MATEMATIKA, 2005, Jilid 21, Bil. 1, hlm. 1–13
©Jabatan Matematik, UTM.

Three-Soliton Solutions of The Kadomtsev-Petviashvili Equation

Tiong Wei King, Ong Chee Tiong & Mukheta Isa
Department of Mathematics, Universiti Teknologi Malaysia, Skudai, 81310 Johor Bahru, Malaysia.

Abstract Soliton solutions of the Kadomtsev-Petviashvili (KP) equation which is a two dimensional form of the Korteweg-de Vries (KdV) equation can be obtained by using Hirota Bilinear method. The traditional group-theoretical approach can generate analytic soliton solutions because the KP equation has infinitely many conservation laws. Two-soliton solutions of the KP equation produces a triad, quadruplet and a non-resonant soliton structures in soliton interactions. In three-soliton solutions of the KP equation, we observed two types of interactions patterns namely a triad with a soliton and also a quadruplet with a soliton.

Keywords Soliton, Hirota Bilinear method, Korteweg-de Vries and Kadomtsev-Petviashvili equations.

Abstrak *Penyelesaian soliton persamaan Kadomtsev-Petviashvili yang merupakan persamaan Korteweg-de Vries (KdV) dua dimensi boleh diperolehi dengan menggunakan kaedah Bilinear Hirota. Kaedah tradisi berasaskan teori kumpulan mampu menjana penyelesaian soliton secara analitik kerana persamaan KP mempunyai ketakterhinggaan banyaknya hukum keabadian. Penyelesaian dua-soliton persamaan KP menghasilkan struktur-struktur berbentuk "triad", kuadruplet dan soliton tak beresonan dalam interaksi soliton. Dalam penyelesaian tiga soliton KP, kita dapati dua jenis interaksi iaitu antara "triad" dengan satu soliton dan antara kuadruplet dengan satu soliton.*

Katakunci *Soliton, kaedah Bilinear Hirota, persamaan Korteweg-de Vries dan Kadomtsev-Petviashvili.*

1 Introduction

Resonance is not a new phenomena. In general, whenever a system is capable of oscillating when acted on by a periodic series of impulses having a frequency equal or nearly equally to its natural frequencies, the system will be set into oscillations with a relatively large amplitude. This phenomenon is called resonance and the system is said to resonate with the applied impulses.

Soliton resonance occurs when the interacting region between the incident solitons and the centered-shifted solitons after interaction is essentially itself a single soliton leads to a very simple conceptual picture of the interaction process. This pattern of interacting soliton is the *resonant soliton* associated with the two incident solitons,[2][3].

In this paper, we will find the solution for two and three solitons in the KP equation. Since the KP equation is a two dimensional of the Korteweg-de Vries (KdV) equation, thus there exists solitary waves running in different direction and many interaction patterns can be observed from the interaction of two and three KP solitons.

2 Two-Soliton Solutions of the KP Equation

In general the KP equation can be written as

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+6\,u\,\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right)=3\,\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

and can be simply written as

$$(u_t+6\,u\,u_x+u_{xxx})_x=3\,u_{yy}. \tag{2}$$

We will consider the linearized form of Equation (2), then we have plane-wave solutions whose phase variable $kx+my-\omega t$ satisfied the dispersion relation

$$\omega=-\left(\frac{3\,m^2}{k}+k^3\right). \tag{3}$$

N-soliton solution for Equation (2) had been solved by Satsuma (1976) by using Hirota Bilinear method (1971) as

$$\begin{aligned} u(x,y,t) &= 2\frac{\partial^2}{\partial x^2}\ln f && (4)\\ &= 2\left[\frac{f_{xx}\,f-f_x^2}{f^2}\right] && (5)\end{aligned}$$

with the function $f(x,y,t)$ given by

$$f(x,y,t)=\left|\delta_{ij}+\frac{a_i}{l_i+n_j}\exp(\eta_i)\right| \tag{6}$$

where

$$\delta_{ij}=\begin{cases}1 & i=j,\\ 0 & i\neq j,\end{cases}$$

and

$$\eta_i = k_i x+m_i y-\omega_i t \tag{7}$$

with

$$k_i = l_i + n_i, \tag{8}$$
$$m_i = n_i^2 - l_i^2, \tag{9}$$
$$\omega_i = -\frac{3m_i^2}{k_i} - k_i^3 \tag{10}$$

and

$$\varepsilon_i = \frac{a_i}{l_i + n_i} \quad \text{where } a_i \text{ is constant} \quad \text{for} \quad i, j = 1 \text{ and } 2. \tag{11}$$

