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Boundary Integral Equations with the Generalized Neumann Kernel for the Neumann Problem

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Abstract This paper presents a new method to solve the interior and the exterior Neumann problems in simply connected regions with smooth boundaries. The method is based on two uniquely solvable Fredholm integral equations of the second kind with the generalized Neumann kernel. Numerical examples reveal that the present method offers an effective numerical method for the Neumann problems when the boundaries are sufficiently smooth.

Keywords Neumann problem, Fredholm integral equation, Generalized Neumann kernel.

1 Introduction

The boundary integral equation method is a classical method for solving the Neumann problem (see [1, 2, 5, 6]). The classical boundary integral equations for the Neumann problems are the two second kind Fredholm integral equations with the Neumann kernel. These integral equations are derived by representing the solutions of the Neumann problems as the potential of a single layer. However, the integral equation for the interior Neumann problem is not uniquely solvable. Furthermore, extra calculations are required for determining the boundary values of the solutions of the Neumann problems from the solutions of the integral equation (see e.g., [1, pp. 315], [2], [6, pp. 72–73] and [5, p. 282]).

In this paper we continue our research on boundary integral equations with the generalized Neumann kernel for the elliptic boundary value problems. We shall extend the results of [8, 10, 11] from the case of the Dirichlet and the Riemann-Hilbert problem to the Neumann problem. Two Fredholm integral equations of the second kind with the generalized Neumann kernel will be derived for the interior and the exterior Neumann problems. The derived integral equations are uniquely solvable which can provide us with the boundary values of the solution of the Neumann problem without any extra calculations. The later properties mean the presented integral equations have advantages over the classical integral equations mentioned above.

This paper is organized as follows: After the presentation of some auxiliary material in Section 2, we recall in Section 3 the integral equations for the interior and the exterior Dirichlet problems. In Sections 4 and 5, we derive two uniquely solvable Fredholm integral equations of the second kind with the generalized Neumann kernel for the interior and the exterior Neumann problems, respectively. In Section 6 we discuss the question of how to treat the integral equations numerically and in Section 7 we give a few examples. A short conclusions will be given in Section 8.

2 Auxiliary Material

Let Ω be a bounded simply connected Jordan region with $0 \in \Omega$. The boundary $\Gamma := \partial\Omega$ is assumed to have a positively oriented parametrization $\eta(t)$ where $\eta(t)$ is a 2π-periodic twice continuously differentiable function with $\dot\eta(t) = \frac{d\eta}{dt} \neq 0$. The parameter t need not be the arc length parameter. The exterior of Γ is denoted by Ω^-.

For a fixed α with $0 < \alpha < 1$, the Hölder space H^α consists of all 2π-periodic real functions which are uniformly Hölder continuous with exponent α. It becomes a Banach space when provided with the usual Hölder norm. A Hölder continuous function $\hat h$ on Γ can be interpreted via $h(t) := \hat h(\eta(t))$ as a Hölder continuous function h of the parameter t and vice versa.

Let $A(t)$ be a continuously differentiable $2\pi-$periodic function with $A \neq 0$. We define two real functions N and M by

$$N(\tau,t) := \frac{1}{\pi}\operatorname{Im}\left(\frac{A(\tau)}{A(t)}\frac{\dot\eta(t)}{\eta(t)-\eta(\tau)}\right), \tag{1}$$

$$M(\tau,t) := \frac{1}{\pi}\operatorname{Re}\left(\frac{A(\tau)}{A(t)}\frac{\dot\eta(t)}{\eta(t)-\eta(\tau)}\right). \tag{2}$$

The kernel $N(\tau,t)$ is called the *generalized Neumann kernel* formed with A and η [8, 10]. When $A = 1$, the kernel N is the Neumann kernel which aries frequently in the integral equations for potential theory and conformal mapping (see [5, p. 286]).

