Matematika, 2003, Jilid 19, bil. 2, hlm. 83–89
©Jabatan Matematik, UTM.

On the Stability of Fully Implicit Block Backward Differentiation Formulae

[1]**Zarina Bibi Ibrahim**, [3]**Rozita Johari** & [2]**Fudziah Ismail**
Department of Mathematics
Faculty of Science & Environmental Studies
Universiti Putra Malaysia, 43400 UPM, Serdang, Selangor
[1]zarinabb@fsas.upm.edu.my
[2]fudziah@fsas.upm.edu.my
[3]rozita@fsktm.upm.edu.my

Mohamed Suleiman
Lembaga Akreditasi Negara
Tingkat 3B, 11B, 14A & 14B, Menara PKNS-PJ
No. 17, Jalan Yong Shook Lin
46050 Petaling Jaya, Selangor Darul Ehsan

Abstract This paper focuses on obtaining stability regions of numerical methods for ordinary differential equations (ODEs). In particular, the aim of this paper is to construct a stability region for method of Two Point Block Backward Differentiation Formulae (BDF). For the method to be of practical importance, the stability region must cover the whole of the negative half-plane.

Keywords Stability region, block, backward differentiation formulae.

1 Introduction

Many numerical techniques are available for the solution of initial value problems (IVPs) and these techniques depend on many factors including speed of convergence, computational expense, data-storage requirements, accuracy, and stability. Shampine and Watts[1], Chu and Hamilton [2] both suggest that the stability problem appears to be the most serious limitation of block methods. Our aim is to investigate the linear stability properties of this block BDF. Below we give some basic definition of stability of a multistep method given in Lambert [3].

Definition 1.1

The linear multistep method or ***linear k-step method*** can be represented in standard form by an equation

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{1}$$

where $y_{n+j} \approx y\left(x_{n+j}\right)$ and $f_{n+j} \equiv f\left(x_{n+j}, y_{n+j}\right)$, coefficients α_j, β_j are suitably chosen constants subject to conditions $\alpha_k = 1$, $|\alpha_0| + |\beta_0| \neq 0$ and k is defined as the order of the particular method employed.

Definition 1.2

The ***first characteristic polynomial***, ρ associated with the general method (1), where it is the polynomial of degree k whose coefficients are α_j and the ***second characteristic polynomial*** σ whose coefficients are β_j are defined by

$$\left.\begin{aligned} \rho(\zeta) &= \sum_{j=0}^{k} \alpha_j \zeta^j \\ \sigma(\zeta) &= \sum_{j=0}^{k} \beta_j \zeta^j \end{aligned}\right\} \tag{2}$$

where $\zeta \in \mathbb{C}$ is a dummy variable. Stability is determined by the location of the roots of the characteristic polynomials.

Definition 1.3

The linear multistep method (1) is said to satisfy the ***root condition*** if all of the roots of the first characteristic polynomial *have modulus less than or equal to unity*, and those of modulus unity are ***simple***. The method (1) is said to be ***zero-stable*** if it satisfies the root condition.

We emphasize the fact that all the roots of the first characteristic polynomial must lie in or on the unit circle and there must be no multiple roots on the unit circle.

Definition 1.4

The linear multistep method (1) is said to be ***absolutely stable*** in a region R for a given $h\lambda$ if for that $h\lambda$, all the roots r_s of the stability polynomial $\pi\left(r, h\lambda\right) = \rho(r) - h\lambda\sigma(r) = 0$, satisfy $|r_s| < 1$, $s = 1, 2, \ldots, k$.

Definition 1.5

A method is said to be ***A-stable*** if all numerical approximations tend to zero as $n \to \infty$ when it is applied to the differential equation $y' = \lambda y$ with a fixed positive h and a (complex) constant λ with a negative real part.

2 Stability Theory of Block Numerical Methods for ODEs

In this section, we introduce the basic definition of a block method described by Fatunla [6].

Definition 2.1

Let Y_m and F_m be vectors defined by

$$\left.\begin{array}{rcl} Y_m &=& [y_n, y_{n+1}, y_{n+2}, \ldots, y_{n+r-1}]^t \\ F_m &=& [f_n, f_{n+1}, f_{n+2}, \ldots, f_{n+r-1}]^t \end{array}\right\} \quad (3)$$

respectively.

Then a general k-block, r-point method is a matrix finite difference equation of the form

$$Y_m = \sum_{i=1}^{k} A_i Y_{m-i} + h \sum_{i=0}^{k} B_i F_{m-i} \quad (4)$$

where all A_i's and B_i's are properly chosen $r \times r$ matrix coefficients and $m = 0, 1, 2, \ldots$ represents the block number, $n = mr$ the first step number in the m-th block and r is the proposed block size.

We defined zero stable for block methods according to Chu and Hamilton [3] as follows.

