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The Controversy of Linearity in Shapley's Axioms

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Abstract The Shapley value is a unique function which obeys three axioms, which are anonymous, pareto optimal, and linear. This is a solution concept for cooperative games with transferable utility. But when the set of games which is considered is the set of simple games, the condition of linearity has no sense. In this paper, we introduce the concept of monotonicity to answer the controversy in linearity.

Keywords Simple games, power distribution, Shapley value, monotonicity.

1 Introduction

A point valued solution concept essentially defines a function on the set of games which yields a unique outcome for every game. Obviously, there are infinitely many possible functions. One way to select a particular function is to specify a list of properties or axioms which the function must satisfy and which are sufficient to produce a unique function. This is the approach underlying the Shapley value ([5]).

Shapley ([6]) approaches his value axiomatically. He proceeds from a set of three axioms (anonymous, pareto optimal, linear), having simple intuitive interpretations, which suffice to determine a unique function. This function has come to be called the Shapley value.

Shapley's result is still discussed, mainly in reference to the underlying concept of linearity, which specifies how the values of different games must be related to one another and which is the driving force behind Shapley's demonstration of the uniqueness of his value. The standard objection to this axiom has its advocates especially among social scientists and is the following: When the set of games which is considered is the set of simple games, the condition of linearity has no sense. We would like to introduce the concept of monotonicity on this domain that maybe can help solving the problem.

2 The Shapley Value

We first, define some terms. Recall that a permutation Π of a finite set S is a one-to-one mapping from S onto itself. Given any game (T, ν), we define the permuted game $(\Pi T, \Pi\nu)$ by

$$\Pi\nu(S) = \nu(\Pi S),\ \forall S. \tag{1}$$

A permuted game is the game obtained by relabelling all the players. Similarly, for any allocation x,

$$\Pi x[i] = x[\Pi i]. \tag{2}$$

We say that a solution is anonymous (symmetric) if the names of the players do not matter, that is it is invariant to a relabelling of the players. Formally, a point-valued solution concept Φ is anonymous if

$$\Phi[\Pi\nu] = \Pi\Phi[\nu]. \tag{3}$$

We say that a player is a null player if his contribution to every coalition is zero, that is

$$\nu[S \cup \{i\}] = \nu[S], \ \forall S. \tag{4}$$

We denote the space of all games by ς. The symbol $\varsigma^n \subset \Re^{2^n}$ is the set of n-person games, i.e. such games where the cardinality of the set of players is finite and equal to n.

Definition 1. The value of the game $\nu \in \varsigma$ is the function $f : \varsigma^n \to \Re^n_{+0}$ defined $\forall n \in \mathcal{Z}^+$.

Shapley approaches his value axiomatically. He gives two definitions as mention in definition 2 and definition 3.

Definition 2. A carrier for a game ν is a coalition T such that $\forall S : \nu(S) = \nu(S \cap T)$.

Definition 2 states that any player who does not belong to a carrier is a dummy, i.e., incapable of contributing anything to any coalition.

Definition 3. Let $\Pi(U)$ denote the set of permutations of U - that is, the one-to-one mappings of U onto itself. If $\pi \in \Pi(U)$, then, writing πS for the image of S under π, we may define the function $\pi\nu$ by $\pi\nu(\pi S) = \nu(S), \ \forall S \in 2^U$, where $S \in 2^U$ is $S : U \to U$.

Effectively, the game $\pi\nu$ is nothing other than the game ν, with the roles of the players interchanged by the permutation π. With these two definitions, it is possible to give axiomatic treatment.

Definition 4 (Shapley's axioms). A value is a point valued solution concept $F[\nu]$ which is

$$\text{Linear } \Phi[\nu + w] = \Phi[\nu] + \Phi[w] \tag{5}$$

$$\text{Anonymous } \Phi[\Pi\nu] = \Pi\Phi[\nu] \tag{6}$$

$$\text{Pareto Optimal } \Phi[\nu][T] = \nu[T] \tag{7}$$

$$\Phi[\nu][i] = 0 \text{ for every null player.} \tag{8}$$

Theorem 1 *Axioms (5)-(8) are sufficient to determine a unique value Φ for all games.*

Proof. The easiest way to understand why this function exists and is unique is to think of a characteristic function v as a vector with $2^{|U|} - 1$ components, one for each nonempty member of the power set 2^U. (For simplicity of this explanation, take the universe U of players to be finite. This will be the case which will be considered in the entire article.) Then the set ς of all characteristic function games is the subset of Euclidean space of dimension $2^{|U|} - 1$. The additivity axiom says that if we know a value function of some set of games that constitute an additive basis for ς, then we can determine the value for any game.