Lemma 1 ([10]) *a) The kernel $N(\tau,t)$ is continuous with*

$$N(t,t) = \frac{1}{\pi}\operatorname{Im}\left(\frac{1}{2}\frac{\ddot\eta(t)}{\dot\eta(t)} - \frac{\dot A(t)}{A(t)}\right). \tag{3}$$

b) The kernel $M(\tau,t)$ has the representation

$$M(\tau,t) = -\frac{1}{2\pi}\cot\frac{\tau-t}{2} + M_1(\tau,t) \tag{4}$$

with a continuous kernel M_1 which takes on the diagonal the values

$$M_1(t,t) = \frac{1}{\pi}\operatorname{Re}\left(\frac{1}{2}\frac{\ddot\eta(t)}{\dot\eta(t)} - \frac{\dot A(t)}{A(t)}\right). \tag{5}$$

Let $\mathcal{N}$ and $\mathcal{M}_1$ be the Fredholm integral operators associate with the continuous kernels N and M_1, i.e.,

$$(\mathcal{N}\mu)(\tau) := \int_0^{2\pi} N(\tau,t)\mu(t)dt, \tag{6}$$

$$(\mathcal{M}_1\mu)(\tau) := \int_0^{2\pi} M_1(\tau,t)\mu(t)dt. \tag{7}$$

Let also $\mathcal{M}$ and $\mathcal{K}$ be the singular integral operators

$$(\mathcal{M}\mu)(\tau) := \int_0^{2\pi} M(\tau,t)\mu(t)dt, \tag{8}$$

$$(\mathcal{K}\mu)(\tau) := \frac{1}{2\pi}\int_0^{2\pi} \mu(t)\cot\frac{\tau-t}{2}dt. \tag{9}$$

The integral in (8) and (9) is a principal value integral. The operator $\mathcal{K}$ is known as the conjugation operator. It is also known as the Hilbert transform (see e.g., [5, p. 107]).

The operators $\mathcal{N}$, $\mathcal{M}$, $\mathcal{M}_1$ and $\mathcal{K}$ are bounded in H^α and map H^α into H^α (see e.g., [10, 9]). It follows from (4) that

$$\mathcal{M} = \mathcal{M}_1 - \mathcal{K}. \tag{10}$$

The eigenproblem of the generalized Neumann kernel has been studied in [8, 10]. In this paper, we shall consider only the generalized Neumann kernel with $A = 1$ and $A = \eta$.

Lemma 2 ([8, 10]) (a) *Let N be the generalized Neumann kernel formed with $A = 1$ and η. Then $\lambda = -1$ is not an eigenvalue of N.*
(b) *Let N be the generalized Neumann kernel formed with $A = \eta$ and η. Then $\lambda = 1$ is not an eigenvalue N.*

2.1 The Dirichlet Problem

Let u be a real function defined in the domain Ω and let $z = x + \mathrm{i}y \in \Omega$. In this paper, for simplicity, we shall write $u(z)$ instead of $u(x,y)$. The interior and the exterior Dirichlet problem are defined as follows.
Interior Dirichlet problem. Let $\gamma \in H^\alpha$ be a given function. Find the function u harmonic in Ω, Hölder continuous on Γ and satisfies on the boundary Γ with boundary condition

$$u(\eta(t)) = \gamma(t), \quad \eta(t) \in \Gamma. \tag{11}$$

Lemma 3 ([1, p. 308]). *The interior Dirichlet problem* (11) *is uniquely solvable.*

Exterior Dirichlet Problem. Let $\gamma \in H^\alpha$ be a given function. Find the function u harmonic in Ω^-, Hölder continuous on Γ, $u(z)$ bounded when $|z| \to \infty$ and satisfies on the boundary Γ with boundary condition

$$u(\eta(t)) = \gamma(t), \quad \eta(t) \in \Gamma. \tag{12}$$

Lemma 4 ([1, p. 312]). *The exterior Dirichlet problem* (12) *is uniquely solvable.*

2.2 The Neumann Problem

Interior Neumann problem. Let $\mathbf{n}$ be the exterior normal to Γ and let $\gamma \in H^\alpha$ be a given function such that

$$\int_0^{2\pi} \gamma(\tau)|\dot{\eta}(\tau)|d\tau = 0. \tag{13}$$

Find the function u harmonic in Ω, Hölder continuous on Γ and satisfies on the boundary Γ with boundary condition

$$\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\eta(t)} = \gamma(t), \quad \eta(t) \in \Gamma. \tag{14}$$

The interior Neumann problem (14) is uniquely solvable up to an additive real constant [1, p. 308]. This arbitrary real constant can be specified by assuming $u(0) = 0$ as in the following lemma.

