

## New Algebraic Method for Solving the Axial $N$ - Index Transportation Problem Based on the Kronecker Product

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**Abstract** A new algebraic method to solve the axial two and three - index transportation problem that is based on the Kronecker product of matrices is suggested. It is applied to solve the axial  $N(N \geq 2)$  - index transportation problem, as a general case. Further, the optimal feasible solution of these problem is obtained. The proposed method is considered algebraic, simple, robust and easily programmable.

**Keywords** Transportation problem, Kronecker product, partitioned matrix.

### 1 Introduction

The transportation problem is a special case of linear programming problem. It deals with the shipment of goods from a set of sources to a collection of destination centers at a minimum cost. This problem has applications in applied science, in economic planning, and in many areas of the industry. There is a rich literature dealing with the balanced transportation problem which is also called the two-index transportation problem solved by various methods such as: Northwest corner method, least cost method, Vogel's method, and some other methods [ 4, 5].

The axial three-index transportation problem, which was considered by Schell [7] and has been studied by various researchers [1,2,6,7] is a generalization of the axial two-index transportation problem. Bammi [2] formulated the transportation problem with generalized-indices and proved a theorem on the number of its basic variables. Korsnikov and Burkard [6] showed some theorems concerning the regularity, representability and the index of such problems. Recently Bulut [3] investigated some algebraic properties of the singular value decomposition of basic variables and singular values in the two-index transportation problem. Also Aysun Bulut [1] generalized the results in [3] and investigated the further relations between the planar and axial transportation problems.

In this paper, we suggest a new efficient algebraic method to solve the axial  $N$  -index transportation problem that is based on the Kronecker product of matrices. The method is illustrated by formulating the problem as a linear programming involving the Kronecker product of matrices and partitioned these matrices in order to get simple equations. By solving these equations, we obtain the initial feasible solution of the problem, then we obtain the optimal solution.

The following notations are used in this paper:

- $R^{m \times n}$  - the set of all  $m \times n$  real matrices.
- $A^T$  - the transpose of matrix  $A$ .
- $\text{rank}(A)$  - the rank of matrix  $A$ .
- $A \otimes B$  - the Kronecker product of matrices  $A$  and  $B$ .
- $I_m$  - the  $m \times m$  identity matrix.
- $1_m$  - the  $1 \times m$  vector whose entries are all 1.

For any two matrices  $A = [a_{ij}] \in R^{m \times n}$  and  $B = [b_{kl}] \in R^{p \times q}$ , the Kronecker product of matrices  $A$  and  $B$  is defined as the partitioned matrix  $A \otimes B = [a_{ij}B] \in R^{mp \times nq}$ .

## 2 Problem Statement

Consider a commodity which is available at  $m$  sources (numbered  $1, 2, \dots, m$ ) and is required at  $n$  destination centers (numbered  $1, 2, \dots, n$ ). Let  $a_i$  be the amount of the commodity available at source  $i$ ,  $b_j$  the amount of commodity required at destination center  $j$ , and  $c_{ij}$  the cost of transporting one unit of the commodity from source  $i$  to destination center  $j$ . Now if  $x_{ij}$  is the amount of the commodity to be transported from source  $i$  to destination center  $j$ , the  $\sum_{j=1}^n x_{ij}$  the total amount transported from source  $i$ ,  $\sum_{i=1}^m x_{ij}$  the total amount received at destination center  $j$ , and  $\sum_{i=1}^m a_i$  the total amount available at all the  $m$  sources, and  $\sum_{j=1}^n b_j$  the total amount required at all the  $n$  destination centers and  $\sum_{i=1}^m \sum_{j=1}^n c_{ij}x_{ij}$  will be the total cost. Thus the problem is determined by transporting a commodity which is available at  $m$  sources to  $n$  destination centers so that the objective function:

$$f(s) = \sum_{i=1}^m \sum_{j=1}^n c_{ij}x_{ij} \quad (1)$$

is to be minimized with the following assumptions:

$$(i) \quad \sum_{j=1}^n x_{ij} \leq a_i, \quad i = 1, 2, \dots, m \quad (2)$$

$$(ii) \quad \sum_{i=1}^m x_{ij} \leq b_j, \quad j = 1, 2, \dots, n \quad (3)$$

$$(iii) \quad x_{ij} \geq 0, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n \quad (4)$$