Thus for two-soliton solutions of the KP equation,

$$f(x, y, t) = \underbrace{1}_{(1)} + \underbrace{\varepsilon_1 \exp(\eta_1)}_{(2)} + \underbrace{\varepsilon_2 \exp(\eta_2)}_{(3)} + \underbrace{A_{12}\varepsilon_1\varepsilon_2 \exp(\eta_1 + \eta_2)}_{(4)} \tag{12}$$

where

$$A_{12} = \frac{(n_1 - n_2)(l_1 - l_2)}{(l_1 + n_2)(l_2 + n_1)}, \tag{13}$$
$$\varepsilon_1 = \frac{a_1}{l_1 + n_1}, \tag{14}$$
$$\varepsilon_2 = \frac{a_2}{l_2 + n_2}. \tag{15}$$

There are three types of resonance in the KP solitons interaction which are full resonance, partial resonance and non-resonance. Resonance process will only occurs when the value of A_{12} is zero. This is because as A_{12} tends to zero, then $\ln A_{12}$ will tends to $\pm\infty$. Therefore the parametric values of n_1, n_2, l_1 and l_2 will then determined the resonant structure. If we fix the values of l_1 and l_2, then n_1 and n_2 will then determined the value of A_{12}.

2.1 A Triad (Full Resonance)

To have a full resonance, we need to have a situation where $n_1 = n_2$ or $l_1 = l_2$ so that A_{12} will be zero. Therefore we shall fixed the values of l_1 and l_2 but $l_1 \neq l_2$ and consider the case where $n_1 = n_2$. Therefore Equation (12) will becomes

$$f = \underbrace{1}_{(1)} + \underbrace{\varepsilon_1 \exp(\eta_1)}_{(2)} + \underbrace{\varepsilon_2 \exp(\eta_2)}_{(3)} \tag{16}$$

Any combination between any component of the function $f(x, y, t)$ will form one soliton. We can observed that the first soliton, S_1, second soliton, S_2 and the resonant soliton, S_{12} can be represented by symbol (12), (13) and (23) respectively.

$$\text{Soliton } 12, S_1 \ : \ f = 1 + \varepsilon_1 \exp(\eta_1) \tag{17}$$
$$\text{Soliton } 13, S_2 \ : \ f = 1 + \varepsilon_2 \exp(\eta_2) \tag{18}$$
$$\text{Soliton } 23, S_{12} \ : \ f = \varepsilon_1 \exp(\eta_1) + \varepsilon_2 \exp(\eta_2) \tag{19}$$

When the term $\ln A_{12}$ tends to $-\infty$, the length of resonant soliton S_{12} will increase and

this lead us to a *triad* as shown in Figure 1.

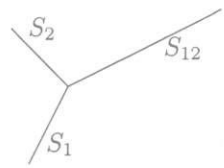

Figure 1: A Triad

2.2 A Quadruplet (Partially Resonance)

In this case, we will consider the case of $n_1 \approx n_2$ so that the values of A_{12} will be so close to zero so that we will have a partially resonance solitons. Thus we have Equation (12) again with $A_{12} \approx 0$. The value of A_{12} will determined the length of resonant soliton S_{12}. In this case we will have an interaction pattern called *quadruplet* as shown in Figure 2 which represent the partially resonant soliton. The bigger the value of A_{12}, the bigger will be $\ln A_{12}$, so the length of interacting soliton will be longer.

