Matematika, 2000, Jilid 16, bil. 1, hlm. 31–40

Interpolation on the Triangle and Pappus' Theorem

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Abstract The purpose of this paper is to discuss a two-dimensional geometrical construction whose validity is justified by Pappus' theorem. An application to polynomial interpolation on the three-pencil mesh expressed in homogeneous coordinate is also given.

Keywords Homogeneous coordinates, Pappus' theorem, three-pencil mesh, polynomial interpolation.

Abstrak *Tujuan kertas ini ialah untuk membincangkan suatu pembinaan geometri bermatra dua yang mana kesahannya dijustifikasikan oleh teorem Pappus. Penggunaannya kepada interpolasi polinomial pada jaringan tiga pensil yang dinyatakan dalam koordinat homogen turut diberikan.*

Katakunci *Koordinat homogen, teorem Pappus, jaringan tiga pensel, interpolasi polinomial.*

1 Introduction

In more recent times, the combination of geometry and analysis has created the lively new area of fractals and, in more applied topics such as finite element method and CAGD (computer aided geometric design). The purpose of this paper is to strike a small blow for geometry by discussing a two-dimensional geometrical construction whose validity is justified by Pappus' theorem and which has an application on the theory of interpolation. We begin the next section with a description of the use of homogeneous coordinates and, as an application, give a proof of Pappus' theorem.

2 Pappus' Theorem

In homogeneous coordinates we use a *triple* of numbers (x, y, z), not all zeros, to denote a point in the plane. If $\lambda \neq 0$, (x, y, z) and $(\lambda x, \lambda y, \lambda z)$ represent the same point and, if $z \neq 0$, the point (x, y, z) coincides with the point $(x/z, y/z)$ in the Euclidean plane. The line with equation $ax + by + c = 0$ in the Euclidean plane has equation $ax + by + cz = 0$ in homogeneous coordinates. Consider now the linear transformation

$$\begin{bmatrix} \xi \\ \mu \\ \zeta \end{bmatrix} = \boldsymbol{M} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{1}$$

Under this transformation, the points (1,0,0), (0,1,0) and (0,0,1), which we denote by X, Y and Z, become the points whose coordinates are given by the columns of the matrix $\boldsymbol{M}$. These three transformed points will not lie in a straight line if $\boldsymbol{M}$ is non-singular. Conversely, given any three points not lying in a straight line, we form a (non-singular) matrix $\boldsymbol{M}$ whose columns are defined by the coordinates of these points. Then the transformation matrix $\boldsymbol{M}^{-1}$ will map the three points into X, Y and Z respectively. We also note that under any such non-singular transformation, straight lines are mapped into straight lines.

Pappus' theorem is said to be the foundation of modern projective geometry. We now recall the statement of the theorem and give its proof using homogeneous coordinates. (See Figure 1).

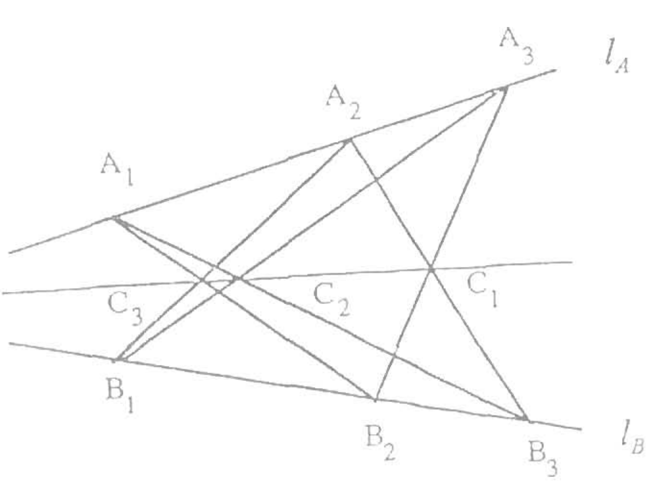

Figure 1: Pappus' Theorem

Theorem 2.1 Pappus' Theorem. *Consider two straight lines l_A and l_B lying in a plane. Let A_1, A_2 and A_3 on l_A and let B_1, B_2 and B_3 lie on l_B. We now construct three further points as follows: let C_1 be the point of intersection of the lines A_2B_3 and A_3B_2 and (cyclically) let C_2 be the intersection of A_3B_1 and A_1B_3 and C_3 be the point of intersection of A_1B_2 and A_2B_1. Then C_1, C_2 and C_3 lie in a straight line.*