Definition 2.2

The Block Method (4) is said to be ***zero-stable*** if the roots $R_j, j = 1(1)k$ of the first characteristic polynomial $\rho(R) = \det\left[\sum_{i=0}^{k} A_i R^{k-i}\right] = 0, A_o = -I$, satisfies $|R_j| \le 1$. If one of the roots is $+1$, we call this root the ***principal root*** of $\rho(R)$.

Consequently, we will extend the approach to formulas that has been derived by Zarina [12] called the ***Two Point Block Backward Differentiation Formula***. These formulas are given by

$$\left.\begin{array}{rcl} y_{n+1} &=& -\frac{1}{3}y_{n-1} + 2y_n - \frac{2}{3}y_{n+2} + 2hf_{n+1} \\ y_{n+2} &=& \frac{2}{11}y_{n-1} - \frac{9}{11}y_n + \frac{18}{11}y_{n+1} + \frac{6}{11}hf_{n+2} \end{array}\right\} \quad (5)$$

The linear stability properties (5) are determined through application of the standard linear test problem

$$y' = \lambda y, \quad \lambda < 0, \quad \lambda \text{ complex} \quad (6)$$

Application of (6) to (5) where $f(x, y) = \lambda y$, then gives the following

$$\left.\begin{array}{rcl} y_{n+1} &=& -\frac{1}{3}y_{n-1} + 2y_n - \frac{2}{3}y_{n+2} + 2\lambda h y_{n+1} \\ y_{n+2} &=& \frac{2}{11}y_{n-1} - \frac{9}{11}y_n + \frac{18}{11}y_{n+1} + \frac{6}{11}\lambda h y_{n+2} \end{array}\right\} \quad (7)$$

Hence

$$\left.\begin{aligned} y_{n+1} - 2\lambda h y_{n+1} &= -\frac{1}{3}y_{n-1} + 2y_n - \frac{2}{3}y_{n+2} \\ y_{n+2} - \frac{6}{11}\lambda h y_{n+2} &= \frac{2}{11}y_{n-1} - \frac{9}{11}y_n + \frac{18}{11}y_{n+1} \end{aligned}\right\} \tag{8}$$

We write (8) in matrix-vector form as

$$\begin{bmatrix} 1-2\lambda h & 0 \\ 0 & -\frac{6}{11}\lambda h \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & 2 \\ \frac{2}{11} & -\frac{9}{11} \end{bmatrix}\begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} 0 & -\frac{2}{3} \\ \frac{18}{11} & 0 \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} \tag{9}$$

Setting $\hat{h} = \lambda h$, and rearrange (9) will give

$$\begin{bmatrix} \left(1-2\hat{h}\right) & \frac{2}{3} \\ -\frac{8}{11} & \left(1-\frac{6}{11}\hat{h}\right) \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & 2 \\ \frac{2}{11} & -\frac{9}{11} \end{bmatrix}\begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} \tag{10}$$

which is equivalent to

$$AY_m = BY_{m-1} \tag{11}$$

with the matrix coefficients specified as

$$A = \begin{bmatrix} \left(1-2\hat{h}\right) & \frac{2}{3} \\ -\frac{8}{11} & \left(1-\frac{6}{11}\hat{h}\right) \end{bmatrix}, Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix}, B = \begin{bmatrix} -\frac{1}{3} & 2 \\ \frac{2}{11} & -\frac{9}{11} \end{bmatrix}, Y_{m-1} = \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix}.$$

The first characteristic polynomial of the Block Method (5) is given by

$$\begin{aligned} \rho(t) &= \det\left[tA - B\right] \\ &= \det\left[t\begin{pmatrix} 1-2\hat{h} & \frac{2}{3} \\ -\frac{8}{11} & \left(1-\frac{6}{11}\hat{h}\right) \end{pmatrix} - \begin{pmatrix} -\frac{1}{3} & 2 \\ \frac{2}{11} & -\frac{9}{11} \end{pmatrix}\right] \\ &= \det\begin{pmatrix} t\left(1-2\hat{h}\right)+\frac{1}{3} & \frac{2t}{3}-2 \\ -\frac{18t}{11}-\frac{2}{11} & t\left(1-\frac{6}{11}\hat{h}\right)+\frac{9}{11} \end{pmatrix} \\ &= -\frac{1}{11} - 2t - \frac{20\hat{h}t}{11} + \frac{23t^2}{11} - \frac{28\hat{h}t^2}{11} + \frac{12\hat{h}^2t^2}{11} \end{aligned}$$

By solving $\rho(t) = \det\left[tA - B\right] = 0$, the following stability polynomial is obtained,

$$-\frac{1}{11} - 2t - \frac{20\hat{h}t}{11} + \frac{23t^2}{11} - \frac{28\hat{h}t^2}{11} + \frac{12\hat{h}^2t^2}{11} = 0 \tag{12}$$

whose stability region is depicted in Section 3.