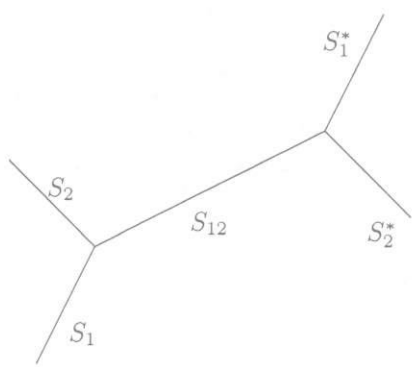

Figure 2: A Quadruplet

From Figure 2, when S_1 and S_2 come into interaction it will produce a resonant soliton S_{12} and later break up again to form S_1^* and S_2^* which is actually soliton S_1 and S_2 respectively but of course with some phase shift.

2.3 A Cross (Non-Resonance)

For the non-resonance case, we will take $n_1 \neq n_2$ and choose the suitable parametric values so that $A_{12} \neq 0$. This means that the line S_{12} is no longer exists. This phenomena is called non-resonant soliton interactions. In this case, S_1 will be centered along the line $k_1x + m_1y = 0$ whereas S_2 will be centered along the line $k_2x + m_2y = 0$ with

$$\tan\alpha = \frac{p_1 - p_2}{1 + p_1 p_2}, \qquad \text{where} \quad p_1 = -\frac{m_1}{k_1}, \quad \text{and} \quad p_2 = -\frac{m_2}{k_2}. \tag{20}$$

In this case α is the angle of interaction between S_1 and S_2 and p_1, p_2 are the gradients of the lines respectively.

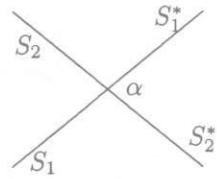

Figure 3: A Cross

3 Three-Soliton Solutions of the KP Equation

By using Equation (2) again and the Hirota Bilinear method, we obtain the function $f(x, y, t)$ for three solitons solution of the KP equation with $N = 3$ as below.

$$\begin{aligned} f &= \underbrace{1}_{(1)} + \underbrace{\varepsilon_1 \exp(\eta_1)}_{(2)} + \underbrace{\varepsilon_2 \exp(\eta_2)}_{(3)} + \underbrace{\varepsilon_3 \exp(\eta_3)}_{(4)} \\ &+ \underbrace{A_{12}\varepsilon_1\varepsilon_2 \exp(\eta_1 + \eta_2)}_{(5)} + \underbrace{A_{13}\varepsilon_1\varepsilon_3 \exp(\eta_1 + \eta_3)}_{(6)} \\ &+ \underbrace{A_{23}\varepsilon_2\varepsilon_3 \exp(\eta_2 + \eta_3)}_{(7)} + \underbrace{A_{123}\varepsilon_1\varepsilon_2\varepsilon_3 \exp(\eta_1 + \eta_2 + \eta_3)}_{(8)} \end{aligned} \tag{21}$$

where

$$A_{12} = \frac{(l_1 - l_2)(n_1 - n_2)}{(l_1 + n_2)(l_2 + n_1)}, \tag{22}$$

$$A_{13} = \frac{(l_1 - l_3)(n_1 - n_3)}{(l_1 + n_3)(l_3 + n_1)}, \tag{23}$$

$$A_{23} = \frac{(l_2 - l_3)(n_2 - n_3)}{(l_2 + n_3)(l_3 + n_2)}, \tag{24}$$

$$
\begin{aligned}
A_{123} &= \left[\frac{a_1a_2a_3}{(l_1+n_1)(l_2+n_2)(l_3+n_3)} - \frac{a_1a_2a_3}{(l_1+n_1)(l_3+n_2)(l_2+n_3)}\right. \\
&- \frac{a_1a_2a_3}{(l_1+n_2)(l_2+n_1)(l_3+n_3)} + \frac{a_1a_2a_3}{(l_1+n_2)(l_3+n_2)(l_2+n_3)} \\
&+ \left.\frac{a_1a_2a_3}{(l_1+n_3)(l_2+n_1)(l_3+n_2)} - \frac{a_1a_2a_3}{(l_1+n_3)(l_3+n_1)(l_2+n_2)}\right]. \qquad (25)
\end{aligned}
$$