Notice the equation (2) states that we do not send more than the available quantity  $a_i$ . Similarly equation (3) states that the destination center  $j$  receives its minimum requirement  $b_j$ , and  $x_{ij}$ ,  $c_{ij}$ ,  $a_i$  and  $b_j$  are all positive integer numbers.

Therefore, the transportation problem has a feasible solution if and only if

$$\sum_{i=1}^m a_i \geq \sum_{j=1}^n b_j \quad (5)$$

This case is called the “unbalanced transportation problem”.

Any transportation problem in which the problem data satisfies equation (5) has at least one optimal solution  $x_{ij}$  where  $0 \leq x_{ij} \leq \min(a_i, b_j)$ .

If equation (5) is satisfied as equality, the transportation problem is called the “axial two-index transportation problem”. This problem can be formulated as a linear programming problem with:

$$\min_{x_{ij}} \left( \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \right)$$

subject to:

$$(i) \quad \sum_{j=1}^n x_{ij} = a_i \quad , \quad i = 1, 2, \dots, m \quad (6)$$

$$(ii) \quad \sum_{i=1}^m x_{ij} = b_j \quad , \quad j = 1, 2, \dots, n \quad (7)$$

$$(iii) \quad x_{ij} \geq 0, \text{ for all } i \text{ and } j, \text{ where } \sum_{i=1}^m a_i = \sum_{j=1}^n b_j \quad (8)$$

In section 4, we will discuss the general case of the axial  $N(N \geq 2)$ -index transportation problem.

### 3 Algebraic Method for Solving the Axial 2 -Index Transportation Problem Based on the Kronecker Product of Matrices

Let us assume that the axial two-index transportation problem has  $m$  sources and  $n$  destination centers. This problem can be formulated as a linear programming based on the Kronecker product with:

$$\min_x \{cx : Ax = g, 1_m a = 1_n b, x \geq 0\}. \quad (9)$$

Here,

$$A = \left[ \begin{array}{cccccc} I_m & I_m & \dots & \dots & I_m \\ 1_m & 0_m & \dots & \dots & 0_m \\ 0_m & 1_m & \dots & \dots & 0_m \\ 0_m & 0_m & \dots & \dots & 1_m \end{array} \right] \quad \left. \begin{array}{l} \} m - rows \\ \} n - rows \end{array} \right\} = \left[ \begin{array}{c} 1_n \otimes I_m \\ I_n \otimes 1_m \end{array} \right] \quad \left. \begin{array}{l} \} m - sources \\ \} n - destinations \end{array} \right\} \quad (10)$$

$$1_m = [1, 1, \dots, 1] \in R^{1 \times m} \quad , \quad 1_n = [1, 1, \dots, 1] \in R^{1 \times n}$$

$$a = [a_1, a_2, \dots, a_m]^T \in R^{m \times 1} \quad , \quad b = [b_1, b_2, \dots, b_n]^T \in R^{n \times 1}$$

$$\begin{aligned}
c &= [c_{11}, c_{21}, \dots, c_{m1}, c_{22}, \dots, c_{m2}, \dots, c_{1n}, \dots, c_{mn}] \in R^{1 \times mn} \\
x &= [x_{11}, x_{21}, \dots, x_{m1}, x_{22}, \dots, x_{m2}, \dots, x_{1n}, \dots, x_{mn}]^T \in R^{mn \times 1} \\
g &= [a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n]^T \in R^{(m+n) \times 1}
\end{aligned}$$

