On Supper Continuity of Topological Spaces

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Abstract
In 1968, Volicko defined the concept of supper-continuity between topological spaces. In this paper supper-continuity is characterised relating to several other well-known variations of continuity, and sufficient and necessary conditions for any of them to imply supper-continuous are also provided. Moreover, many constructions of supper-continuity are studied.

Keywords
Continuity, supper-closed, characterization, closure.

1 Introduction

Several generalizations of continuity were studied in order to provide new characterizations and decompositions of continuous maps. Levine [5] introduced weak continuity; Fomin [2] and Andrew and Whittlesy [1] introduced, independently, the notion of closure continuity. Since then, closure continuity was studied by Saleemi, Shahzad and Alghamdi [9] and they were the authors who provided sufficient conditions for almost continuous maps in the sense of Husain which implies closure continuity. Later Saleh [10] provided several decompositions of closure continuity.

Let \((X, \mathcal{I})\) be a topological space and \(A \subseteq X\). Following Volicko [11], define the supper-closure and the supper-interior of \(A\) by \(A^+ = \{x \in X : \overline{U} \cap A \neq \emptyset\ \text{for every open set } U \text{ containing } x\}\) and \(A^- = \{x \in A : \overline{U} \subseteq A \ \text{for some open set } U \text{ containing } x\}\), respectively. Thus \(A\) is supper-closed if \(A^+ = A\) and supper-open if \(A^- = A\). Equivalently, \(A\) is supper-open if and only if \(X \setminus A\) is supper-closed. A map \(f\) from a topological space \(X\) into a topological space \(Y\) is supper-continuous if the inverse image of every closed subset of \(Y\) is supper-closed in \(X\) or equivalently, the inverse image of every open subset of \(Y\) is supper-open in \(X\). In this paper, we characterize supper-continuity, relate it to several other well-known variations of continuity and provide sufficient and necessary conditions for any of these to be super-continuous. Moreover, we study many constructions of supper-continuity. All throughout this paper, \(X, Y, Z\) and \(W\) will stand for arbitrary topological spaces unless otherwise mentioned.

2 Supper-continuity versus other variations of continuity

We begin this section by recalling a few definitions. A map \(f : X \rightarrow Y\) is strongly-continuous if \(f(\overline{A}) \subseteq f(A)\) for every subset \(A\) of \(X\), see Levine [6]. A map \(f : X \rightarrow Y\) is closure-continuous if it is closure continuous at every point of its domain, that is for every
Consider the topological spaces \( X \subseteq Y \subseteq \mathbb{A} \subseteq \mathbb{A}^+ \). Thus using the well-known fact that semi-continuity needs not imply continuity, we conclude \( X \) is semi-continuous but the converse need not be true, see for example Levine [4].

Example 1 Consider the topological space \( X = \{a, b, c\} \) and \( \mathfrak{T} = \{\emptyset, X, \{a, b\}\} \). Set \( A = \{c\}. \) Then \( A \) is closed, hence semi-closed, but not supper-closed.

Similarly, every supper-open set is both open and semi-open, but the converse need not be true as shown in the following example.

Example 2 Consider the topological spaces \( X = \{a, b, c\} \) and \( \mathfrak{T}_X = \{\emptyset, X, \{a\}, \{a, b\}\} \) and \( \mathfrak{T}_Y = \{\emptyset, Y, \{a\}\} \). Then the identity map \( f \) from \( X \) onto \( Y \) is continuous. Since \( \{c\} \) is closed in \( Y \) and not supper-closed in \( X \), then \( f \) is not supper-continuous.

Theorem 1 Strong continuity implies continuity.

Proof. Let \( f : X \to Y \) be strongly continuous. Then for every \( A \subseteq X \), \( f(A) \subseteq f(A) \) and since \( f(A) \subseteq f(A) \), \( f(A) \subseteq f(A) \) and hence \( f \) is continuous.