The above computational of A_{12}, A_{13}, A_{23} and A_{123} were obtain by simple permutations of the parametric values. Any two combinations of the components of $f(x, y, t)$ will give a soliton solution. Therefore for each soliton we have,

$$
\begin{aligned}
\text{Soliton 1,} \quad S_1 &: (12),\ (35),\ (46),\ (78); && (26)\\
\text{Soliton 2,} \quad S_2 &: (13),\ (25),\ (47),\ (68); && (27)\\
\text{Soliton 3,} \quad S_3 &: (14),\ (26),\ (37),\ (88); && (28)\\
\text{Resonant Soliton,} \quad S_{12} &: (15),\ (23),\ (48),\ (67); && (29)\\
\text{Resonant Soliton,} \quad S_{13} &: (16),\ (24),\ (38),\ (57); && (30)\\
\text{Resonant Soliton,} \quad S_{23} &: (17),\ (28)\ (34),\ (56). && (31)
\end{aligned}
$$

To explain this, let us take the combination of the terms (4) and (6), then we have

$$
\begin{aligned}
f &= \varepsilon_3 \exp(\eta_3) + A_{13}\varepsilon_1\varepsilon_3 \exp(\eta_1+\eta_3) && (32)\\
&= \varepsilon_3 \exp(\eta_3)\left[1 + A_{13}\varepsilon_1 \exp(\eta_1)\right]. && (33)
\end{aligned}
$$

We know that the term $\exp(\eta_3)$ is linear in x, hence it will vanish when we substitute Equation (33) into Equation (4). Thus $f = 1 + A_{13}\varepsilon_1 \exp(\eta_1)$ which is the $f(x, y, t)$ of soliton 1. For the rest of soliton like S_1, S_2, S_3, S_{12}, S_{13} and S_{23}, the same procedure apply.

4 Soliton Interaction Patterns

In three soliton interactions, we will have two type of interactions that is interactions between a triad and a soliton and interactions between a quadruplet and a soliton. We will start with the interactions between a triad with a soliton and later interactions between a quadruplet and a soliton.

4.1 Interactions Patterns of A Triad And A Soliton

For this case the values of the parameters were chosen as follow where S_1 and soliton S_2 will be in full resonance, $(n_1 = n_2 = 3)$ and we also choose n_3 to be very close to n_1, n_2 so that it will then go into resonance process with the other two solitons.

$$
\begin{aligned}
n_1 &= 3 \qquad n_2 = 3 \qquad n_3 = 3 + 10^{-11}\\
l_1 &= -2 \qquad l_2 = 3 \qquad l_3 = 2 && (34)
\end{aligned}
$$

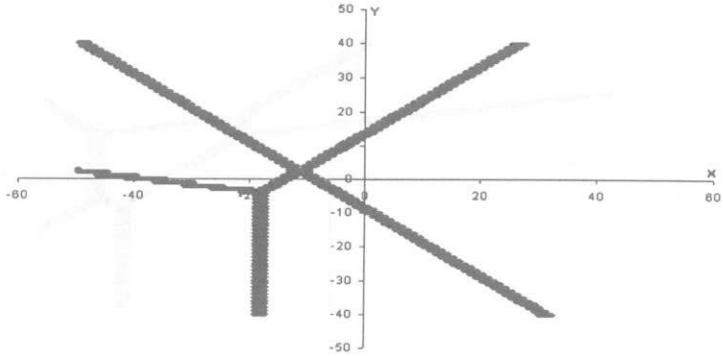

Figure 4: Interaction patterns of a Triad and a Soliton at $t = 0.5$

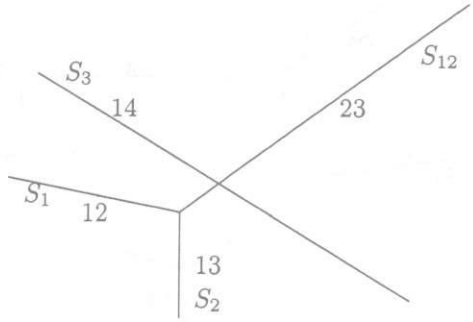

Figure 5: Geometrical representation for the interaction patterns of a Triad and a Soliton at $t = 0.5$