To determine for zero stable, we substitute $\hat{h} = h\lambda = 0$ to the equation (12). We get

$$-\frac{1}{11} - 2t + \frac{23t^2}{11} = 0$$

Hence,

$$23t^2 - 22t - 1 = 0$$

$$(-23t - 1)\,(-t + 1) = 0$$

This will yield

$$t = \frac{1}{23}, \quad t = 1.$$

Thus the linear multistep method is zero stable. Since one of the roots is $+1$, we call this root principal root and label it $t_1(= +1)$.

3 Stability Region

The absolute stability region R associated with the block method (5) is the set
$R = \{h\lambda$: for that $h\lambda$ where the roots of the stability polynomial (12) are of moduli less than one$\}$.

Below we present the stability region R which corresponds to the block BDF. The stability region R (shaded area) is shown in Figure 1. The stability region is drawn in the $h\lambda$ plane and hence it takes all the values of $h\lambda$.

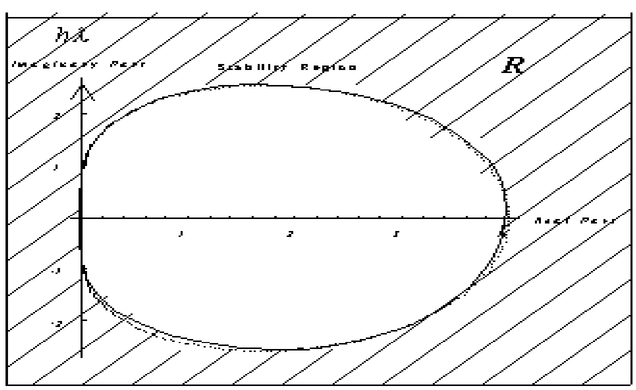

Figure 1: Stability Region for Two Point Block BDF

The roots for the shaded region of the stability polynomial lie inside the complex unit circle and those on its boundary are of moduli one. Hence the Two Point Block Backward Differentiation method is stable in the entire plane except in the white circle. In conclusion, since all region in the left half plane is in the stability region, the method is also A-stable.

Table 1

Test problem	**Step size**	**Maximum error**
Problem 1 $y' = f(x, y) = -10y; y(2) = e^{-20}; 2 \leq x \leq 5$ Solution: $y(x) = e^{-10x}$	0.2	1.97877×10^{-5}
Problem 2 $y' = f(x, y) = -20y + 20; y(1) = 1 + e^{-20}; 1 \leq x \leq 3$ Solution: $y(x) = 1 + e^{-20x}$	0.2	1.45322×10^{-3}

4 Numerical Results

In this section, we give the results for the stiff problems we have tested. If these stiff problems are solved using Euler method, the restriction on the stepsize h is $|h\lambda| < 2$. Hence the problem cannot be solved with Euler for $h = 0.2$ for both problem. These problems are solved by the given Block Method with $h = 0.2$ The numerical results are tabulated in Table 1.

Based on the numerical results, it can be concluded that the Block Method is suitable for stiff problems because of its A stability property.

References

[1] L.F. Shampine & H.A. Watts, *Block implicit one-step methods*, Math. Comp. 23, 731-740, (1969).

[2] M.T. Chu & H. Hamilton, *Parallel solution of ODE's by multi-block methods*,SIAM J. Sci. Stat. Comput. 8, 342-353, (1987).

[3] J.D. Lambert, *Numerical methods for Ordinary Differential Equations: The Initial Value Problems,* John Wiley & Sons, (1991).

[4] J.C. Butcher, *A modified multistep method for the numerical integration of ordinary differential equations*, J. Ass. Comput. Mach. 12, 124-135, (1965).

[5] J.B. Rosser, *A Runge-Kutta for all seasons*, SIAM. Rev. 9, 417-452, (1967).

[6] Fatunla, *Block Methods for Second Order ODEs*, Intern. J. Comp. Math. 41,55-63, (1990).

[7] H.Tam & R.Skeel, *Stability of Parallel Explicit ODE Methods*, SIAM J. NUMER. ANAL 30, 1179-1192,(1993).

[8] R. Alexander, *Stability and error bounds for numerical methods for stiff initial value problems*, Annals of Numerical Mathematics.1-15,(1994).

[9] D. Voss & S. Abbas, *Block Predictor-Corrector Schemes for the Parallel Solution of ODEs*, Comp. Math. Applic. 33,65-72, (1997).

[10] L.S. Lee, *Two and Three Point Block Methods For Solving First Order Ordinary Differential Equations In Parallel,* MSc thesis, Universiti Putra Malaysia, (2000).

[11] Z.B. Omar, *Developing Parallel Block Methods for solving Higher Order ODEs directly,* PhD Thesis, Universiti Putra Malaysia, (1999).

[12] Zarina Bibi, I., Suleiman, M., Fudziah, I., *Fully Implicit Two Point Block Backward Difference Formula for Solving a First Order Initial Value Problems*, Science Putra Research Buletin 11(2): 14-17,(2003).