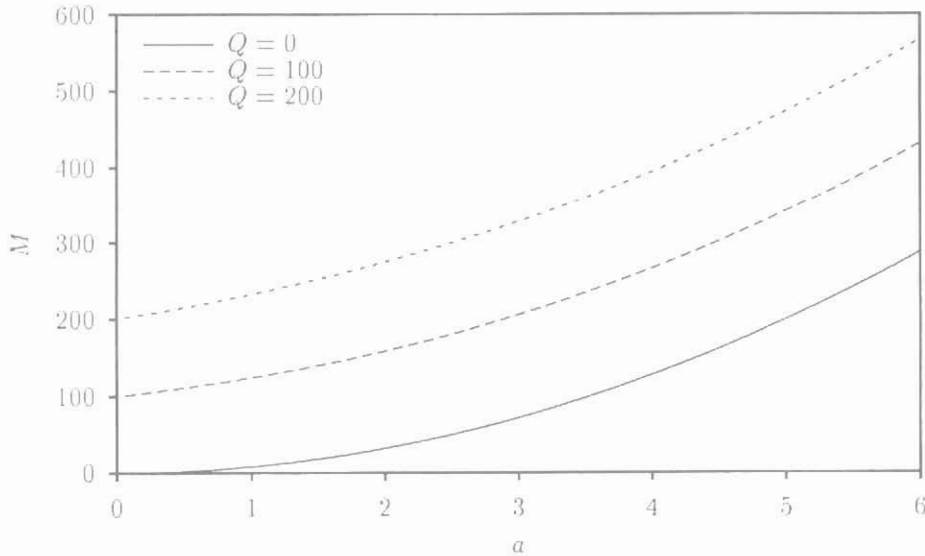

Figure 1: Marginal Stability Curves for the Onset of Steady Convection for Various Values of Q in the Case $B_i = 0$.

and since $A_3 \neq 0$ we deduce that

$$M = [4a + Qa^{-1} + 2(4a^2 + Q)^{\frac{1}{2}}](a + B_i). \tag{32}$$

Note that this expression for the steady marginal curve is independent of C_r, B_o and, of course, P_1 and P_2. Setting $Q = 0$ and $B_i = 0$ in expression (32) we recover the solution first obtained by Scanlon and Segel [10], while setting $Q = 0$ with $B_i \neq 0$ we recover the expression obtained by Garcia-Ybarra and Velarde [2], $M = 8a(a + B_i)$. Evidently the critical values are $M_c = 0$ and $a_c = 0$ which do not depend on B_i. Thus the zero-magnetic field model predicts that no matter how small the temperature gradient might be and regardless of the values of B_i, steady convection sets in. The marginal stability curves for the onset of steady convection for various values of Q in the case $B_i = 0$ and $B_i \neq 0$ are plotted in Figures 1 and 2 respectively. The marginal stability curves separate regions of stable modes with $\mathrm{Re}(s) < 0$ (regions below the curves) from those of unstable modes with $\mathrm{Re}(s) > 0$ (regions above the curve). Both Figures 1 and 2 show that the effect of increasing the magnetic field strength Q is to stabilise the layer in the case of $B_i = 0$ and $B_i = 1$ respectively. In the limit $a \to 0$, $M = \frac{QB_i}{a} + Q^{\frac{1}{2}}(Q^{\frac{1}{2}} + 2B_i) + O(a)$, while in the limit $a \to \infty$, $M = 8a(a + B_i) + \frac{3}{2}Q(1 + \frac{B_i}{a}) + O(\frac{1}{a^2})$. The case $B_i = 0$ for the insulating boundary condition (Figure 1) is particular in that the critical wavenumber a_c is zero and the critical value of M is Q. In a real experiment the depth of the layer would be large rather than infinite and so this result means that in practice the critical wavelength $2\pi/a_c$ will always be comparable to the depth of the layer, i.e. the presence of the lower boundary will always play a significant role in determining the exact conditions for the onset of steady convection.

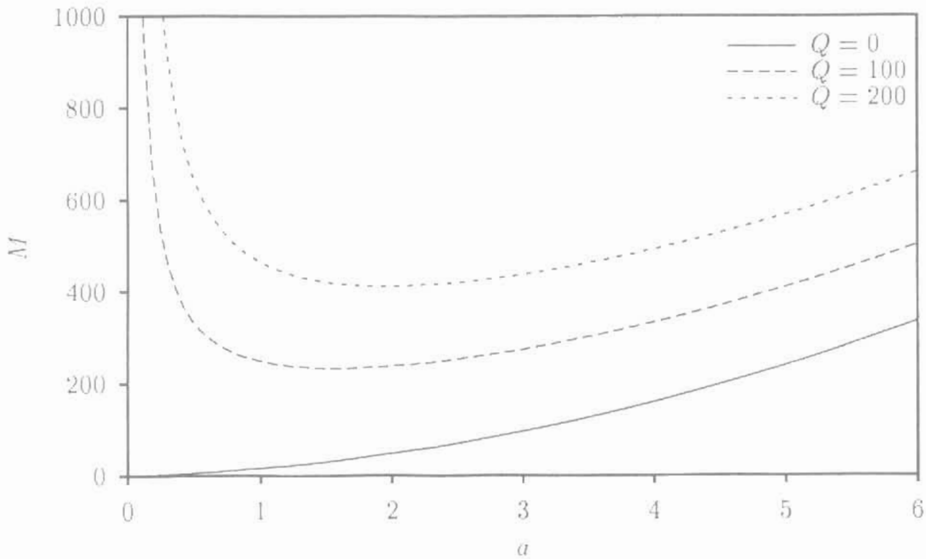

Figure 2: Marginal Stability Curves for the Onset of Steady Convection for Various Values of Q in the Case $B_i = 1$.

However, this is *not* always the case for the onset of steady convection in the case $B_i \neq 0$ and $Q \neq 0$. Figure 2 shows that convection first occurs at non-zero values of a in the case $B_i = 1$ and $Q \neq 0$. We note that in the limit of $Q \to \infty$, $M = (a + B_i)[\frac{Q}{a} + 2Q^{\frac{1}{2}} + O(1)]$.

5 Conclusions

In this paper we analysed the effect of a uniform vertical magnetic field on the onset of steady Marangoni convection in a semi-infinitely deep layer of quiescent fluid. We obtained an explicit expression for the marginal stability curve for the onset of steady convection. We showed that the effect of increasing the magnetic field is to stabilise the layer.

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