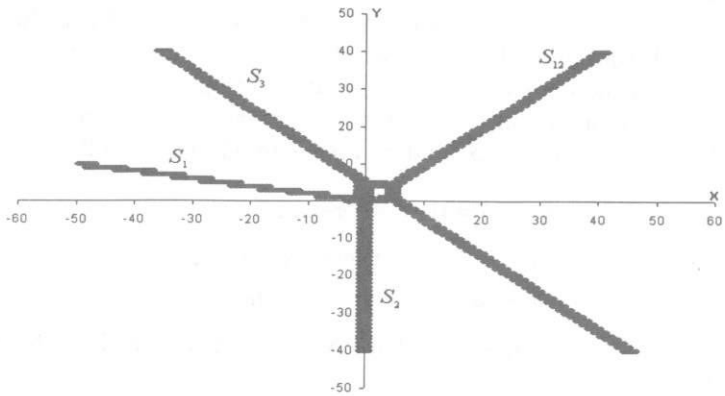

Figure 6: Interaction patterns of a Triad and a Soliton at $t = 0$

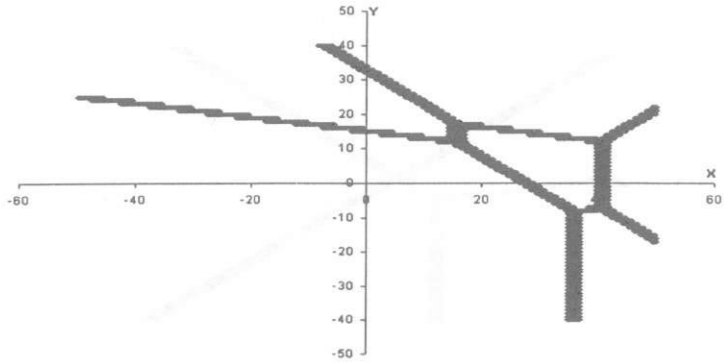

Figure 7: Interaction patterns of a Triad and a Soliton at $t = -1$

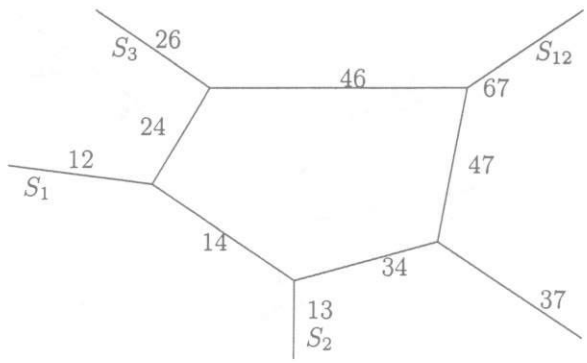

Figure 8: Geometrical representation for the interaction patterns of a Triad and a Soliton at $t = -1$

We can observed that the interaction between a triad and a soliton is from $t = 0.5$ to $t = -1$ and is not from negative value of t to positive value of t. This is due to the positive dispersion of the KP equation. If we considered the negative dispersion of the KP equation, then all interactions generated will be from negative t to positive t.

Before we discuss more about the interaction patterns, let us determine the position of S_1, S_2, S_3 and S_{12}. At $t = 0$, S_1, S_2 and S_3 are centered along the line $k_i x + m_i y = 0$.

$$\begin{aligned} S_1 &: \quad x + 5y = 0 \qquad \text{therefore} \quad y = -\frac{1}{5}x \\ S_2 &: \quad 6x + 0 = 0 \qquad \text{therefore} \quad x = 0 \\ S_3 &: \quad (5 + 10^{-11})x + 5y = 0 \quad \text{therefore} \quad y \approx -x \end{aligned} \tag{35}$$

For this case, we had choose $n_1 = n_2$. Thus S_1 will interacts with S_2 to produce S_{12}, as shown in Figure 4.