A set of games that will permit us to accomplish this is the set consisting of the games ν_R, defined for each subset R of U by

$$\nu_R(S) = \begin{cases} 1, & \text{if } \quad R \subset S_i, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Any player not in R is a dummy player in this game. Since the players in R are all symmetric, the anonymous axiom (6) requires that $\Phi_i(\nu_R) = \Phi_j(\nu_R)\,\forall i,j \in R$. Since R is the carrier of the game ν_R, the pareto optimal axiom (7) allows us to conclude that $\Phi_i(\nu_R) = \frac{1}{r}\,\forall i \in R$, where $r = |R|$. (For any finite coalition S, we will denote by s the number of players in S.) Thus the value is uniquely defined for all games of the form ν_R or, for that matter, for games of the form $\alpha\nu_R$ for any number α. We point out merely that, as game are essentially real-valued non-negative functions, it is possible to talk of a number times a game, or, as above, of the sum of two or more games. The difference of two games is not necessary a game.

But the games ν_R form a basis for the superset of all games, because there are $2^{|U|} - 1$ of them, one for each nonempty subset R of U, and because they are linearly independent. Therefore any game ν can be written as the sum of games of the form $\alpha_R\nu_R$. And so, the linearity axiom (5) implies that there is a unique value obeying Shapley's axioms defined on the space of all games. □

Shapley expressed this unique value Φ explicitly:

$$\Phi_i(\nu) = \sum_{S \subset T} \gamma s(\nu[S] - \nu[S - \{i\}]) \tag{10}$$

where $\gamma s = \dfrac{(|S|-1)!(n-|S|)!}{n!}$ ([3]).

3 The Conception of Monotonicity

Before discussing the conception of monotonicity, we start with the concept of simple games. We state the definition 5, of non-zero n-person simple game first, then follow by three definitions of monotonicity concept.

Definition 5. If for non-zero n-person simple game ν, an $(n+1)$-dimensional non-negative vector $r_\nu = (\rho_1, ..., \rho_n, q)$ exists such that

$$\sum_{i=1}^{n} \rho_i = 1, \;\; 0 < q \leq 1, \tag{11}$$

$$(\nu(S) = 1) \Leftrightarrow \sum_{i \in S} \rho_i \geq q \tag{12}$$

where a non-zero game ν is a game for which exists at least one coalition S such that $\nu(S) \neq 0$, then we call the simple game ν a weighted game. The number ρ_i is called the weight of player i; the number q is called quota; the vector r_ν is called the representation of the game ν.

Remark 1. Usually in literature ([4], [2]), condition (11) is not required. But if for game ν there is a vector $r'_v = (\rho'_1, ..., \rho'_n, q')$ which satisfies (12) but does not satisfy condition (11), it is enough to define

$$\theta = \sum_{i=1}^{n} \rho'_i \wedge (q = \frac{q'}{\theta}) \wedge (\rho_i = \frac{\rho'_i}{\theta} \quad , \quad \forall i \ . \tag{13}$$

Then vector $r_v = (\rho_1, ..., \rho_n, q)$ satisfies conditions (11) and (12). Thus ν is a weighted game according to our definition.

Although a weighted game is a very small subset of simple games, most of the common social problems can be described by them (e.g., the corporation with n share-holders each have ρ_i shares of stock each). On the other side, some very trivial simple games are not weighted games. The six-player game with players $\{1, 2, 3, A, B, C\}$ for which the winning coalition contains at least two players from the set $\{1, 2, 3\}$ and (at least) two players from the set $\{A, B, C\}$ cannot be represented as the weighted game. The proof is direct. Considering the symmetry of the problem, the weights of players have to be equal to each other. Contradiction: the two members coalition $\{1, 2\}$ is the blocking coalition while the coalition $\{1, A\}$ is not (The coalition S is blocking iff $\nu\left(\frac{U}{S}\right) = 0$).