Note that if we rearrange  $x$  as:

$$x = [x_{11}, x_{12}, \dots, x_{1n}, x_{22}, \dots, x_{2n}, \dots, x_{m1}, \dots, x_{mn}]^T \in R^{mn \times 1}$$

then the matrix  $A$  in (10) will have the following form :

$$A_{\approx} = \left[ \begin{array}{cccccc} 1_n & 0_n & . & . & . & 0_n \\ 0_n & 1_n & . & . & . & 0_n \\ 0_n & 0_n & . & . & . & 1_n \\ \hline I_n & I_n & . & . & . & I_n \end{array} \right] \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \begin{array}{l} m - \text{rows} \\ \\ n - \text{rows} \end{array} = \left[ \begin{array}{c} I_m \otimes 1_n \\ 1_m \otimes I_n \end{array} \right] \left. \begin{array}{l} \\ \end{array} \right\} \begin{array}{l} m - \text{sources} \\ n - \text{destinations} \end{array} \quad (11)$$

Notice that the matrix  $A$  defined in (10) has the following properties:

- (i) The order of  $A$  is  $(m+n) \times mn$ .
- (ii)  $A$  is  $\{0, 1\}$  matrix, i.e., every entry of  $A$  is either 0 or 1.
- (iii) Every column of  $A$  has precisely 2 ones.
- (iv)  $A$  is a totally unimodular matrix, i.e., every square submatrix of  $A$  has determinant  $-1$  or  $1$  or  $0$ .
- (v) The rows of  $A$  are partitioned into two disjoint sets  $T_1$  and  $T_2$ , where  $T_1$  is  $m$ -sources which contains  $(n)$  ones and  $T_2$  is  $n$ -destination centers which contains  $(m)$  ones and the repetition of number 1 in  $T_1$  equals the repetition of number 1 in  $T_2$ .

We note also that the matrix  $A_{\approx}$  defined in equation (11) satisfies the properties (i) to (v).

### 3.1 Theorem

Let  $A$  and  $A_{\approx}$  be the matrices defined in (10) and in (11), respectively. Then the  $(m+n)$  rows of  $A$  and  $A_{\approx}$  are linearly dependent.

#### Proof

Let  $a^{(i)}, (i = 1, 2, \dots, m)$  be the  $i$  row in  $T_1$  and  $b^{(j)}, (j = 1, 2, \dots, n)$  be the  $j$  row in  $T_2$ . But  $a^{(i)} \in R^{1 \times mn}$  contains  $n$  ones and  $b^{(j)} \in R^{1 \times mn}$  contains  $(m)$  ones. Since the repetition of number 1 in  $T_1$  equals the repetition of number 1 in  $T_2$ , then

$$\sum_{i=1}^m a^{(i)} - \sum_{j=1}^n b^{(j)} = 0.$$

Hence the rows of  $A$  are linearly dependent.

### 3.2 Theorem

Let  $A$  and  $A_{\approx}$  be the matrices defined in (10) and in (11), respectively. Then

$$\text{rank}(A) = \text{rank}(A_{\approx}) = m + n - 1.$$

#### Proof

Let  $x$  be rearranged as

$$x = [x_{11}, x_{12}, \dots, x_{1n}, x_{21}, x_{22}, \dots, x_{2n}, \dots, x_{m1}, \dots, x_{mn}]^T.$$

Then by Theorem (3.1), we have

$$\text{rank}(A) = \text{rank}(A_{\approx}) < m + n.$$

Now consider the  $(m + n - 1)$  columns as:

$$a_{1n}, a_{2n}, \dots, a_{mn}, a_{11}, a_{12}, \dots, a_{1(n-1)}.$$

Deleting the last row from  $A_{\approx}$  and rearranging the entries, gives a square triangular matrix  $D$  of order

$$(m + n - 1) \times (m + n - 1)$$

and all diagonal entries are 1. The general form of  $D$  is:

$$D = \begin{bmatrix} I_m & F \\ O & I_{n-1} \end{bmatrix}, \text{ where } F = \begin{bmatrix} 1_{n-1} \\ O \end{bmatrix}.$$

Hence  $\det(D) = 1 \neq 0$  and  $\text{rank}(D) = \text{rank}(A) = \text{rank}(A_{\approx}) = m + n - 1$ .