Lemma 5 ([1, p. 313]). *The interior Neumann problem* (14) *with the condition* $u(0) = 0$ *is uniquely solvable.*

Exterior Neumann Problem. Let $\mathbf{n}$ be the exterior normal to Γ and let $\gamma \in H^\alpha$ be a given function such that

$$\int_0^{2\pi} \gamma(\tau)|\dot{\eta}(\tau)|d\tau = 0. \tag{15}$$

Find the function u harmonic in Ω^-, Hölder continuous on Γ, $u(z) = O\left(|z|^{-1}\right)$ as $z \to \infty$ and satisfies on the boundary Γ with boundary condition

$$\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\eta(t)} = \gamma(t), \quad \eta(t) \in \Gamma. \tag{16}$$

Lemma 6 ([1, p. 313]). *The exterior Neumann problem* (16) *is uniquely solvable.*

3 Integral Equations for the Dirichlet Problem

In this section, we shall review the boundary integral equations derived in [10] for the interior and exterior Dirichlet problems.

Suppose that u is the unique solution of the interior Dirichlet problem. Since it is harmonic in Ω, u has a harmonic conjugate in Ω. We denote to the boundary values of this harmonic conjugate by μ. Then $\gamma + \mathrm{i}\mu$ are boundary values of a function f analytic in Ω, i.e.,

$$f^+(\eta(t)) := \gamma(t) + \mathrm{i}\mu(t), \quad \eta(t) \in \Gamma. \tag{17}$$

The function $f(z)$ is unique up to an additive imaginary constant which can be determined by assuming $f(0)$ is real. Hence, from Theorem 11(a) in [10], we have the following lemma.

Lemma 7 ([10]) *Let μ be the unique solution of the integral equation*

$$\mu - \mathcal{N}\mu = -\mathcal{M}\gamma \tag{18}$$

where the kernels of the operators $\mathcal{N}$ and $\mathcal{M}$ are formed with $A = \eta$. Then the function $f^+ = \gamma + \mathrm{i}\mu$ is a boundary value of an analytic function f in Ω with $\operatorname{Im} f(0) = 0$.

By the Cauchy integral formula, the function $f(z)$ stated in the above lemma is given by

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{\gamma + \mathrm{i}\mu}{\eta - z} d\eta, \quad z \in \Omega. \tag{19}$$

By obtaining the unique analytic function f, the unique solution of the interior Dirichlet problem (11) is given in $\Omega \cup \Gamma$ by $u(z) = \operatorname{Re} f(z)$.

Next, we shall review an integral equation for the exterior Dirichlet problem. Similarly to the interior problem, the function u has also a harmonic conjugate in Ω with boundary values will be denoted by μ. Then $\gamma + \mathrm{i}\mu$ with boundary values of a unique analytic function g in Ω,

$$g^{-}(\eta(t)) := \gamma(t) + \mathrm{i}\mu(t), \quad \eta(t) \in \Gamma, \tag{20}$$

with $g(z) = c + O(z^{-1})$ near ∞ with real c.

Let the function G be defined in Ω^{-} by

$$G(z) := \frac{g(z)}{z}, \tag{21}$$

then G is analytic in Ω^{-} with $G(z) = c/z + O(z^{-2})$ near ∞ with real c. The boundary values of the function G are given by

$$\eta G^{-} = \gamma + \mathrm{i}\mu. \tag{22}$$

Lemma 8 ([10]) *Let μ be the unique solution of the integral equation*

$$\mu + \mathcal{N}\mu = \mathcal{M}\gamma \tag{23}$$

where the kernels of the operators $\mathcal{N}$ and $\mathcal{M}$ are formed with $A = 1$. Then the function $\eta G^{-} = \gamma + \mathrm{i}\mu$ is a boundary value of an analytic function G in Ω^{-} with $G(z) = c/z + O(z^{-2})$ near ∞ with real c

Since $G(\infty) = 0$, hence the Cauchy integral formula implies that the function G stated in the above lemma is given by

$$G(z) = -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{\gamma + \mathrm{i}\mu}{\eta} \frac{d\eta}{\eta - z}, \quad z \in \Omega^{-}. \tag{24}$$

Thus, by (21), the function g is given by

$$\begin{aligned} g(z) &= -\frac{z}{2\pi \mathrm{i}} \int_{\Gamma} \frac{\gamma + \mathrm{i}\mu}{\eta} \frac{d\eta}{\eta - z} \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{\gamma + \mathrm{i}\mu}{\eta} d\eta - \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} (\gamma + \mathrm{i}\mu) \frac{d\eta}{\eta - z} \end{aligned} \tag{25}$$

Consequently, the unique solution of the exterior Dirichlet problem (12) is given in $\Omega^{-} \cup \Gamma$ by $u(z) = \operatorname{Re} g(z)$.