The more precise description of the space description of the space of weighted games can be simply given: Vectors $(\rho_1, ..., \rho_n)$ create a non-negative part of a unit sphere S^n_{+0} in the n-dimensional space with an additive norm. For 2-players games S^n_{+0} is the abscissa AB, where $A = (0, 1)$ and $B = (1, 0)$, for 3-players games we get the triangular ABC, where $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$ etc. The whole set of all representations of n-person weighted games is equal to the n-dimensional convex manifold $W^n = S^n_{+0} \times (0, 1\rangle$.

The set of all representations of a particular game ν is a convex subset of W^n. The proof of this statement is straightforward: any linear combination of r^1_ν and r^2_ν is a representation of the game ν. Thus the set W^n is compound from the convex disjoint subsets W^n_v which represent the individual n-person weighted games. The most natural mapping between the set of all n-person weighted games and the set of $(n+1)$ dimensional vectors can be given: let us represent game ν by the centre of gravity of W^n_v. Although this mapping seems to be the most reasonable from the geometrical point of view it cannot be taken as a voting power index-it does not satisfy the pareto optimal condition (7).

On the other side the existence of a representation r_ν of the game ν such that $(\rho^\nu_i = 0)$ iff i is dummy can simply be proven. When we note the set of such representation by D^n_ν, and we represent the game ν by the centre of gravity of D^n_ν, we receive functions $\gamma_i(v)$ which satisfy the axioms (6,7). We will call $\gamma(\nu)$ the geometrical voting power index.

In order to find the axioms which will restrict the space of functions $f(\nu) : \varsigma \to \Re^n_{+0}$, we have to formulate of monotonicity concept that suitable for the whole space games.

Definition 6. The voting power index ϕ satisfies the local topological monotonic property on the domain of all games if

$$\left.\begin{array}{l}(\nu(S\backslash\{i\}) \leq \nu(S\backslash\{j\})\forall S, \quad \text{such} \quad \text{that} \quad i,j \in S \\ \wedge(\exists S' \ , \quad i,j \in S' \ , \quad \nu(S'\backslash\{i\}) < \nu(S'\backslash\{j\}))\end{array}\right\} \Rightarrow \phi_i(\nu) > \phi_j(\nu)$$
$$(\nu(S\backslash\{i\}) = \nu(S\backslash\{j\})\forall S, \quad \text{such} \quad \text{that} \quad i,j \in S \quad)) \Rightarrow \phi_i(\nu) = \phi_j(\nu) \tag{14}$$

Remark 2. Here, we merely point out that if the index satisfies the efficiency axiom, it is an imputation. If

$$
\begin{array}{l}
(\forall S)(i,j\in S)(\nu(S\backslash\{i\})\leq\nu(S\backslash\{j\})\Leftrightarrow(\forall S)(i,j\notin S)(\nu(S\cup\{i\})\geq\nu(S\cup\{j\})\\
(\forall S)(i,j\in S)(\nu(S\backslash\{i\})=\nu(S\backslash\{j\})\Leftrightarrow(\forall S)(i,j\notin S)(\nu(S\cup\{i\})=\nu(S\cup\{j\})\\
(\exists S')(i,j\in S')(v(S'\backslash\{i\})<\nu(S'\backslash\{j\})\Leftrightarrow(\exists\tilde{S})(i,j\notin\tilde{S})(\nu(\tilde{S}\cup\{i\})>\nu(\tilde{S}\cup\{j\})
\end{array}
$$

Proof.

$$
\begin{array}{l}
(\forall S\ \text{such}\quad\text{that}\ i,j\notin S)(\exists\tilde{S}=S\cup\{i,j\})\\
\nu(S\cup\{i\})=\nu(\tilde{S}\backslash\{j\})\geq\nu(\tilde{S}\backslash\{i\})=\nu(S\cup\{j\}).\\
(\forall S\ \text{such}\quad\text{that}\ i,j\notin S)(\exists\tilde{S}=S\cup\{i,j\})\\
\nu(S\cup\{i\})=\nu(\tilde{S}\backslash\{j\})=\nu(\tilde{S}\backslash\{i\})=v(S\cup\{j\}).\\
(\tilde{S}=S'\backslash\{i,j\})v(S'\backslash\{i\})=\nu(\tilde{S}\cup\{j\})<\nu(\tilde{S}\cup\{i\})=v(S'\backslash\{j\}).\square
\end{array}
$$

