MATEMATIKA, 2005, Jilid 21, Bil. 1, hlm. 43–49

On Supper Continuity of Topological Spaces

Talal Ali Al-Hawary
Department of Mathematics & Statistics, Mu'tah University
P. O. Box 6, Karak–Jordan. talal@mutah.edu.jo

Abstract In 1968, Volicko defined the concept of supper-continuity between topological spaces. In this paper supper-continuity is characterised relating to several other well-known variations of continuity, and sufficient and necessary conditions for any of them to imply supper-continuous are also provided. Moreover, many constructions of super-continuity are studied.

Keywords Continuity, supper-closed, characterization, closure.

1 Introduction

Several generalizations of continuity were studied in order to provide new characterizations and decompositions of continuous maps. Levine [5] introduced weak continuity; Fomin [2] and Andrew and Whittlesy [1] introduced, independently, the notion of closure continuity. Since then, closure continuity was studied by Saleemi, Shahzad and Alghamdi [9] and they were the authors who provided sufficient conditions for almost continuous maps in the sense of Husain which implies closure continuity. Later Saleh [10] provided several decompositions of closure continuity.

Let $(X, \mathfrak{T})$ be a topological space and $A \subseteq X$. Following Volicko [11], define the *supper-closure* and the *supper-interior* of A by $A^+ = \{x \in X : \bar{U} \cap A \neq \varnothing$ for every open set U containing $x\}$ and $A^- = \{x \in A : \bar{U} \subseteq A$ for some open set U containing $x\}$, respectively. Thus A is *supper-closed* if $A^+ = A$ and *supper-open* if $A^- = A$. Equivalently, A is supper-open if and only if $X \backslash A$ is supper-closed. A map f from a topological space X into a topological space Y is supper-continuous if the inverse image of every closed subset of Y is supper-closed in X or equivalently, the inverse image of every open subset of Y is supper-open in X. In this paper, we characterize supper-continuity, relate it to several other well-known variations of continuity and provide sufficient and necessary conditions for any of these to be super-continuous. Moreover, we study many constructions of supper-continuity. All throughout this paper, X, Y, Z and W will stand for arbitrary topological spaces unless otherwise mentioned.

2 Supper-continuity versus other variations of continuity

We begin this section by recalling a few definitions. A map $f : X \to Y$ is *strongly-continuous* if $f(\bar{A}) \subseteq f(A)$ for every subset A of X, see Levine [6]. A map $f : X \to Y$ is *closure-continuous* if it is closure continuous at every point of its domain, that is for every

$x \in X$ and every open set V in Y such that $f(x) \in V$, there exists an open set U in X containing x and satisfies $f(\bar{U}) \subseteq \bar{V}$, see Fomin [2]. A subset A of X is called *semi-open* if there exists an open set U such that $U \subseteq A \subseteq \bar{U}$. The *semi-closure* of $\underline{A}$ is defined analogously to $\bar{A}$ and A is *semi-closed* if $\underline{A}= A$. A map $f : X \to Y$ is *semi-continuous* if the inverse image of every open subset of Y is semi-open in X. Clearly, every continuous map is semi-continuous but the converse needs not be true, see for example Levine [4].

Note that $A \subseteq \underline{A} \subseteq \bar{A} \subseteq A^+$. Therefore, any supper-closed set is both closed and semi-closed but the converse need not be true as shown in the following example.

Example 1 *Consider the topological space* $X = \{a, b, c\}$ *and* $\mathfrak{T} = \{\varnothing, X, \{a, b\}\}$. *Set* $A = \{c\}$. *Then* A *is closed, hence semi-closed, but not supper-closed.*

Similarly, every supper-open set is both open and semi-open, but the converse need not be true. It follows that every supper-continuous map is both continuous and semi-continuous. In the next example, we show that continuity need not imply supper-continuity. Thus using the well-known fact that semi-continuity needs not imply continuity, we conclude that semi-continuity needs not imply supper-continuity.