In the following two sections, we shall use the Cauchy-Riemann equations to reduce the interior and the exterior Neumann problems to equivalent Dirichlet problems (see e.g., Mikhlin [7, p. 153]). Then, we shall use the integral equations (18) and (23) to derive boundary integral equations for the Neumann problems.

4 An Integral Equation for the Interior Neumann Problem

In the parametric representation $\eta(t)$, $0 \le t \le 2\pi$ of Γ, we assumed t is not the arc-length parameter. Let $s = s(t)$ be the arc-length of the curve Γ from $\eta(0)$ to $\eta(t)$, i.e.,

$$s(t) = \int_0^t |\dot{\eta}(\tau)| d\tau, \quad 0 \le t \le 2\pi. \tag{26}$$

Then

$$\frac{ds}{dt} = |\dot{\eta}(t)|. \tag{27}$$

Since $\dot{\eta}(t) \neq 0$, hence the function $s = s(t)$ has an inverse function $t = t(s)$ for all $0 \le s \le L$ where $L = \int_0^{2\pi} |\dot{\eta}(\tau)| d\tau$ is the length of Γ.

Suppose that u is a solution of the exterior Neumann problem. Since u is harmonic function in Ω, then u has a harmonic conjugate v in Ω. Denote to the boundary values of the functions u and v by $\tilde{\gamma}$ and $\tilde{\mu}$, respectively, i.e.,

$$\tilde{\gamma}(t) := u(\eta(t)), \quad \tilde{\mu}(t) := v(\eta(t)), \quad \eta(t) \in \Gamma, \quad 0 \le t \le 2\pi.$$

Hence, $\tilde{\gamma} + \mathrm{i}\tilde{\mu}$ is a boundary value of a function f analytic in Ω, i.e.,

$$f^+(\eta(t)) = \tilde{\gamma}(t) + \mathrm{i}\tilde{\mu}(t). \tag{28}$$

Then by the Cauchy-Riemann equations (see e.g., [3, p. 27]), we have

$$\left.\frac{\partial v}{\partial s}\right|_{\eta(t)} = \left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\eta(t)} = \gamma(t), \quad t = t(s). \tag{29}$$

Using the chain rule, we have

$$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial s}\frac{ds}{dt}. \tag{30}$$

Consequently, by (27), we obtain

$$\left.\frac{\partial v}{\partial t}\right|_{\eta(t)} = \gamma(t)|\dot{\eta}(t)|. \tag{31}$$

Hence, for $0 \le \tau \le 2\pi$, we have

$$\tilde{\mu}(\tau) = v(\eta(\tau)) = v(\eta(0)) + \int_0^\tau \left.\frac{\partial v}{\partial t}\right|_{\eta(t)} dt = v(\eta(0)) + \int_0^\tau \gamma(t)|\dot{\eta}(t)| dt. \tag{32}$$

Let the real functions $\varphi(t)$ and $\mu(t)$ be defined on $[0, 2\pi]$ by

$$\varphi(t) \;:=\; \int_0^t \gamma(\tau)|\dot{\eta}(\tau)| d\tau, \tag{33}$$

$$\mu(t) \;:=\; -\tilde{\gamma}(t) + \operatorname{Re} f(0), \tag{34}$$

and let the complex-valued function $F(z)$ be defined on Ω by

$$F(z) := -\mathrm{i}f(z) + \mathrm{i}\operatorname{Re} f(0) - v(\eta(0)). \tag{35}$$

Then F is analytic in Ω with $\operatorname{Im} F(0) = 0$.

From (32) and (33), we have $\tilde{\mu}(t) = \varphi(t) + v(\eta(0))$. Hence, the boundary values of the function f are given by

$$f^{+} = -\mu + \operatorname{Re} f(0) + \mathrm{i}\varphi + \mathrm{i}v(\eta(0)). \tag{36}$$

Consequently, the boundary values of the function F are given by

$$F^{+} = \varphi + \mathrm{i}\mu. \tag{37}$$

The function F is analytic in Ω with $\operatorname{Im} F(0) = 0$ and $F^{+} = \varphi + \mathrm{i}\mu$. Hence, by Lemma 7, the function μ is the unique solution of the integral equation

$$\mu - \mathcal{N}\mu = -\mathcal{M}\varphi \tag{38}$$

where the kernels of the operators $\mathcal{N}$ and $\mathcal{M}$ are formed with $A = \eta$.