Definition 7. The voting power index ϕ satisfies the global topological monotonic property on the domain of all games whenever $\forall$ n-person games u,ν such that,

$$\exists k\in\{1,...,n\}:((k\in S\Rightarrow\nu(S)\leq u(S))\wedge(k\notin S\Rightarrow\nu(S)\geq u(S))), \tag{15}$$

where $u=(\rho_1^u,...,\rho_n^u,q)$, $\nu=(\rho_1^v,...,\rho_n^v,q)$, then the following inequality holds

$$\phi_k(u)\geq\phi_k(v). \tag{16}$$

Definition 8. The voting power index ϕ is topologically monotonic on the domain of all games if it is locally and globally topologically monotonic on the domain of all games.

Theorem 2 *Each index $\phi(\nu)$ is locally topologically monotonic on the domain of weighted games ν if and only if it is locally weakly monotonic on this domain.*

Proof. For all weighted games ν (using Remark 2):

$$(\nu(S\backslash\{i\})<\nu(S\backslash\{j\}))\Leftrightarrow((\nu(S\backslash\{i\})=0\wedge\nu(S\backslash\{j\})=1\wedge\nu(S)=1)\Leftrightarrow$$

$$\Leftrightarrow(S\in C_i(\nu)\wedge S\notin C_j(\nu))$$

$$(\forall S)(i,j\in S)(\nu(S\backslash\{i\})=\nu(S\backslash\{j\})\Leftrightarrow$$

$$\Leftrightarrow\left\{\begin{array}{c}(\forall S)(i,j\in S)(\nu(S\backslash\{i\})=\nu(S\backslash\{j\})\\(\forall S(i,j\notin S)(\nu(S\cup\{i\})=\nu(S\cup\{j\})\end{array}\right\}\Leftrightarrow$$

$$\Leftrightarrow\left\{\begin{array}{c}(\forall S)(i,j\in S)(S\in C_j(\nu))\Leftrightarrow(S\in C_i(\nu))\\(\forall S)(i,j\notin S)([S\cup\{j\}]\in C_j(\nu))\Leftrightarrow([S\cup\{i\}]\in C_i(\nu))\end{array}\right.$$

$$(\forall S)(i,j\in S)(\nu(S\backslash\{i\})\leq\nu(S\backslash\{j\})\Leftrightarrow$$

$$\Leftrightarrow\left\{\begin{array}{c}(\forall S)(i,j\in S)(\nu(S\backslash\{i\})\leq\nu(S\backslash\{j\})\\(\forall S)(i,j\notin S)(\nu(S\cup\{i\})\geq\nu(S\cup\{j\})\end{array}\right\}\Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (\forall S)(i,j \in S)(S \in C_j(\nu)) \Rightarrow (S \in C_i(\nu)) \\ (\forall S)(i,j \notin S)([S \cup \{j\}] \in C_j(\nu)) \Rightarrow ([S \cup \{i\}] \in C_i(\nu)) \end{cases}$$

It can be easily seen that the assumptions of (14) is equivalent on the domain of weighted games. □

Theorem 3 *Each index $\phi(\nu)$ which is globally topologically monotonic on the domain of weighted games v is globally weakly monotonic on this domain.*

Proof. Direct: All pairs of the games u, ν which satisfy the assumptions of global weak monotonicity $((\rho_k^u \geq \rho_k^\nu) \wedge (\rho_j^u \leq \rho_j^\nu,\ \forall j \neq k))$ satisfy the assumptions of local topological monotonicity (15):

$$(\rho_i^u \leq \rho_i^\nu)(\forall i \neq k) \Rightarrow \left(\sum_{i \in S} \rho_i^u - q \leq \sum_{i \in S} \rho_i^\nu - q\right)(\forall S : k \notin S) \Rightarrow$$

$$\Rightarrow \begin{cases} (u(S) \leq \nu(S))(\forall S : k \notin S) \\ \left(\sum\limits_{i \in S} \rho_i^u - q \geq \sum\limits_{i \in S} \rho_i^\nu - q\right)(\forall S : k \in S) \Rightarrow (u(S) \geq \nu(S))(\forall S : k \in S). \end{cases} \tag{17}$$