Therefore, strong continuity implies semi-continuity while if semi-continuity implies strong continuity, it implies continuity, a contradiction. Example 2 indicates that continuity need not imply strong-continuity since \( \{b\} = \{b, c\} \) is not a subset of \( \{b\} \). Next, we show that supper-continuity and strong continuity are independent notions.

Example 3 Consider the topological spaces \( X = \{a, b, c\} \) and \( \mathfrak{T}_X = \{\emptyset, X, \{a\}, \{a, b\}\} \) and \( \mathfrak{T}_Y = \{\emptyset, Y, \{a\}\} \). Then the map \( f : X \to Y \) defined by \( f(a) = f(c) = b \) and \( f(b) = a \) is clearly supper-continuous. As \( f(\{b\}) = \{a, b\} \notin f(\{b\}) \), \( f \) is not strongly-continuous.

Example 4 Consider the topological spaces \( X = \{a, b, c\} \) and \( \mathfrak{T}_X = \{\emptyset, X, \{a\}, \{a, b\}\} \) and \( \mathfrak{T}_Y = \{\emptyset, Y, \{a\}\} \). Then the map \( f : X \to Y \) defined by \( f(a) = a \) and \( f(b) = f(c) = b \) is clearly strongly-continuous. As \( \{b\} \) is closed in \( Y \) and as \( (f^{-1}(\{b\}))^+ = \{c, b\} \neq f^{-1}(\{b\}) \), \( f \) is not supper-continuous.

We now prove our first main result.

Theorem 2 A surjective strongly-continuous map is supper-continuous.

Proof. Let \( f : X \to Y \) be a surjective strongly-continuous map. For every closed subset \( F \subseteq Y \), \( f^{-1}(F) \subseteq (f^{-1}(F))^+ \) and \( f^{-1}(F) \). Since \( f \) is continuous, \( f^{-1}(F) \) is closed in \( X \) and \( x \in X \setminus f^{-1}(F) \) which is open. As \( x \in f^{-1}(F)^+ \), \( f^{-1}(F)) \cap X \setminus f^{-1}(F) \neq \emptyset. \) Thus there exists \( z \in (f^{-1}(F)) \cap X \setminus f^{-1}(F). \) Hence \( f(z) \in F \cap f(X \setminus f^{-1}(F)) \subseteq F \cap f(X \setminus f^{-1}(F)) \).
Let $f$ is strongly-continuous. Thus as $f$ is surjective, $f(z) \in F \cap (Y \setminus F) = \emptyset$ which is impossible. Therefore, $(f^{-1}(F))^+ \subseteq f^{-1}(F) \subseteq (f^{-1}(F))^+$ and so $f$ is supper-continuous.

**Theorem 3** Every supper-continuous map is closure-continuous.

**Proof.** Let $f : X \to Y$ be a supper-continuous map. Let $x \in X$ and $V$ be an open set in $Y$ such that $f(x) \in V$. As $f$ is continuous, there exists open $U$ in $X$ such that $x \in U$ and $f(U) \subseteq V$. Hence $f(U) \subseteq V$. Again as $f$ is continuous, $f(U) \subseteq f(U)$ and hence $f(U) \subseteq V$. Therefore $f$ is closure-continuous.

We recall the following example from Saleh [10].

**Example 5** Consider the topological space $Y = \{a, b, c, d\}$ and

$$\mathcal{T} = \{\emptyset, Y, \{a, b\}, \{b\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\},$$

$\mathbb{R}$ with the usual topology and the map $f : \mathbb{R} \to Y$ defined by $f(x) = b$ if $x \in \mathbb{Q}$ and $f(x) = d$ if $x \notin \mathbb{Q}$. Then $f$ is closure-continuous but is not continuous.

Since every supper-continuous is continuous, it follows that a closure-continuous map needs not be supper-continuous.