Proof The following proof is based on that given in Maxwell [5]. In any homogeneous coordinate system, suppose that l_A and l_B intersect at a point P and let Q and R denote other points on l_B and l_A, respectively. Now we choose a new homogeneous coordinate system obtained by mapping P, Q and R onto X, Y, and Z, respectively. Thus the two straight lines l_A and l_B will be denoted by $y = 0$ and $z = 0$, respectively. Then we may denote the six points A_i and B_i as follows:

$$\begin{aligned} A_i &= (\alpha_i, 0, 1), \qquad i = 1, 2, 3, \\ B_i &= (\beta_i, 1, 0), \qquad i = 1, 2, 3. \end{aligned}$$

The equation of A_2B_3 is $x - \beta_3 y - \alpha_2 z = 0$ and similarly the equation of A_3B_2 is $x - \beta_2 y - \alpha_3 z = 0$. Thus the point of intersection of A_2B_3 and A_3B_2 is

$$C_1 = (\alpha_2\beta_2 - \alpha_3\beta_3, \alpha_2 - \alpha_3, \beta_2 - \beta_3).$$

Similarly, using the cyclic order, we see that the coordinates of C_2 and C_3 are

$$C_2 = (\alpha_3\beta_3 - \alpha_1\beta_1, \alpha_3 - \alpha_1, \beta_3 - \beta_1)$$

and

$$C_3 = (\alpha_1\beta_1 - \alpha_2\beta_2, \alpha_1 - \alpha_2, \beta_1 - \beta_2).$$

We now note that

$$\det \begin{bmatrix} \alpha_2\beta_2 - \alpha_3\beta_3 & \alpha_2 - \alpha_3 & \beta_2 - \beta_3 \\ \alpha_3\beta_3 - \alpha_1\beta_1 & \alpha_3 - \alpha_1 & \beta_3 - \beta_1 \\ \alpha_1\beta_1 - \alpha_2\beta_2 & \alpha_1 - \alpha_2 & \beta_1 - \beta_2 \end{bmatrix} = 0.$$

and the points C_1, C_2 and C_3 which are represented by the row vectors, lie in a straight line. A little calculation shows that equation of this line is $ax + by + cz = 0$, where

$$\begin{aligned} a &= (\alpha_2\beta_3 - \alpha_3\beta_2) + (\alpha_3\beta_1 - \alpha_1\beta_3) + (\alpha_1\beta_2 - \alpha_2\beta_1), \\ b &= \alpha_1\beta_1(\beta_3 - \beta_2) + \alpha_2\beta_2(\beta_1 - \beta_3) + \alpha_3\beta_3(\beta_2 - \beta_1), \\ c &= \alpha_1\beta_1(\alpha_2 - \alpha_3) + \alpha_2\beta_2(\alpha_3 - \alpha_1) + \alpha_3\beta_3(\alpha_1 - \alpha_2). \end{aligned}$$

If the two lines l_A and l_B are parallel, we can represent them in homogeneous coordinates by the pair of equations

$$\begin{aligned} ax + by + cz &= 0 \\ ax + by + c'z &= 0 \end{aligned}$$

with $c \neq c'$. In this case we say that they "meet" at the $(-b, a, 0)$, say P. The proof is then completed similarly as above. □

In Pappus' theorem the set of points $\{C_1, C_2, C_3\}$ is created by applying the construction described in the statement of the theorem to the sets of points $\{A_1, A_2, A_3\}$ and $\{B_1, B_2, B_3\}$. We remark that there is a full symmetry in these three sets of points in the sense that, beginning with any two of them, we can generate the third by the Pappus construction.

3 Three-pencil Mesh

In this section we describe the construction of the "three-pencil mesh" proposed by Lee and Phillips [2]. We begin with any three vertices X, Y and Z not lying on a straight line and draw one line through each of the vertices, creating the three points $P_{0,0}, P_{1,0}$ and $P_{0,1}$, as in Figure 2. We then draw a second line through X, joining X to $P_{0,1}$, and a second line through Z, joining Z to $P_{1,0}$, letting two new lines intersect at $P_{1,1}$. We then complete this second stage by drawing a second line through Y, by joining it to $P_{1,1}$ and we also label $P_{0,2}$ and $P_{2,0}$.