Example 2 *Consider the topological spaces* $X = \{a, b, c\} = Y$, $\mathfrak{T}_X = \{\varnothing, X, \{a\}, \{a, b\}\}$ *and* $\mathfrak{T}_Y = \{\varnothing, Y, \{a, b\}\}$. *Then the identity map* f *from* X *onto* Y *is continuous. Since* $\{c\}$ *is closed in* Y *and not supper-closed in* X, *then* f *is not supper-continuous.*

Theorem 1 *Strong continuity implies continuity.*
Proof. *Let* $f : X \to Y$ *be strongly continuous. Then for every* $A \subseteq X$, $f(\bar{A}) \subseteq f(A)$ *and since* $f(A) \subseteq \overline{f(A)}$, $f(\bar{A}) \subseteq \overline{f(A)}$ *and hence* f *is continuous.*

Therefore, strong continuity implies semi-continuity while if semi-continuity implies strong continuity, it implies continuity, a contradiction. Example 2 indicates that continuity needs not imply strong-continuity since $\overline{\{b\}} = \{b, c\}$ is not a subset of $\{b\}$. Next, we show that supper-continuity and strong continuity are independent notions.

Example 3 *Consider the topological spaces* $X = \{a, b, c\} = Y$, $\mathfrak{T}_X = \{\varnothing, X, \{a\}, \{a, b\}\}$ *and* $\mathfrak{T}_Y = \{\varnothing, Y\}$. *Then the map* $f : X \to Y$ *defined by* $f(a) = f(c) = b$ *and* $f(b) = a$ *is clearly supper-continuous. As* $f(\overline{\{b\}}) = \{a, b\} \nsubseteq f(\{b\})$, f *is not strongly-continuous.*

Example 4 *Consider the topological spaces* $X = \{a, b, c\} = Y$, $\mathfrak{T}_X = \{\varnothing, X, \{a\}, \{c, b\}\}$ *and* $\mathfrak{T}_Y = \{\varnothing, Y, \{c\}\}$. *Then the map* $f : X \to Y$ *defined by* $f(a) = a$ *and* $f(b) = f(c) = b$ *is clearly strongly-continuous. As* $\{b\}$ *is closed in* Y *and as* $(f^{-1}(\{b\}))^+ = \{c, b\} \neq f^{-1}(\{b\})$, f *is not supper-continuous.*

We now prove our first main result.

Theorem 2 *A surjective strongly-continuous map is supper-continuous.*
Proof. *Let* $f : X \to Y$ *be a surjective strongly-continuous map. For every closed subset* $F \subseteq Y$, $f^{-1}(F) \subseteq (f^{-1}(F))^+$. *Suppose there exists* $x \in (f^{-1}(F))^+ \backslash f^{-1}(F)$. *Since* f *is continuous,* $f^{-1}(F)$ *is closed in* X *and* $x \in X \backslash f^{-1}(F)$ *which is open. As* $x \in (f^{-1}(F))^+$, $(f^{-1}(F)) \cap \overline{X \backslash f^{-1}(F)} \neq \varnothing$. *Thus there exists* $z \in (f^{-1}(F)) \cap \overline{X \backslash f^{-1}(F)}$. *Hence*

$$f(z) \in F \cap f(\overline{X \backslash f^{-1}(F)}) \subseteq F \cap f(X \backslash f^{-1}(F)$$

since f is strongly-continuous. Thus as f is surjective, $f(z) \in F \cap (Y \backslash F) = \varnothing$ which is impossible. Therefore, $(f^{-1}(F))^+ \subseteq f^{-1}(F) \subseteq (f^{-1}(F))^+$ and so f is supper-continuous.

Theorem 3 *Every supper-continuous map is closure-continuous.*
Proof. *Let $f : X \to Y$ be a supper-continuous map. Let $x \in X$ and V be an open set in Y such that $f(x) \in V$. As f is continuous, there exists open U in X such that $x \in U$ and $f(U) \subseteq V$. Thus $\overline{f(U)} \subseteq \overline{V}$. Again as f is continuous, $f(\overline{U}) \subseteq \overline{f(U)}$ and hence $f(\overline{U}) \subseteq \overline{V}$. Therefore f is closure-continuous.*

We recall the following example from Saleh [10].

Example 5 *Consider the topological space $Y = \{a, b, c, d\}$ and*

$$\mathfrak{T} = \{\varnothing, Y, \{a, b\}, \{b\}, \{d\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\},$$

$\mathbb{R}$ with the usual topology and the map $f : \mathbb{R} \to Y$ defined by $f(x) = b$ if $x \in \mathbb{Q}$ and $f(x) = d$ if $x \notin \mathbb{Q}$. Then f is closure-continuous but is not continuous.

Since every supper-continuous is continuous, it follows that a closure-continuous map needs not be supper-continuous.