From the above interaction pattern, we can observed clearly that this interaction is between a triad and a soliton at $t = 0.5$. If t increase, the patterns will still remain the same but the distance between S_3 with S_1 and S_2 will be even further away.

When $t = 0$ which is from Figure 6, S_3 starts to interact with S_1 and S_2 at the beginning point of the resonant soliton and produce a square pattern. This square becomes bigger and bigger and finally becomes a pentagon as time increase. The pentagon will not change anymore to other form. However it is getting bigger as t decreases. From Figure 7, we can observed that there were two quadruplets and a triad in that pattern.

These interactions patterns can be represented geometrically in Figure 5 and 8. By referring to Figure 5, we see that S_1 (12) interacts with S_2 (13) to produce a resonant soliton S_{12} (23) and later interacts with S_3 (14). Since S_3 (14) has no term that is of common to S_{12} (23), so there is no interaction and they merely collide and cross over. But later as time passed, S_3 (14) will meet S_1 (12) and S_2 (13) and we get a square and later becomes a pentagon.

By referring to Figure 8, we can see that pentagon has all its sides representing a soliton formed by S_1 (46), S_{13} (24), S_3 (14), S_{23} (34) and S_2 (47). If we zoomed in, we realized that there are two quadruplets which is Q_{13} formed by S_1 and S_3 and Q_{23} formed by S_2 and S_3 and also a triad, T_{12} formed by S_1 and S_2.

4.2 Interaction Pattern of A Quadruplet And A Soliton

The interactions of a quadruplet and soliton can be observe when we choose the values as follow where S_1 and S_2 is partially resonance, $(n_1 \approx n_2)$ and we also choose n_3 to be very close to n_1, n_2 so that it will then go into resonance process with the other two soliton.

$$\begin{aligned} n_1 &= 3 + 10^{-11} \qquad n_2 = 3 + 10^{-15} \qquad n_3 = 3 \\ l_1 &= -2 \qquad l_2 = 3 \qquad l_3 = 2 \end{aligned} \tag{36}$$

Now we determine the position of each solitons first. At $t = 0$, every solitons will be centered along the line $k_i x + m_i y = 0$.

$$\begin{aligned} S_1 &: (1 + 10^{-11})x + 5y = 0 \quad \text{therefore} \quad y \approx -\frac{1}{5}x \\ S_2 &: (6 + 10^{-15})x + 0 = 0 \qquad \text{therefore} \quad x = 0 \\ S_3 &: 5x + 5y = 0 \qquad \text{therefore} \quad y = -x \end{aligned} \tag{37}$$

The interaction of a quadruplet and a soliton at $t = 0$ is given Figure 12. Figure 9 to Figure 15 shows the interacting patterns of a quadruplet and a soliton at different value of t.

From the above Figures, we can observed obviously that there is a quadruplet formed by S_1 and S_2 and it will interacts with a soliton at $t = 2$. Since $n_1 \approx n_2 \approx n_3$, therefore S_3 will interacts with both S_1 and S_2 to produce another two quadruplets. So at the beginning of the interactions, we have three quadruplets. We named it as Q_{12}, Q_{13} and Q_{23}.

As the value of t decreases, S_3 will moves towards Q_{12} until it interacts with S_{12}. However there is much interaction happens. Later, the length of both S_{13} and S_{23} become shorter and shorter. After that, there is some interactions at the center or at the origin of the axis. It starts to break up into a pentagon at first and later change into a hexagon. If the value of t continues changing, the size of the hexagon will becomes bigger and bigger.