Since  $\text{rank}(A) = m + n - 1$  and  $1_m a = 1_n b$ , the system  $Ax = g$  defined in the equation (9) can be solved in terms of  $(m + n - 1)$  linearly independent equations. Now, after we delete the first row from matrix  $A$  and rearrange the columns of  $A$ , we can rewrite (9) in the following form:

$$\min_x \{Lx = h : 1_m a = 1_n b, x \geq 0\}, \quad (12)$$

where

$$L = \begin{matrix} & n & n(m-1) \\ m-1 & \begin{bmatrix} 0 & (1_n \otimes I_{m-1}) \\ I_n & (I_n \otimes 1_{m-1}) \end{bmatrix} \end{matrix} \in R^{(m+n-1) \times mn}$$

$$h = [a_2, a_3, \dots, a_m, b_1, b_2, \dots, b_n]^T \in R^{(m+n-1) \times 1}$$

$$x = [x_{11}, x_{12}, \dots, x_{1n}, x_{22}, \dots, x_{2n}, \dots, x_{m1}, \dots, x_{mn}]^T \in R^{mn \times 1}$$

The  $(m + n - 1) \times mn$  matrix  $L$  of  $\text{rank}(L) = m + n - 1$  is called the “simplest matrix of matrix  $A$  and  $A_{\approx}$ ”.

The axial two-index transportation problem always has a feasible solution and hence an optimal solution. Now in order to obtain an initial basic feasible solution of the axial two-index transportation problem depending on the idea of the Kronecker product of matrices,

we solve the system  $Lx = h$  defined in (12) by rewriting it as the following form:

$$\begin{matrix} m-1 \\ n \end{matrix} \begin{bmatrix} n & n(m-1) \\ 0 & (1_n \otimes I_{m-1}) \\ I_n & (I_n \otimes 1_{m-1}) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ y_1 \\ y_2 \end{bmatrix}_{n(m-1)}^n = \begin{bmatrix} 1 \\ h_1 \\ h_2 \end{bmatrix}_{n(m-1)}^{m-1} \quad (13)$$

Here,

$$\begin{aligned} y_1 &= [x_{11}, x_{12}, \dots, x_{1n}]^T \in R^{n \times 1} \\ y_2 &= [x_{21}, x_{22}, \dots, x_{2n}, \dots, x_{m1}, \dots, x_{mn}]^T \in R^{n(m-1) \times 1} \\ h_1 &= [a_2, a_3, \dots, a_m]^T \in R^{(m-1) \times 1} \\ h_2 &= [b_1, b_2, \dots, b_n]^T \in R^{n \times 1} \end{aligned}$$

Now, the equation (13) gives:

$$[1_n \otimes I_{m-1}]y_2 = h_1 \quad (14)$$

$$y_1 + [I_n \otimes 1_{m-1}]y_2 = h_2 \quad (15)$$

where (14) is a system which contains  $(m-1)$  equations and  $n(m-1)$  variables and it is easy to solve it by determining the values of  $x_{ij}$  in  $y_2$  which satisfies of the following subjects:

$$\sum_{j=1}^n x_{ij} = a_i, \quad i = 1, 2, \dots, m, \quad \sum_{i=1}^m x_{ij} = b_j, \quad j = 1, 2, \dots, n$$

and taking the other variables are zeros. Also (15) is a system which contains at most  $(m+n-1)$  nonzero variables. Since  $y_2$  is known by solving (14), then it easy to find  $y_1$  as

$$y_1 = h_2 - [I_n \otimes 1_{m-1}] y_2 \quad (16)$$

By this method, we obtain an initial feasible solution of axial two-index transportation problem and then the optimal solution. It is also possible to use this method to obtain an initial feasible solution for un balanced transportation problem by changing it to the axial transportation problem and then obtain the optimal solution.