By solving the integral equation (38) for μ, then from (34), (33) and (36), we obtain the boundary values of the function f from (36). Let C be the complex constant $C := \operatorname{Re} f(0) + \mathrm{i}v(\eta(0))$, then the function $f(z)$ is given in Ω by

$$f(z) = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma} \frac{-\mu(t) + \mathrm{i}\varphi(t)}{\eta - z} d\eta + C. \tag{39}$$

It clear from (41) that the solution $f(z)$ contains an additive arbitrary complex constant C. We can determined the constant C by assuming $f(0) = 0$, i.e.,

$$C = -\frac{1}{2\pi\mathrm{i}} \int_{\Gamma} \frac{-\mu(t) + \mathrm{i}\varphi(t)}{\eta} d\eta. \tag{40}$$

Hence,

$$f(z) = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma} \frac{-\mu(t) + \mathrm{i}\varphi(t)}{\eta - z} d\eta - \frac{1}{2\pi\mathrm{i}} \int_{\Gamma} \frac{-\mu(t) + \mathrm{i}\varphi(t)}{\eta} d\eta. \tag{41}$$

Then, the unique solution of the interior Neumann problem with the condition $u(0) = 0$ is given in $\Omega \cup \Gamma$ by

$$u(z) = \operatorname{Re} f(z). \tag{42}$$

5 An Integral Equation for the Exterior Neumann Problem

Suppose that u is the solution of the exterior Neumann problem and v is a harmonic conjugate of u in Ω^{-}. Denote the boundary values of the functions u and v by γ and μ, respectively, i.e.,

$$\tilde{\gamma}(t) := u(\eta(t)), \quad \tilde{\mu}(t) := v(\eta(t)), \quad 0 \le t \le 2\pi.$$

Hence, $\tilde{\gamma} + \mathrm{i}\tilde{\mu}$ is a boundary value of a function g analytic in Ω,

$$g^{-}(\eta(t)) = \tilde{\gamma}(t) + \mathrm{i}\tilde{\mu}(t), \tag{43}$$

with $g(z) = \tilde{c} + O(z^{-1})$ near ∞ with a real constant $\tilde{c}$. Since

$$u(z) = \operatorname{Re} g(z) \to 0 \quad \text{when} \quad z \to \infty,$$

hence $\tilde{c} = 0$. Then $g(z) = (c_1 + \mathrm{i}c_2)/z + O(z^{-2})$ near ∞ with real constants c_1 and c_2.

Since the function $g = u + \mathrm{i}v$ is an analytic function, then by the Cauchy-Riemann equations, we have

$$\left.\frac{\partial v}{\partial s}\right|_{\eta(t)} = \left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\eta(t)} = \gamma(t), \quad t = t(s). \tag{44}$$

Using the chain rule and (27), we obtain

$$\left.\frac{\partial v}{\partial t}\right|_{\eta(t)} = \left.\frac{\partial v}{\partial s}\right|_{\eta(t)} \frac{ds}{dt} = \gamma(t)|\dot{\eta}(t)|. \tag{45}$$

Hence, for $0 \leq \tau \leq 2\pi$, we have

$$\tilde{\mu}(t) = v(\eta(\tau)) = v(\eta(0)) + \int_0^\tau \left.\frac{\partial v}{\partial t}\right|_{\eta(t)} dt = v(\eta(0)) + \int_0^\tau \gamma(t)|\dot{\eta}(t)|dt. \tag{46}$$

Let the real functions $\varphi(t)$ and $\mu(t)$ be defined on $[0, 2\pi]$ by

$$\varphi(t) \quad := \quad \int_0^t \gamma(\tau)|\dot{\eta}(\tau)|d\tau, \tag{47}$$

$$\mu(t) \quad := \quad -\tilde{\gamma}(t), \tag{48}$$

and let the complex-valued function $G(z)$ be defined on Ω by

$$G(z) := \frac{-\mathrm{i}g(z) - v(\eta(0))}{z}. \tag{49}$$

Then G is analytic in Ω^- with $G(z) = c/z + O(z^{-2})$ near ∞ with real constant $c = -v(\eta(0))$. The boundary values of the function $G(z)$ are given by

$$\eta G^- = -\mathrm{i}(\tilde{\gamma} + \mathrm{i}\tilde{\mu}) - v(\eta(0)) = \varphi + \mathrm{i}\mu. \tag{50}$$