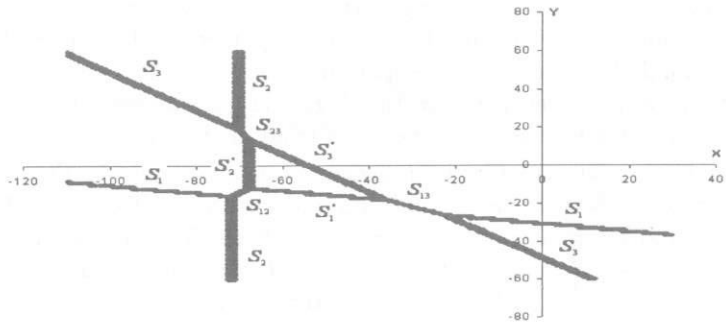

Figure 9: Interaction patterns of a Quadruplet and a Soliton at $t = 2$

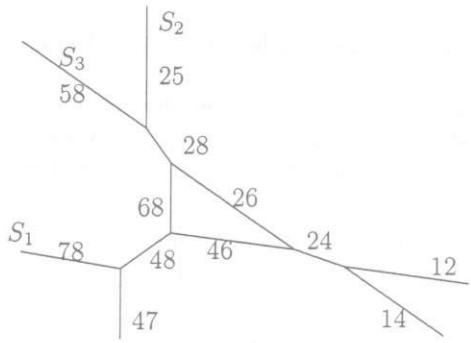

Figure 10: Geometrical representation for the interaction patterns of a Quadruplet and a Soliton at $t = 2$