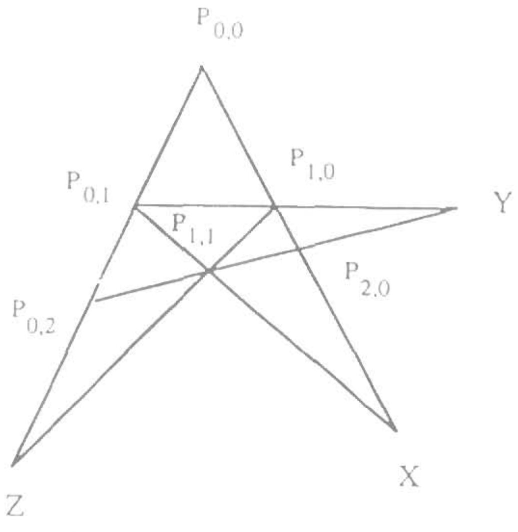

Figure 2: Three-pencil Lattice

The next stage is to draw a third line through each of the vertices X, Y and Z as follows: we join X to $P_{0,2}$ and Z to $P_{2,0}$, creating the two further points $P_{1,2}$ and $P_{2,1}$. We see that the three points $P_{1,2}, P_{2,1}$ and Y lie in a straight line (see Figure 3). Indeed, this follows from Pappus' theorem by taking l_A as the line containing $P_{1,0}$, $P_{2,0}$ and X, and l_B as the line containing $P_{0,1}$, $P_{0,2}$ and Z. Now we let the line joining Y to $P_{1,2}$ and $P_{2,1}$ cut the line $XP_{0,0}$ at $P_{3,0}$ and line $ZP_{0,0}$ at $P_{0,3}$. This completes the third stage, when we have three lines through each of the vertices X,Y and Z and a mesh of 10 points $P_{i,j}$, with $i \geq 0, j \geq 0$ and $i + j \leq 3$. The point $P_{i,j}$ lies on the $(j+1)$th line through X, the $(i+1)$th line through Z, and the $(i+j)$th line through Y.

After k stages we have k lines through each of the vertices X, Y and Z and a mesh of $(k+1)(k+2)/2$ points $P_{i,j}$, with $i \geq 0$, $j \geq 0$ and $i + j \leq k$. Next we join X to $P_{0,k}$

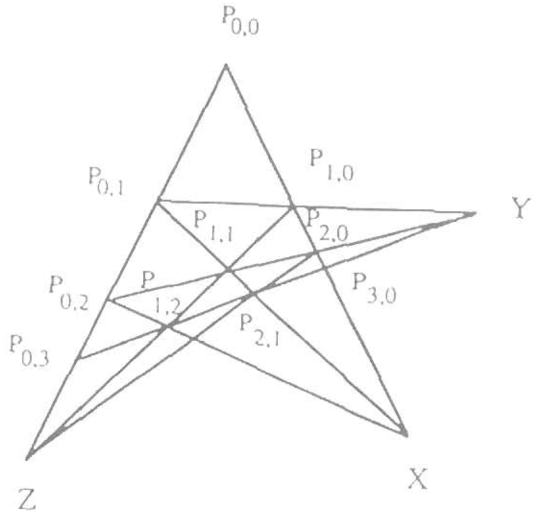

Figure 3: Next Stage in the Construction of a Three-pencil Lattice

and Z to $P_{k,0}$. Then we find, by applying Pappus' theorem k times, that Y is on the same straight line as k points in our construction. We label these as described above, as we also label the two points $P_{k+1,0}$ and $P_{0,k+1}$ where the new line through Y cuts $XP_{0,0}$ and $ZP_{0,0}$ respectively. We terminate the construction process when, for some choice of positive integer n, we have three pencils, each of n lines, and with X, Y and Z being the vertices of these pencils, thus creating a mesh of $(n+1)(n+2)/2$ points.

4 Application to Interpolation

We now show how, for each point $P_{i,j}$ in the mesh and in any given homogeneous coordinate system, we can construct a polynomial $p_{i,j}(x,y,z)$ which has the value 1 at the point $P_{i,j}$ and the value zero at all other points of the mesh. For, with respect to the given coordinate system, let

$$\begin{aligned} u_k(x,y,z) &= 0, \qquad 0 \le k \le n-1, \\ v_k(x,y,z) &= 0, \qquad 0 \le k \le n-1, \\ w_k(x,y,z) &= 0, \qquad 0 \le k \le n-1, \end{aligned}$$

denote the equations of the lines in the three pencils with vertices X, Y and Z respectively, where $u_k = 0$, $v_k = 0$, and $w_k = 0$ are the equations of the $k+1$ - lines in each pencil. Then, for any non-zero constant $\lambda_{i,j}$,we define

$$p_{i,j}(x,y,z) = \lambda_{i,j} \prod_{t=0}^{j-1} u_t(x,y,z) \prod_{t=i+j+1}^{n} v_t(x,y,z) \prod_{t=0}^{i-1} w_t(x,y,z), \tag{2}$$

where an empty product denotes unity. Thus $p_{i,j}(x,y,z)$, being a constant times a product of n linear forms is non-zero at the point $P_{i,j}$ and is zero at all other points of the mesh.