Hence, by Lemma 8, the function μ is the unique solution of the integral equation

$$\mu + \mathcal{N}\mu = \mathcal{M}\varphi \tag{51}$$

where the kernels of the operators $\mathcal{N}$ and $\mathcal{M}$ are formed with $A = 1$.

By solving the integral equation (51) for μ, then from (43), (46), (47) and (48), we obtain the boundary values of the function g,

$$g^-(\eta) = -\mu + \mathrm{i}\varphi + \mathrm{i}v(\eta(0)). \tag{52}$$

Since $g(\infty) = 0$, then, by the Cauchy integral formula (see [4, p. 2]), the function $g(z)$ is given in Ω^- by

$$g(z) = -\frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{-\mu(t) + \varphi(t)}{\eta - z} d\eta. \tag{53}$$

Furthermore, the function g satisfies

$$\frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{g^-(\eta)}{\eta} d\eta = 0.$$

Hence, by (52), we have

$$v(\eta(0)) = -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{\varphi + \mathrm{i}\mu}{\eta} d\eta. \tag{54}$$

It clear from (53) that the solution $g(z)$ is uniquely determined by μ and φ. By knowing the real functions φ, μ, and the real constant $v(\eta(0))$, the boundary values g^- of the function g can be calculated from (52). The function g itself can be calculated by (53). Then, the unique solution of the exterior Neumann problem is given in $\Omega^- \cup \Gamma$ by

$$u(z) = \operatorname{Re} g(z). \tag{55}$$

6 Numerical Implementation

Since the functions A and η are $2\pi-$periodic, the integrals in the integral operator $\mathcal{N}$ can be best discretized on an equidistant grid by the trapezoidal rule, i.e., the integral operator $\mathcal{N}$ is discretized by the Nyström method [1].

Let n be a given integer and define the the n equidistant collocation points t_j by

$$t_j := (j-1)\frac{2\pi}{n}, \quad j = 1, 2, \ldots, n. \tag{56}$$

Then, using the Nyström method with the trapezoidal rule to discretize the integral equations (38) and (51), we obtain the linear systems

$$\mu_n(t_i) - \frac{2\pi}{n} \sum_{j=1}^{n} N(t_i, t_j)\mu_n(t_j) \quad = \quad -(\mathcal{M}\varphi)(t_i), \tag{57}$$

$$\mu_n(t_i) + \frac{2\pi}{n} \sum_{j=1}^{n} N(t_i, t_j)\mu_n(t_j) \quad = \quad (\mathcal{M}\varphi)(t_i), \quad i = 1, 2, \ldots, n, \tag{58}$$

where μ_n is an approximation to μ. Note that the kernels N and M in (57) are formed with $A = \eta$ and formed with $A = 1$ in (58).

For the calculation of $(\mathcal{M}\varphi)(t_i)$ in the right-hand side of (57) and (58), we calculate

$$\varphi(t) = \int_0^t \gamma(\tau)|\dot{\eta}(\tau)|d\tau. \tag{59}$$

Since the integrand $\gamma(\tau)|\dot{\eta}(\tau)|$ is not periodic on $[0, t]$ unless $t = 2\pi$. Hence, to calculate the integral in (59) accurately, we shall use the Gaussian quadrature method. For $t > 0$, we have

$$\varphi(t) = \int_{-1}^{1} \gamma\left(\frac{(\tau+1)t}{2}\right) \left|\dot{\eta}\left(\frac{(\tau+1)t}{2}\right)\right| \frac{t}{2}\, d\tau.$$

Hence, by using the $m + 1$ points Gaussian quadrature method, we obtain

$$\varphi(t) \approx \sum_{i=0}^{m} \frac{\omega_i t}{2} \gamma\left(\frac{(\sigma_i+1)t}{2}\right) \left|\dot{\eta}\left(\frac{(\sigma_i+1)t}{2}\right)\right| \tag{60}$$

where σ_i and ω_i $(i = 0, 1, 2, \ldots, m)$ are the Gaussian abscissas and weights.