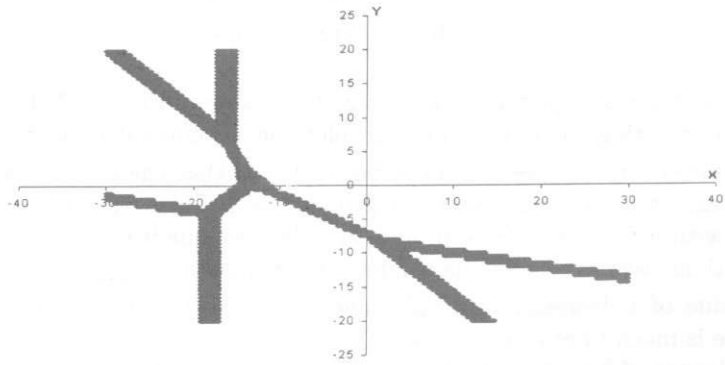

Figure 11: Interaction patterns of a Quadruplet and a Soliton at $t = 0.5$

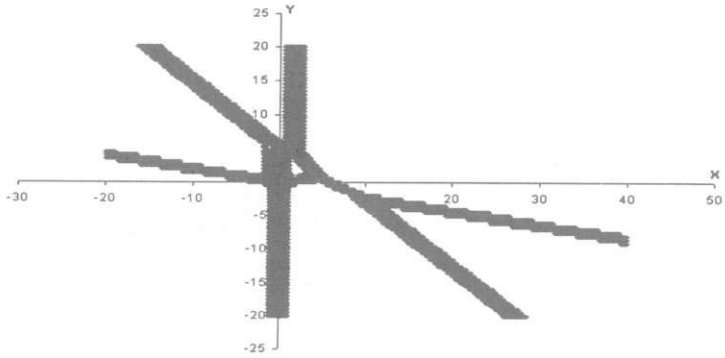

Figure 12: Interaction patterns of a Quadruplet and a Soliton at $t = 0$

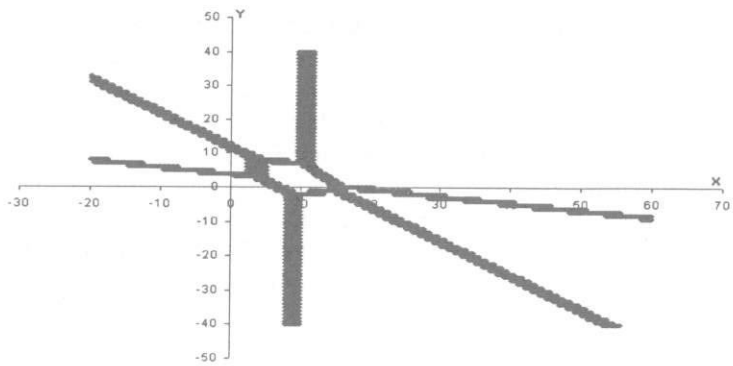

Figure 13: Interaction patterns of a Quadruplet and a Soliton at $t = -0.25$

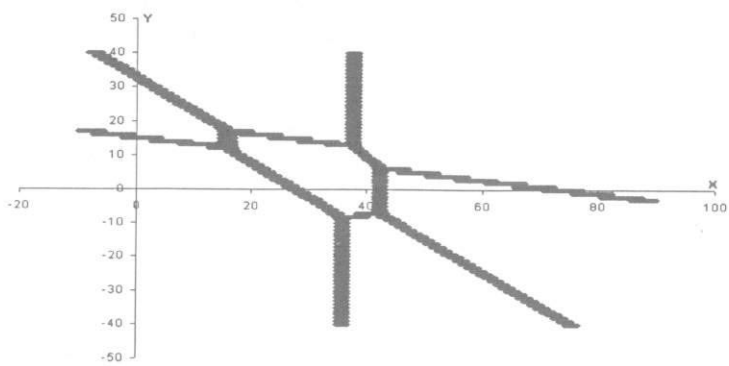

Figure 14: Interaction patterns of a Quadruplet and a Soliton at $t = -1$

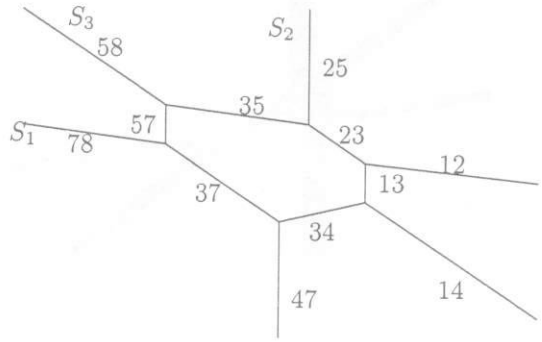

Figure 15: Geometrical representation for the interaction patterns of a Quadruplet and a Soliton at $t = -1$

If we want to explain this interactions by using the combination of terms in the $f(x, y, t)$ function, we still can use Equation (21) and the combination the terms are listed from Equation (26) to Equation (31). This is shown in the Figure 10 and Figure 15.

By referring to Figure 10, we can see that there are three quadruplets which is Q_{12} by S_1 and S_2, Q_{13} by S_1 and S_3 and Q_{23} by S_2 and S_3. As t passed, these three quadruplet will interact among themselves and produce a pentagon and later becomes a hexagon. By referring to Figure 15, we see that hexagon has all its sides representing a soliton formed by S_1 (35), S_{13} (57), S_3 (37), S_{23} (34), S_2 (13) and S_{12} (23). In zooming in, we realized that there still have three quadruplets Q_{12}, Q_{13} and Q_{23}.

5 Conclusion

We have studied the interaction patterns of a triad and a soliton as well as a quadruplet and a soliton in the KP equation. In deriving the solution for three-soliton solutions, we faced some difficulties due to high rank of matrix determinant. Therefore for higher number of soliton, $N = 4, 5, 6, ...$, we need the Wronskian determinant to make our work simpler. This will be our future research and we hope to see more soliton interactions patterns for higher value of N in the KP equation. In this research we use to get analytical solutions because there are infinitely many conservations laws in the KP equation. We hope to develop a Visual Basic program to enable us to observe these interactions patterns automatically as we change the values of the parameters. This sort of user friendly software is important in our future research.

References

[1] D. Anker, and N.C. Freeman, *Interpretation of Three-soliton Interactions in Terms of Resonant Triads*, Journal of Fluid Mech., 87, Part 1, 1978, pg 17-31.

[2] J.W. Miles, *Obliquely Interacting Solitary Waves*, Journal of Fluid Mech., 79, Part 1, 1977, pg 157-169.

[3] J.W. Miles, *Resonantly Interacting Solitary Waves*,Journal of Fluid Mech., 79, Part 1, 1977, pg 171-179.

[4] C.T. Ong, *Various Aspects of Solitons Interactions*, M.Sci. Thesis (Applied Mathematics), Universiti Teknologi Malaysia, 1993.

[5] N.J. Zabusky and M.D. Kruskal, *Interaction of "Solitons" in a Collisionless Plasma and Recurrence of the Initial States*, Phys. Rev. Lett., 15, 1965, pg 240-243.