### 3.3 Example

Assume that the axial two- index transportation problem has three sources and three destination centers and related data are given in the following table:

		<i>Destinations</i>					
		1	2	3			
<i>Sources</i>	1	3	1	4	15	<i>Supplies</i>	
	2	2	1	2	5		
	3	4	5	3	5		
		5	10	10			
		<i>Demands</i>					

In order to find the minimum cost of this problem by using the Kronecker product of matrices, we compute:

$$[1_3 \otimes I_2]y_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \\ x_{31} \\ x_{32} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}.$$

This gives

$$\begin{aligned} x_{21} + x_{23} + x_{32} &= 5 \text{ which implies } x_{21} = 5, x_{23} = 0, x_{32} = 0, \\ x_{22} + x_{31} + x_{33} &= 5 \text{ which implies } x_{22} = 0, x_{31} = 0, x_{33} = 5. \end{aligned}$$

Thus

$$y_2 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}.$$

Next,

$$\begin{aligned} y_1 &= \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = h_2 - [I_3 \otimes 1_2] y_2 \\ &= \begin{bmatrix} 5 \\ 10 \\ 10 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 10 \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}. \end{aligned}$$

Thus  $x_{11} = 0$ ,  $x_{12} = 10$ ,  $x_{13} = 5$ . Hence

$$\min (\text{cost}) = 1 \times 10 + 4 \times 5 + 2 \times 5 + 3 \times 5 = 55.$$

Notice that this is an optimal solution.

#### 4 Algebraic Method for Solving the Axial $N$ -Index Transportation Problem Based on the Kronecker Product

Let us assume that the axial three-index transportation problem has  $m$  factories,  $n$  warehouses and  $p$  wholesale outlets. Further let  $S = \{S_i\}$  be a set of factories,  $D = \{D_j\}$  be a set of warehouses and  $P = \{P_k\}$  be a set of wholesale outlets.

Now let

$$G = (S, D, P, S \times D \times P) \quad (17)$$

such a network which is shown geometrically as a cuboid with three dimensions which are  $S$ ,  $D$  and  $P$ , and is called as an *axial three-index transportation problem of order  $m \times n \times p$* , where the sets  $S$ ,  $D$  and  $P$  denote nodes and  $S \times D \times P$  denotes arcs of the network  $G$ . This problem can be formulated as a linear programming problem based on the Kronecker product with:

$$\min_x \{c^T x : Bx = g, 1_m s = 1_n q = 1_p r, x \geq 0\} \quad (18)$$

Here,

$$B = \begin{bmatrix} I_p \otimes 1_n \otimes 1_m \\ 1_p \otimes I_n \otimes 1_m \\ 1_p \otimes 1_n \otimes I_m \end{bmatrix} = \begin{bmatrix} I_p \otimes 1_{nm} \\ 1_p \otimes A_{\approx} \end{bmatrix}, \quad (19)$$

$$g^T = [r^T, q^T, s^T],$$

$$r^T = [1_m d_1, 1_m d_2, \dots, 1_m d_p], q^T = [1_p b_1, 1_p b_2, \dots, 1_p b_n], s^T = [1_n a_1, 1_n a_2, \dots, 1_n a_m],$$

$$a_i^T = [a_{i1}, \dots, a_{in}], d_i^T = [d_{i1}, \dots, d_{ip}], b_j^T = [b_{j1}, \dots, b_{jp}], i = 1, \dots, m, j = 1, \dots, n,$$

$$x^T = [x_{111}, x_{112}, \dots, x_{11p}, \dots, x_{mn1}, x_{mn2}, \dots, x_{mnp}],$$

$$c^T = [c_{111}, c_{112}, \dots, c_{11p}, \dots, c_{mn1}, c_{mn2}, \dots, c_{mnp}]$$

and  $I_m$  is the  $m \times m$  identity matrix,  $1_m$  is the  $1 \times m$  vector whose all entries are 1,

$$I_{mn} = I_m \otimes I_n \text{ and } 1_{mn} = 1_m \otimes 1_n.$$

The  $(m+n+p) \times mnp$  matrix  $B$  of rank  $m+n+p-2$  is called the *node-arcs incidence matrix* of the network  $G$  defined in (17).