Next, we state several definitions. A map $f : X \to Y$ is called *almost continuous at $x$ in the sense of Singal and Singal* (respectively, *Husain at $x$*) if for each open subset $V \subseteq Y$ containing $f(x)$, there exists an open set $U$ containing $x$ such that $f(U) \subseteq \overline{V}$ (respectively, the closure of $f^{-1}(V)$ is a neighborhood of $x$). A map $f$ is *almost continuous in the sense of Singal and Singal* (simply, *a.c.S.* (respectively, *Husain* (simply, *a.c.H.*))) if it is a.c.S. (respectively, a.c.H.) at each point $x \in X$. A map $f$ is called *almost continuous in the sense of Stallings* (simply, *a.c.St.*) if given any open set $W \subseteq X \times Y$ containing the graph of $f$, there exists a continuous map $g : X \to Y$ such that the graph of $g$ is a subset of $W$. A map $f$ is called *weakly continuous at $x$* if for each open subset $V \subseteq Y$ containing $f(x)$, there exists an open set $U$ containing $x$ such that $f(U) \subseteq \overline{V}$. A map $f$ is *weakly continuous* (simply *w.c.*) if it is weakly continuous at each $x \in X$. A map $f$ is said to be *w*.c. if for each open subset $V \subseteq Y$, we have $f^{-1}(\text{Bd}(V))$ is closed in $X$, where $\text{Bd}(V)$ is the boundary of $V$. For the preceeding definitions, see for example Long and Carnahan [7] and Noiri [8]. It was shown that continuity implies all these five variations, but none of them implies continuity. For the proof of the following three results, see Saleh [10].

**Theorem 4** Let $f : X \to Y$ be an a.c.H. Then

1. $f$ is closure-continuous iff $f^{-1}(\overline{V}) \subseteq f^{-1}(\overline{V})$ for every open subset $V \subseteq Y$.
2. If $f$ is w.c., then $f$ is closure-continuous.

**Theorem 5** An open a.c.H. map $f : X \to Y$ is closure-continuous iff $\overline{f^{-1}(V)} = f^{-1}(\overline{V})$ for every open subset $V \subseteq Y$.

**Theorem 6** Every w.c. map is a.c.H.

Next, we give similar results for supper-continuity.
Theorem 7 Every supper-continuous map is a.c.S., a.c.H., a.c.St., w.c. and w*.c.

Proof. This follows from the fact that every supper-continuous map is continuous.

None of the converses of Theorem 7 is true as shown next.

Example 6 Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = -x$ otherwise. It was shown in Long and Carnahan [7] that $f$ is a.c.H. but not continuous. Thus $f$ is not supper-continuous.

Example 7 Consider the cofinite topology on $\mathbb{R}$ and identity map from $\mathbb{R}$ onto $\mathbb{R}$. This map is continuous and hence a.c.S., but it is not supper-continuous.

Example 8 Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$. It was shown in Long and Carnahan [7] that $f$ is a.c.St. but not continuous. Thus $f$ is not supper-continuous.

Example 9 Consider the topological spaces $X = Y = \{a, b\}$, $\mathcal{T}_X = \{\emptyset, X, \{a\}\}$ and $\mathcal{T}_Y = \{\emptyset, Y, \{b\}\}$. Then the identity map from $X$ into $Y$ is w.c., but not continuous at $b$ and hence not supper-continuous.

Example 10 Consider the topological spaces $X = Y = \{a, b\}$, $\mathcal{T}_X = \{\emptyset, X\}$ and $\mathcal{T}_Y = \{\emptyset, Y, \{a\}, \{b\}\}$. Then the identity map from $X$ into $Y$ is w*.c., but not continuous at $a$ and hence not supper-continuous.

3 On constructions of supper-continuity

In this section we present some results.

Lemma 1 If $A$ and $B$ are supper-closed sets in $X$, then $A \times B$ is supper-closed.