Next, we state several definitions. A map $f : X \to Y$ is called *almost continuous at x in the sense of Singal and Singal* (respectively, *Husain at x*) if for each open subset $V \subseteq Y$ containing $f(x)$, there exists an open set U containing x such that $f(U) \subseteq \overset{o}{\overline{V}}$ (respectively, the closure of $f^{-1}(V)$ is a neighborhood of x). A map f is *almost continuous in the sense of Singal and Singal* (simply, *a.c.S.*) (respectively, *Husain* (simply, *a.c.H.*)) if it is a.c.S. (respectively, a.c.H.) at each point $x \in X$. A map f is called *almost continuous in the sense of Stallings* (simply, *a.c.St.*) if given any open set $W \subseteq X \times Y$ containing the graph of f, there exists a continuous map $g : X \to Y$ such that the graph of g is a subset of W. A map f is called *weakly continuous at x* if for each open subset $V \subseteq Y$ containing $f(x)$, there exists an open set U containing x such that $f(U) \subseteq \overline{V}$. A map f is *weakly continuous* (simply *w.c.*) if it is weakly continuous at each $x \in X$. A map f is said to be *w*.c.* if for each open subset $V \subseteq Y$, we have $f^{-1}(Bd(V))$ is closed in X, where $Bd(V)$ is the boundary of V. For the preceeding definitions, see for example Long and Carnahan [7] and Noiri [8]. It was shown that continuity implies all these five variations, but none of them implies continuity. For the proof of the following three results, see Saleh [10].

Theorem 4 *Let $f : X \to Y$ be an a.c.H. Then*
(1) f is closure-continuous iff $\overline{f^{-1}(V)} \subseteq f^{-1}(\overline{V})$ for every open subset $V \subseteq Y$.
(2) If f is w.c., then f is closure-continuous.

Theorem 5 *An open a.c.H. map $f : X \to Y$ is closure-continuous iff $\overline{f^{-1}(V)} = f^{-1}(\overline{V})$ for every open subset $V \subseteq Y$.*

Theorem 6 *Every w.c. map is a.c.H.*

Next, we give similar results for supper-continuity.

Theorem 7 *Every supper-continuous map is a.c.S., a.c.H., a.c.St., w.c. and w^*.c.*
Proof. *This follows from the fact that every supper-continuous map is continuous.*

None of the converses of Theorem 7 is true as shown next.

Example 6 *Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = -x$ otherwise. It was shown in Long and Carnahan [7] that f is a.c.H. but not continuous. Thus f is not supper-continuous.*

Example 7 *Consider the cofinite topology on $\mathbb{R}$ and identity map from $\mathbb{R}$ onto $\mathbb{R}$. This map is continuous and hence a.c.S., but it is not supper-continuous.*

Example 8 *Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$. It was shown in Long and Carnahan [7] that f is a.c.St. but not continuous. Thus f is not supper-continuous.*

Example 9 *Consider the topological spaces $X = Y = \{a, b\}$, $\mathfrak{T}_X = \{\varnothing, X, \{a\}\}$ and $\mathfrak{T}_Y = \{\varnothing, Y, \{b\}\}$. Then the identity map from X into Y is w.c., but not continuous at b and hence not supper-continuous.*

Example 10 *Consider the topological spaces $X = Y = \{a, b\}$, $\mathfrak{T}_X = \{\varnothing, X\}$ and $\mathfrak{T}_Y = \{\varnothing, Y, \{a\}, \{b\}\}$. Then the identity map from X into Y is w^*.c., but not continuous at a and hence not supper-continuous.*

3 On constructions of supper-continuity

In this section we present some results.

Lemma 1 *If A and B are supper-closed sets in X, then $A \times B$ is supper-closed.*
Proof. *Let $A = A^+$ and $B = B^+$. Clearly $A \times B \subseteq (A \times B)^+$. On the other hand, for every $(x, y) \in (A \times B)^+$ and for every open sets U and V in X such that $(x, y) \in U \times V$, $\overline{U \times V} \cap (A \times B) \neq \varnothing$. Hence $\overline{U} \times \overline{V} \cap (A \times B) \neq \varnothing$ and so $\overline{U} \cap A \neq \varnothing$ and $\overline{V} \cap B \neq \varnothing$. Then $(x, y) \in A^+ \times B^+$.*

Corollary 1 *If $f : X \to Y$ and $g : Z \to W$ are supper-continuous, then $f \times g$ is supper-continuous.*

The proof of the following result follows immediately from the definition.