Note that the  $(n+m) \times nm$  matrix

$$A_{\approx} = \begin{bmatrix} I_n \otimes 1_m \\ 1_n \otimes I_m \end{bmatrix} \quad (20)$$

of rank  $m+n-1$  is the coefficient matrix of the axial two-index transportation problem with  $n$  sources (origins) and  $m$  destination centers as defined in (17). By using the same technique for solving the axial two-index transportation problem, we can also solve the axial three-index transportation problem. Since

$$\text{rank}(B) = m+n+p-2 \text{ and } 1_m s = 1_n q = 1_p r,$$

then the system  $Bx = g$  defined in (18) can be solved in terms  $m+n+p-2$  of linearly independent equations. Hence, we can delete any two rows from matrix  $B$  and rearrange its columns; we can rewrite (18) as in the following form:

$$\min_x \{c^T x : Tx = h, 1_m s = 1_n q = 1_p r, x \geq 0\} \quad (21)$$

Here,

$$T = \begin{matrix} p-1 \\ m+n-1 \end{matrix} \begin{bmatrix} mn & nm(p-1) \\ 0 & 1_{nm} \otimes I_{p-1} \\ L & L \otimes 1_{p-1} \end{bmatrix} \in R^{(m+n-2) \times mn}$$

$$L = \begin{matrix} n-1 \\ m \end{matrix} \begin{bmatrix} m & m(n-1) \\ 0 & (1_m \otimes I_{n-1}) \\ I_m & (I_m \otimes 1_{n-1}) \end{bmatrix} \in R^{(m+n-1) \times mn}$$

Note the matrix  $L$  is the simplest matrix of matrix  $A_{\approx}$  defined in (20).

To solve an axial three-index transportation problem defined in (21), we re-partitioned the system  $Tx = h$  in the form:

$$\begin{matrix} p-1 \\ m+n-1 \end{matrix} \begin{bmatrix} mn & nm(p-1) \\ 0 & 1_{nm} \otimes I_{p-1} \\ L & L \otimes 1_{p-1} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ z_1 \\ z_2 \end{bmatrix} \begin{matrix} nm \\ nm(p-1) \end{matrix} = \begin{bmatrix} 1 \\ h_1 \\ h_2 \end{bmatrix} \begin{matrix} p-1 \\ m+n-1 \end{matrix} \quad (22)$$

Now, (22) gives (23) and (24):

$$[1_{nm} \otimes I_{p-1}]z_2 = h_1. \quad (23)$$

This is a system which contains  $(p-1)$  equations and  $nm(p-1)$  variables.

To solve (23), we determine the values of  $x_{ijk}$  in  $z_2$  which satisfies  $1_m s = 1_n q = 1_p r$  and take the values of other variables are zeros.

$$Lz_1 + [L \otimes 1_{p-1}]z_2 = h_2. \quad (24)$$

We can prove that this is a system which contains at most  $m+n+p-2$  nonzero of variables and it is easy to find the other variables  $x_{ijk}$  in  $z_1$ .

By using this method, we can also obtain an initial basic feasible solution of the axial three-index transportation problem.

We now consider the axial  $N(N \geq 2)$  - index transportation problem of order  $m_1 \times m_2 \times \dots \times m_N$ , that can be formulated as linear programming based on the Kronecker product with:

$$\min_x \{c^T x : Dx = g, x \geq 0\} \quad (25)$$

where,

$$\begin{aligned} D &= \begin{bmatrix} I_{m_N} \otimes 1_{m_{N-1}} \otimes 1_{m_{N-2}} \otimes \dots \otimes 1_{m_2} \otimes 1_{m_1} \\ 1_{m_N} \otimes I_{m_{N-1}} \otimes 1_{m_{N-2}} \otimes \dots \otimes 1_{m_2} \otimes 1_{m_1} \\ \vdots \\ 1_{m_N} \otimes 1_{m_{N-1}} \otimes 1_{m_{N-2}} \otimes \dots \otimes 1_{m_2} \otimes I_{m_1} \end{bmatrix} \\ &= \begin{bmatrix} I_{m_N} \otimes 1_{m_{N-1}m_{N-2}\dots m_2 m_1} \\ 1_{m_N} \otimes W_{\approx} \end{bmatrix} \\ g^T &= [g_1^T, g_2^T, \dots, g_N^T] \\ 1_{m_{N-1}m_{N-2}\dots m_2 m_1} &= 1_{m_{N-1}} \otimes 1_{m_{N-2}} \otimes \dots \otimes 1_{m_2} \otimes 1_{m_1} \end{aligned} \quad (26)$$

The  $(m_1 + m_2 + \dots + m_N) \times (m_1 m_2 \dots m_N)$  matrix  $D$  of

$$\text{rank}(D) = (m_1 + m_2 + \dots + m_N + 1 - N)$$

is called the *node-arcs incidence matrix* of the network

$$G_{\#} = (Q_1, Q_2, \dots, Q_N, Q_1 \times Q_2 \times \dots \times Q_N).$$

Note that the  $(m_1 + m_2 + \cdots + m_{N-1}) \times (m_1 m_2 \cdots m_{N-1})$  matrix

$$W_{\approx} = \begin{bmatrix} I_{m_{N-1}} \otimes 1_{m_{N-2}} \otimes \cdots \otimes 1_{m_2} \otimes 1_{m_1} \\ \vdots \\ 1_{m_{N-1}} \otimes 1_{m_{N-2}} \otimes \cdots \otimes 1_{m_2} \otimes I_{m_1} \end{bmatrix} \quad (27)$$

of rank  $(m_1 + m_2 + \cdots + m_N + 2 - N)$  is the coefficient matrix of the axial  $(N - 1)$  -index transportation problem.

Since  $\text{rank}(D) = (m_1 + m_2 + \cdots + m_N + 1 - N)$ , by the same technique which we used to solve the axial three-index transportation problem, we solve the axial  $N$  - index transportation problem defined in (25) by rewriting it in the following form:

$$\min_x \{c^T x : Mx = h, x \geq 0\}. \quad (28)$$

Here,

$$M = \begin{bmatrix} 0 & 1_{m_{N-1}m_{N-2}\cdots m_2m_1} \otimes I_{N-1} \\ K & K \otimes 1_{N-1} \end{bmatrix} \in R^{(m_1+m_2+\cdots+m_N+1-N) \times (m_1m_2\cdots m_N)}$$

and  $K$  is the simplest matrix of matrix  $W_{\approx}$  defined in (27). We next re-partition the system  $Mx = h$  defined in (28) in the form:

$$\begin{bmatrix} 0 & 1_{m_{N-1}m_{N-2}\cdots m_2m_1} \otimes I_{N-1} \\ K & K \otimes 1_{N-1} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \quad (29)$$

Now, (29) can be written as (30) and (31):

$$[1_{m_N m_{N-1} \cdots m_2 m_1} \otimes I_{N-1}]v_2 = h_1 \quad (30)$$

$$Kv_1 + [K \otimes 1_{N-1}]v_2 = h_2 \quad (31)$$

By solving the equations defined in (30) and (31), we obtain an initial (optimal) feasible solution of the axial  $N$  -index transportation problem. These results on the axial transportation problem can be applied to the study of the solution of the planar transportation problem, as a general case [1, 10], which can be obtained from the matrix  $D$  defined in (26) replacing  $1_n$  by  $I_n$  and  $I_n$  by  $1_n$ .

## 5 Conclusion

We have presented an efficient algebraic method to obtain initial basic feasible solutions of the axial two and three-index transportation problem based on the Kronecker product of matrices, and we have applied this method to solve the axial  $N$  - index transportation problem. This method is comparable other methods. As a matter of fact it can be considered superior, since it is simple, easily programmable and robust.

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