since the union of the straight lines corresponding to the linear forms which appear in (2) contains every point of the mesh except for $P_{i,j}$. Finally we choose the scaling factor $\lambda_{i,j}$ so that $p_{i,j}(x,y,z)$ has the value 1 at $P_{i,j}$. Thus, if $f(x,y,z)$ is any function defined on the triangle with vertices at $P_{0,0}$, $P_{n,0}$ and $P_{0,n}$, and $f_{i,j}$ denotes the value of this function at the point $P_{i,j}$, the polynomial

$$\sum_{i,j} f_{i,j} P_{i,j}(x,y,z) \tag{3}$$

will interpolate the function at all $(n+1)(n+2)/2$ points of the three-pencil mesh. The above summation is to be taken over all $i \geq 0$, $j \geq 0$, with $i+j \leq n$.

5 The Mesh Expressed in Homogeneous Coordinates

Let us consider any point (x_0, y_0, z_0) with x_0, y_0, z_0 all non-zero. Then we may make a transformation of the form (1), where

$$\boldsymbol{M} = \begin{bmatrix} 1/x_0 & 0 & 0 \\ 0 & 1/y_0 & 0 \\ 0 & 0 & 1/z_0 \end{bmatrix}$$

We see that under this transformation (1,0,0), (0,1,1) and (0,0,1) are mapped onto themselves and the point (x_0, y_0, z_0) is mapped to (1,1,1). Thus, to describe the above three-pencil mesh we can choose a coordinate system so that X, Y and Z are denoted by (1,0,0), (0,1,0) and (0,0,1) respectively, and $P_{0,0}$ is (1,1,1). Then the line $u_0(x,y,z)=0$ through X is $y-z=0$, and the line $w_0(x,y,z)=0$ through Z is $x-y=0$. We now need only to fix $P_{0,1}$, say, then the whole three-pencil mesh is determined. Let us therefore choose $P_{0,1}$ as the point $(q,q,1)$, for some positive q. Then this fixes the line $v_0(x,y,z)=0$ as $x-qz=0$ and $P_{1,0}$ is $(q,1,1)$. We now follow the construction through algebraically and an induction argument shows that $P_{i,j}$ is $(q^{i+j}, q^j, 1)$ and the three pencil of lines are

$$\begin{aligned} u_k(x,y,z) &= y - q^k = 0, \qquad 0 \leq k \leq n-1, \\ v_k(x,y,z) &= x - q^{k+1}z = 0, \qquad 0 \leq k \leq n-1, \\ w_k(x,y,z) &= x - q^k y = 0, \qquad 0 \leq k \leq n-1. \end{aligned}$$

The point $P_{i,j}$ is the point of intersection of the three lines

$$u_j(x,y,z)=0, \quad v_{i+j-1}(x,y,z)=0, \quad w_i(x,y,z)=0.$$

Provided that $q>0$, all $(n+1)(n+2)/2$ points $P_{i,j}$, for $i,j \geq 0$, $i+j \leq n$, lie in the triangle with vertices $P_{0,0}$, $P_{n,0}$, and $P_{0,n}$.

6 Transformation to the Euclidean Plane

Still working in homogeneous coordinates, we now apply a linear transformation, represented by the matrix A, mapping the vertices $P_{0,0}$, $P_{n,0}$, and $P_{0,n}$ as follows:

$$
\begin{aligned}
P_{0,0} &= (1,1,1) \rightarrow (0,0,a_0) &(4)\\
P_{n,0} &= (q^n,1,1) \rightarrow (a_1,0,a_1) &(5)\\
P_{0,n} &= (q^n,q^n,1) \rightarrow (0,a_2,a_2) &(6)
\end{aligned}
$$

where $a_0, a_1, a_2 \neq 0$. The transformation matrix is thus defined by

$$
\begin{bmatrix} 0 & 0 & a_0 \\ a_1 & 0 & a_1 \\ 0 & a_2 & a_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ q^n & 1 & 1 \\ q^n & q^n & 1 \end{bmatrix} A,
$$