Proof. Let $A = A^+$ and $B = B^+$. Clearly $A \times B \subseteq (A \times B)^+$. On the other hand, for every $(x, y) \in (A \times B)^+$ and for every open sets $U$ and $V$ in $X$ such that $(x, y) \in U \times V$, $U \times V \cap (A \times B) \neq \emptyset$. Hence $U \times V \cap (A \times B) \neq \emptyset$ and so $U \cap A \neq \emptyset$ and $V \cap B \neq \emptyset$. Then $(x, y) \in A^+ \times B^+$.

Corollary 1 If $f : X \to Y$ and $g : Z \to W$ are supper-continuous, then $f \times g$ is supper-continuous.

The proof of the following result follows immediately from the definition.

Lemma 2 If $A \subseteq B$, then $A^+ \subseteq B^+$.

Theorem 8 If $f : X \to Y$ is supper-continuous and $A$ is a supper-closed subset of $X$, then $f|_A$ is supper-continuous.

Proof. For every open subset $V \subseteq Y$, $(f^{-1}(V))^+ = f^{-1}(V)$. Hence

$$(f|_A^{-1}(V))^+ = (A \cap f^{-1}(V))^+$$

and by Lemma 2 $(f|_A^{-1}(V))^+ \subseteq A \cap f^{-1}(V) = f|_A^{-1}(V) \subseteq (f|_A^{-1}(V))^+$. 

Theorem 9  If $f : X \to Y$ is supper-continuous and $g : Y \to Z$ is continuous, then $g \circ f$ is supper-continuous.

Proof. For every closed subset $F \subseteq Z$, $g^{-1}(F)$ is closed in $Y$ since $g$ is continuous. Since $f$ is supper-continuous, $(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F))$ is supper-closed in $X$.

Corollary 2  The composition of two supper-continuous maps is supper-continuous.

Lemma 3  If $A$ and $B$ are supper-closed sets in $X$, then $A \cup B$ is supper-closed.

Proof. Let $A = A^+$ and $B = B^+$. Clearly $A \cup B \subseteq (A \cup B)^+$. On the other hand, for every $x \in (A \cup B)^+$ and for every open set $U$ in $X$ containing $x$, $\overline{U} \cap (A \cup B) \neq \emptyset$. Hence $\overline{U} \cap A \neq \emptyset$ or $\overline{U} \cap B \neq \emptyset$. Thus $x \in A^+ \cup B^+$.

Corollary 3  If $f : X \to Y$ and $g : Z \to W$ are supper-continuous maps such that $f|_{X \cap Z} = g|_{X \cap Z}$, then $f \cup g$ is supper-continuous.

Let $A \subseteq X$. If there exists a supper-continuous map $f : X \to A$ such that $f|_{A} = id_{A}$, then $A$ is called a supper-retraction of $X$.

Theorem 10  If $A$ is a supper-retraction of a $T_2$ space $X$, then $A$ is closed.

Proof. Let $f : X \to A$ be a supper-continuous map such that $f|_{A} = id_{A}$. For every $x \in X \setminus A$, $f(x) \in A$ and $f(x) \neq x$ and as $X$ is $T_2$, there exists disjoint open sets $U$ and $V$ such that $x \in U$ and $f(x) \in V$. Since $f$ is continuous, there exists an open set $W$ such that $x \in W \subseteq U$ and $f(W) \subseteq V$. If $y \in W \cap A$, then $f(y) \in V$ and since $y \in A$, $y = f(y) \in V$. On the other hand, $y \in W$ and hence $y \in U \cap V = \emptyset$, a contradiction. Thus $W \cap A = \emptyset$ and so $x \in W \subseteq X \setminus A$. Therefore $X \setminus A$ is open. Thus $A$ is closed.

4 Characterizations of supper-continuity

Lemma 4  Let $(X, \mathcal{A})$ be a regular space. Then $A^+ = \overline{A}$ for all $A \subseteq X$.