Lemma 2 *If $A \subseteq B$, then $A^+ \subseteq B^+$.*

Theorem 8 *If $f : X \to Y$ is supper-continuous and A is a supper-closed subset of X, then $f|_A$ is supper-continuous.*
Proof. *For every open subset $V \subseteq Y$, $(f^{-1}(V))^+ = f^{-1}(V)$. Hence*

$$(f|_A^{-1}(V))^+ = (A \cap f^{-1}(V))^+$$

and by Lemma 2 $(f|_A^{-1}(V))^+ \subseteq A \cap f^{-1}(V) = f|_A^{-1}(V) \subseteq (f|_A^{-1}(V))^+$.

Theorem 9 *If $f : X \to Y$ is supper-continuous and $g : Y \to Z$ is continuous, then $g \circ f$ is supper-continuous.*
Proof. *For every closed subset $F \subseteq Z$, $g^{-1}(F)$ is closed in Y since g is continuous. Since f is supper-continuous, $(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F))$ is supper-closed in X.*

Corollary 2 *The composition of two supper-continuous maps is supper-continuous.*

Lemma 3 *If A and B are supper-closed sets in X, then $A \cup B$ is supper-closed.*
Proof. *Let $A = A^+$ and $B = B^+$. Clearly $A \cup B \subseteq (A \cup B)^+$. On the other hand, for every $x \in (A \cup B)^+$ and for every open set U in X containing x, $\overline{U} \cap (A \cup B) \neq \varnothing$. Hence $\overline{U} \cap A \neq \varnothing$ or $\overline{U} \cap B \neq \varnothing$. Thus $x \in A^+ \cup B^+$.*

Corollary 3 *If $f : X \to Y$ and $g : Z \to W$ are supper-continuous maps such that $f|_{X\cap Z} = g|_{X\cap Z}$, then $f \cup g$ is supper-continuous.*

Let $A \subseteq X$. If there exists a supper-continuous map $f : X \to A$ such that $f|_A = id_A$, then A is called a *supper-retraction of* X.

Theorem 10 *If A is a supper-retraction of a T_2 space X, then A is closed.*
Proof. *Let $f : X \to A$ be a supper-continuous map such that $f|_A = id_A$. For every $x \in X \backslash A$, $f(x) \in A$ and $f(x) \neq x$ and as X is T_2, there exists disjoint open sets U and V such that $x \in U$ and $f(x) \in V$. Since f is continuous, there exists an open set W such that $x \in W \subseteq U$ and $f(W) \subseteq V$. If $y \in W \cap A$, then $f(y) \in V$ and since $y \in A$, $y = f(y) \in V$. On the other hand, $y \in W$ and hence $y \in U \cap V = \varnothing$, a contradiction. Thus $W \cap A = \varnothing$ and so $x \in W \subseteq X\backslash A$. Therefore $X\backslash A$ is open. Thus A is closed.*

4 Characterizations of supper-continuity

Lemma 4 *Let $(X, \mathfrak{T})$ b a regular space. Then $A^+ = \overline{A}$ for all $A \subseteq X$.*
Proof. *Clearly $\overline{A} \subseteq A^+$. On the other hand, suppose there exists $x \in A^+\backslash\overline{A}$. Then since X is regular, there exists disjoint open sets U and V such that $x \in U$ and $\overline{A} \subseteq V$. Since $x \in A^+$, $\overline{U} \cap A \neq \varnothing$. But since $\overline{U} \cap V = \varnothing$ and $\overline{A} \subseteq V$, $\overline{U} \cap \overline{A} \neq \varnothing$. This implies that $\overline{U} \cap A = \varnothing$, a contradiction.*

Corollary 4 *Every closed subset of a regular space is supper-closed.*
Proof. $A = \overline{A} = A^+$.

Theorem 11 *Let $(X, \mathfrak{T})$ be a regular space. Then a map $f : X \to Y$ is supper-continuous iff f is continuous.*
Proof. *Supper-continuity implies continuity is trivial. Conversely, let f be continuous and let F be a closed subset of Y. Then $f^{-1}(F)$ is closed in X and as X is regular, by Corollary 4, $f^{-1}(F)$ is supper-closed. Therefore, f is supper-continuous.*

Combining Theorem 2 and Theorem 11, we have the following result.