so that

$$
A = (1-q^n)^{-1} \begin{bmatrix} -a_1 & 0 & a_0 - a_1 \\ a_1 & -a_2 & a_1 - a_2 \\ 0 & a_2 & a_2 - q^n a_0 \end{bmatrix},
$$

provided that $q \neq 1$. This transformation maps the point $P_{i,j} = (q^{i+j}, q^j, 1)$ onto the point $P_{i,j} = (x_{ij}, y_{ij}, z_{ij})$, where

$$
\begin{aligned}
x_{ij} &= a_1 (q^j - q^{i+j}) / (1-q^n), &(7)\\
y_{ij} &= a_2 (1-q^j) / (1-q^n), &(8)\\
z_{ij} &= x_{ij} + y_{ij} + a_0 (q^{i+j} - q^n) / (1-q^n). &(9)
\end{aligned}
$$

If $z_{ij} \neq 0$, this corresponds to the point

$$
P_{i,j} = (x_{ij}/z_{ij}, y_{ij}/z_{ij})
$$

in the Euclidean plane. If $a_0 \neq a_1, a_1 \neq a_2$, and $a_2 \neq q^n a_0$, then the above transformation maps the three vertices X, Y and Z respectively onto the points

$$
(1+\alpha, 0), \quad (-\beta, 1+\beta), \quad \text{and} \quad (0, \gamma)
$$

in the Euclidean plane, where $\alpha, \beta, \gamma \neq 0$ and

$$
\left(1+\frac{1}{\alpha}\right)\left(1+\frac{1}{\beta}\right)\left(1+\frac{1}{\gamma}\right) > 0,
$$

and we have

$$
\frac{a_1}{a_0} = 1 + \frac{1}{\alpha}, \frac{a_2}{a_0} = \left(1+\frac{1}{\alpha}\right)\left(1+\frac{1}{\beta}\right), q^n = \left(1+\frac{1}{\alpha}\right)\left(1+\frac{1}{\beta}\right)\left(1+\frac{1}{\gamma}\right). \qquad (10)
$$

Then (10) shows that, corresponding to any such α, β and γ, which fixes the positions of the vertices of the three pencils of lines in the Euclidean plane, we obtain a unique $q > 0$ and unique rations a_1/a_0 and a_2/a_0. Thus, to each choice of positions of the three vertices, we have a unique three-pencil lattice, determined by (7)-(9). For example, with $\alpha = 1$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{3}$ and $n = 4$, we obtain the lattice shown in Figure 4.

Letting $\alpha \rightarrow \infty$ and keeping $\beta = \frac{1}{2}$, $\gamma = \frac{1}{3}$ and $n = 4$ we obtain the lattice in Figure 5, in which one of the three pencils is a system of parallel lines.

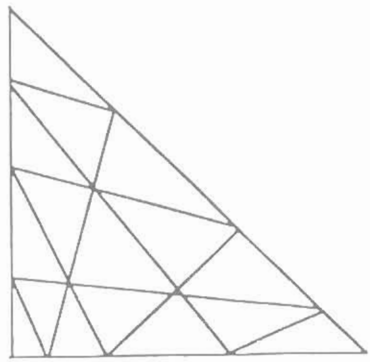

Figure 4: Three-pencil Lattice with All Three Vertices Finite

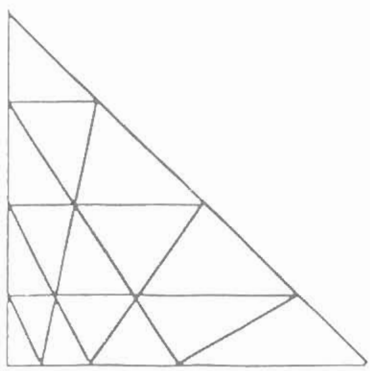

Figure 5: Three-pencil Lattice with Two Finite Vertices and One Pencil of Parallel Lines

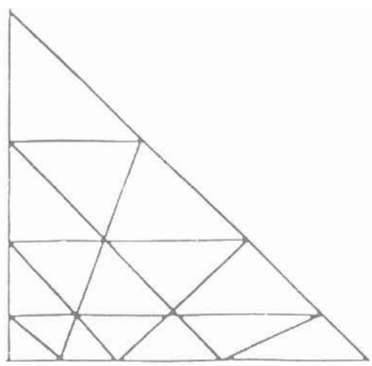

Figure 6: Three-pencil Lattice with One Finite Vertex and Two Pencils of Parallel Lines

If we let both $\alpha \to \infty$ and $\beta \to \infty$ and keep $\gamma = \frac{1}{3}$ and $n = 4$, we obtain the lattice in Figure 6, in which two of the three pencils are systems of parallel lines.