□

Theorem 4 *The Shapley value Φ is weakly monotonic on the space of all games.*

Proof. We can rewrite the Shapley value, defined by equation (10):

$$\begin{aligned} \Phi_i(v) &= \sum_{i \in S \subset N} \alpha_S[\nu(S) - \nu(S \backslash \{i\})] \\ &= \sum_{i,j \in S \subset N} \alpha_S[\nu(S) - \nu(S \backslash \{i\})] + \sum_{S \subset \tilde{N}} \beta_S[\nu(S \cup \{i\}) - \nu(S)], \end{aligned} \tag{18}$$

where

$$\alpha_S = \frac{(s-1)!(n-s)!}{n!}, \qquad \beta_S = \tfrac{s!(n-s-1)!}{n!}, \tag{19}$$

$N \subset U$ is any finite carrier of $\nu, \tilde{N} = N \backslash \{i,j\}$, s is the cardinality of set S, and n is the cardinality of set N. Because the coefficients α_S and β_S do not depend on a particular game, but on the size of coalition, it is enough to compare differences $\nu(S) - \nu(S \backslash \{i\})$ and $\nu(S \cup \{i\}) - \nu(S)$. Local weak monotonicity (using Remark 2):

$$\left.\begin{array}{c} (\nu(S \backslash \{i\}) \leq \nu(S \backslash \{j\}) \forall S \ , \quad \text{such} \quad \text{that} \quad i,j \in S) \\ \wedge(\exists S' \ , \quad i,j \in S' \ , \quad v(S' \backslash \{i\}) < v(S' \backslash \{j\})) \end{array}\right\} \Rightarrow$$

$$\begin{aligned} &\Rightarrow \left\{\begin{array}{c} i,j \in S : \nu(S) - \nu(S \backslash \{i\}) \geq \nu(S) - \nu(S \backslash \{j\}) \\ i,j \notin S : \nu(S \cup \{i\}) - \nu(S) \geq \nu(S \cup \{j\}) - \nu(S) \\ \exists S' \ , \quad i,j \in S' : \nu(S') - v(S' \backslash \{i\}) > \nu(S') - \nu(S' \backslash \{j\}) \end{array}\right\} \Rightarrow \\ &\Rightarrow \Phi_i(\nu) > \Phi_j(\nu). \end{aligned} \tag{20}$$

$$(v(S \backslash \{i\}) = v(S \backslash \{j\}) \forall S, \quad \text{such} \quad \text{that} \quad i,j \in S)) \Rightarrow$$

$$\begin{aligned}&\Rightarrow \left\{\begin{array}{c} i,j \in S : \nu(S) - \nu(S\backslash\{i\}) = \nu(S) - \nu(S\backslash\{j\}) \\ i,j \notin S : \nu(S\cup\{i\}) - \nu(S) = \nu(S\cup\{j\}) - \nu(S) \end{array}\right\} \Rightarrow \\ &\Rightarrow \Phi_i(\nu) = \Phi_j(\nu).\end{aligned} \tag{21}$$

Global weak monotonicity:

$$[\exists k \in \{1,...,n\} : ((k \in S \Rightarrow \nu(S) \leq u(S)) \wedge (k \notin S \Rightarrow \nu(S) \geq u(S)))] \Rightarrow$$

$$\begin{aligned}&\Rightarrow \left\{\begin{array}{c} \forall S, \quad \text{such} \quad \text{that} \quad k \in S : u(S) \geq \nu(S)) \\ u(S\backslash\{k\}) \leq \nu(S\backslash\{k\}) \end{array}\right\} \Rightarrow \\ &\Rightarrow \Phi_k(u) \geq \Phi_k(\nu).\end{aligned} \tag{22}$$

□

4 Conclusion

We have presented several original results in this paper. We proposed to restrict the space of functions which are taken as the solution of cooperative games with transferable utility by Shapley's axioms: we demand that such functions obey pareto optimal, anonymous and weak monotonicity conditions.

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