Proof. Clearly $\overline{A} \subseteq A^+$. On the other hand, suppose there exists $x \in A^+ \setminus \overline{A}$. Then since $X$ is regular, there exists disjoint open sets $U$ and $V$ such that $x \in U$ and $\overline{A} \subseteq V$. Since $x \in A^+$, $U \cap A \neq \emptyset$. But since $U \cap V = \emptyset$ and $\overline{A} \subseteq V$, $U \cap \overline{A} = \emptyset$. This implies that $U \cap A = \emptyset$, a contradiction.

Corollary 4  Every closed subset of a regular space is supper-closed.

Proof. $A = \overline{A} = A^+$.

Theorem 11  Let $(X, \mathcal{A})$ be a regular space. Then a map $f : X \to Y$ is supper-continuous iff $f$ is continuous.

Proof. Supper-continuity implies continuity is trivial. Conversely, let $f$ be continuous and let $F$ be a closed subset of $Y$. Then $f^{-1}(F)$ is closed in $X$ and as $X$ is regular, by Corollary 4, $f^{-1}(F)$ is supper-closed. Therefore, $f$ is supper-continuous.

Combining Theorem 2 and Theorem 11, we have the following result.

Corollary 5  A surjective map $f : X \to Y$ is supper-continuous iff $f$ is strongly-continuous.
Next, we provide a characterization of supper-continuity.

**Theorem 12** $A^+$ is the intersection of the closures of all open supper subsets of $A$.

**Proof.** Suppose there exists $x \in \overline{U}$ for all open sets $U$ such that $A \subseteq U$ and $x \notin A^+$. Then there exists open $V$ such that $x \in V$ and $\overline{V} \cap A = \emptyset$. Thus $A \subseteq X \setminus \overline{V}$ which is open. Hence $x \in X \setminus \overline{V}$. Since $x \in V$, $V \cap X \setminus \overline{V} \neq \emptyset$. Thus $X \setminus \overline{V} \subseteq X \setminus V$ and so $V \notin \overline{V}$, a contradiction.

On the other hand, suppose there exists $x \in A^+$ such that $x \notin \overline{U}$ for some open sets $U$ such that $A \subseteq U$. Then $x \in X \setminus \overline{U}$ which is open and as $x \in A^+$, there exists $y \in A \cap X \setminus \overline{U}$. Since $A \subseteq U$, $U \cap X \setminus \overline{U} \neq \emptyset$. Thus $X \setminus \overline{U} \subseteq X \setminus U$ and so $U \notin \overline{V}$, a contradiction.

If $A$ and $B$ are subsets of $X$, we denote the supper-closure of $B$ in the subspace topology of $A$ by $B^A$. We get the following result.

**Corollary 6** The intersection of any supper-closed set $B$ with any set $A$ is supper-closed in $A$.

**Proof.** By Theorem 12, $(B \cap A)^A = \cap \{ \overline{U} \cap \overline{A} : U \text{ open such that } B \subseteq U \}$ which contains $B \cap A$. If $x \in (B \cap A)^A$, then $x \in \overline{A} \cap \overline{U}$ for all open $U$ such that $B \subseteq U$ and thus $x \in \{ \overline{A} \cap \overline{U} : U \text{ is open containing } B \} = \overline{A} \cap \{ \overline{U} : U \text{ is open containing } B \} = A \cap \overline{B} = A \cap B$. Therefore $(B \cap A)^A \subseteq B \cap A \subseteq (B \cap A)^A$.

Thus the intersection of two supper-closed sets is supper-closed. Finally, we provide a stronger result than Theorem 8.

**Corollary 7** If $f : X \to Y$ is supper-continuous and $A \subseteq X$, then $f|_A$ is supper-continuous.

**Proof.** Let $F$ be a closed subset of $Y$. Then $(f|_A)^{-1}(F) = f^{-1}(F) \cap A$ which is supper-closed by Corollary 6.

5 Conclusion

Several characterizations of supper-continuity are given and the relations of this notion to many other well-known variations of continuity are also provided. Moreover, sufficient and necessary conditions for any of these variations to imply supper-continuity are obtained and constructions of supper-continuity are explored.

**References**