As a further example, we let $\alpha \to \infty$ and $\gamma \to \infty$ and let $\beta = 1/(q^n - 1)$ to give the lattice of points

$$P_{i,j} = \left(\frac{1-q^i}{1-q^n}, 1 - \frac{1-q^{n-j}}{1-q^n} \right), \quad i, j \geq 0, \quad i + j \leq n,$$

which we can also express as the lattice

$$S_q = \left\{ \left(\frac{1-q^i}{1-q^n}, 1 - \frac{1-q^j}{1-q^n} \right) \right\}, \quad 0 \leq i \leq j \leq n. \tag{11}$$

This is illustrated in Figure 7 for the case where $n = 4$ and $q = \frac{2}{3}$. Under the above transformation effected by the matrix A, the points X and Z are sent off to infinity and Y is mapped onto the point $(1/(1-q^n), -q^n/(1-q^n))$, which is the point $(\frac{81}{65}, -\frac{16}{65})$ for the case given in Figure 7.

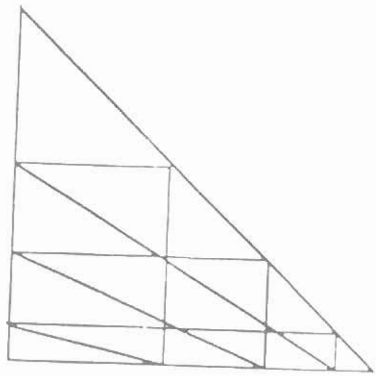

Figure 7: Another Lattice with One Finite Vertex and Two Pencils of Parallel Lines

In the limit as $q \to 1$ the lattice S_q tends to the equally-spaced lattice

$$S_1 = \left\{ \left(\frac{i}{n}, 1 - \frac{j}{n} \right) \right\}, \quad 0 \leq i \leq j \leq n. \tag{12}$$

7 Extensions to Higher Dimensions

All of the lattices discussed in the previous section satisfy what Chung and Yao [1] call the Geometric Characterization (GC) condition: to each point p_i in the lattice S in two-dimensional Euclidean space there correspond n lines $l_{i,1}, l_{i,2}, \ldots, l_{i,n}$ such that $p \in S$ lies in the union of $l_{i,1}, l_{i,2}, \ldots, l_{i,n}$ if and only if $p \neq p_i$. This generalizes to lattices in higher dimensional Euclidean space, as indeed do the lattices which we have constructed in the previous section. For example, for the tetrahedron we can construct a lattice of

$$(n+1)(n+2)(n+3)/3$$

points; there are 4 pencils of planes; each pencil of planes is either parallel or meets in a finite line; to each point p_i in the lattice there correspond n planes $l_{i,1}, l_{i,2}, \ldots, l_{i,n}$ such that $p \in S$ lies in the union of $l_{i,1}, l_{i,2}, \ldots, l_{i,n}$ if and only if $p \neq p_i$, which is the GC condition.

We conclude by giving just one example of such a lattice in three-dimensional Euclidean space. This is the lattice consisting of the set of points

$$p_{i,j,k} = \left(q^j \frac{1-q^i}{1-q^n}, \frac{1-q^j}{1-q^n}, q^{n-k} \frac{1-q^k}{1-q^n} \right), \quad i,j,k \geq 0, \quad i+j+k \leq n.$$

Each point of the lattice lies in one of each of the following four pencils of planes

$$\begin{aligned}
x + y &= \frac{(1-q^v)}{(1-q^n)}, \qquad 0 \leq v \leq n, \\
y &= \frac{(1-q^v)}{(1-q^n)}, \qquad 0 \leq v \leq n, \\
z &= 1 - \frac{(1-q^v)}{(1-q^n)}, \qquad 0 \leq v \leq n, \\
x + y + q^{v-n} z &= \frac{(1-q^v)}{(1-q^n)}, \qquad 0 \leq v \leq n.
\end{aligned}$$

The first three pencils are sets of parallel planes and the fourth is a pencil of planes with common line

$$x + y = \frac{1}{1-q^n}, \qquad z = \frac{-q^n}{1-q^n}.$$

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