Corollary 5 *A surjective map $f : X \to Y$ is supper-continuous iff f is strongly-continuous.*

Next, we provide a characterization of supper-continuity.

Theorem 12 *A^+ is the intersection of the closures of all open supper subsets of A.*

Proof. *Suppose there exists $x \in \overline{U}$ for all open sets U such that $A \subseteq U$ and $x \notin A^+$. Then there exists open V such that $x \in V$ and $\overline{V} \cap A = \varnothing$. Thus $A \subseteq X \backslash \overline{V}$ which is open. Hence $x \in \overline{X \backslash \overline{V}}$. Since $x \in V$, $V \cap X \backslash \overline{V} \neq \varnothing$. Thus $X \backslash \overline{V} \nsubseteq X \backslash V$ and so $V \nsubseteq \overline{V}$, a contradiction.*

On the other hand, suppose there exists $x \in A^+$ such that $x \notin \overline{U}$ for some open sets U such that $A \subseteq U$. Then $x \in X \backslash \overline{U}$ which is open and as $x \in A^+$, there exists $y \in A \cap \overline{X \backslash \overline{U}}$. Since $A \subseteq U$, $U \cap X \backslash \overline{U} \neq \varnothing$. Thus $X \backslash \overline{U} \nsubseteq X \backslash U$ and so $U \nsubseteq \overline{U}$, a contradiction.

If A and B are subsets of X, we denote the supper-closure of B in the subspace topology of A by B^{A+}. We get the following result.

Corollary 6 *The intersection of any supper-closed set B with any set A is supper-closed in A.*

Proof. *By Theorem 12, $(B \cap A)^{A+} = \cap\{\overline{U \cap A} \cap A : U$ open such that $B \subseteq U\}$ which contains $B \cap A$. If $x \in (B \cap A)^{A+}$, then $x \in A \cap \overline{U}$ for all open U such that $B \subseteq U$ and thus $x \in \cap\{A \cap \overline{U} : U$ is open containing $B\} = A \cap \cap\{\overline{U} : U$ is open containing $B\} = A \cap B^{A+} = A \cap B$. Therefore $(B \cap A)^{A+} \subseteq B \cap A \subseteq (B \cap A)^{A+}$.*

Thus the intersection of two supper-closed sets is supper-closed. Finally, we provide a stronger result than Theorem 8.

Corollary 7 *If $f : X \to Y$ is supper-continuous and $A \subseteq X$, then $f|_A$ is supper-continuous.*

Proof. Let F be a closed subset of Y. Then $(f|_A)^{-1}(F) = f^{-1}(F) \cap A$ which is supper-closed by Corollary 6.

5 Conclusion

Several characterizations of supper-continuity are given and the relations of this notion to many other well-known variations of continuity are also provided. Moreover, sufficient and necessary conditions for any of these variations to imply supper-continuity are obtained and constructions of supper-continuity are explored.

References

[1] D. Andrew and E. Whittlesy, *Closure Continuity,* Amer. Math. Month., **73**(1966), 758-759.

[2] S. Fomin, *Extensions of Topological Spaces,* Dokl. Akad. Nauk SSSR (M.S.), **32**(1941), 114-116.

[3] S. Naimpally, *On Strongly Continuous Functions,* Math. Notes, **71**1967, 166-168.

[4] N. Levine, *Semiopen Sets and Semicontinuity in Topological Spaces,* Amer. Math. Month., **70**(1963), 36-44.

[5] N. Levine, *A Decomposition of Continuity in Topological Spaces,* Amer. Math. Month., **68**(1961), 44-66.

[6] N. Levine, *Strong Continuity in Topological Spaces,* Amer. Math. Month., **67**(1960), 269-275.

[7] P. Long and D. Carnahan, *Comparing Almost Continuous Functions,* Amer. Math. Soc., **38 (2)**(1973), 413-418.

[8] T. Noiri, *On Weakly Continuous Mappings*, Proc. Amer. Math. Soc., **46**(1974), 120-124.

[9] B. Saleemi, N. Shahzad and M. Alghamdi, *Almost Continuity VS Closure Continuity,* Archivum Mathematicum (BRNO), **37**(2001), 39-44.

[10] M. Saleh, *On $\theta-$Continuity and Strong $\theta-$Continuity,* Appl. Math. E-Notes, **3**(2003), 42-48.

[11] N. Volicko, *H-Closed Topological Spaces,* Trans. AMS, **78**(1968), 103-118.