Then using (10)

$$(\mathcal{M}\varphi)(t_i) = (\mathcal{M}_1\varphi)(t_i) - (\mathcal{K}\varphi)(t_i), \quad i = 1, 2, 3, \ldots, n.$$

The values $(\mathcal{K}\varphi)(t_i)$ will be calculated by using the FFT and the values $(\mathcal{M}_1\varphi)(t_i)$ will be calculated by using the trapezoidal rule, i.e.,

$$(\mathcal{M}_1\varphi)(t_i) = \frac{2\pi}{n}\sum_{j=1}^{n} M_1(t_i, t_j)\varphi(t_j), \quad i = 1, 2, \ldots, n.$$

Defining the matrix Q by $Q_{ij} = \pi N(t_i, t_j)/n$, and the vectors $\mathbf{x}$, $\mathbf{y}$ by $x_i = \mu_n(t_i)$, $y_i(\mathcal{M}\varphi)(t_i)$, Eqs. (57) and (58) can be rewritten as the two n by n linear systems

$$(I - Q)\mathbf{x} = -\mathbf{y}, \tag{61}$$

$$(I + Q)\mathbf{x} = \mathbf{y}. \tag{62}$$

Since the integral equations (38) and (51) are uniquely solvable, then for n sufficiently large, the linear systems (61) and (62) are uniquely solvable [1, p. 170].

The linear systems (61) and (62) are solved using the MATLABs \ operator that makes use of the Gauss elimination method. By solving the linear systems (61) and (62), we obtain $\mu_n(t_i)$ for $i = 1, 2, \ldots, n$. Then the approximate solution $\mu_n(t)$ can be calculated for all $t \in [0, 2\pi]$ using the Nyström interpolating formula, i.e., the approximate solution $\mu_n(t)$ of the integral equation (38) is given by

$$\mu_n(t) = -(\mathcal{M}\varphi)(t) + \frac{2\pi}{n}\sum_{j=1}^{n} N(t, t_j)\mu_n(t_j), \tag{63}$$

and the approximate solution $\mu(t)$ of the integral equation (51) is given by

$$\mu_n(t) = (\mathcal{M}\varphi)(t) - \frac{2\pi}{n}\sum_{j=1}^{n} N(t, t_j)\mu_n(t_j). \tag{64}$$

7 Examples

For our examples, we use three boundary curves: an ellipse, the ovals of Cassini, and an "amoeba". For the ellipse, the boundary has the parameterization

$$\Gamma_1: \quad \eta(t) = \cos t + \mathrm{i}5\sin t, \quad 0 \le t \le 2\pi,$$

(see Figure 1). For the ovals of Cassini, the boundary parameterization is

$$\Gamma_2: \quad \eta(t) = R(t)\,\mathrm{e}^{\mathrm{i}t}, \quad 0 \le t \le 2\pi,$$

where $R(t) = 2.5 + 2\cos 2t$ (see Figure 2). The parameterization of the amoeba boundary is

$$\Gamma_3: \quad \eta(t) = R(t)\,\mathrm{e}^{\mathrm{i}t}, \quad 0 \le t \le 2\pi,$$

with $R(t) = \mathrm{e}^{\cos t}\cos^2 2t + \mathrm{e}^{\sin t}\sin^2 2t$ (see Figure 3).

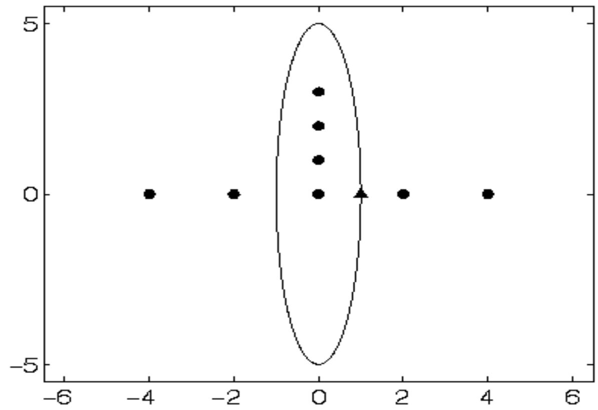

Figure 1: The Curve Γ_1, the Interior and Exterior Test Points.

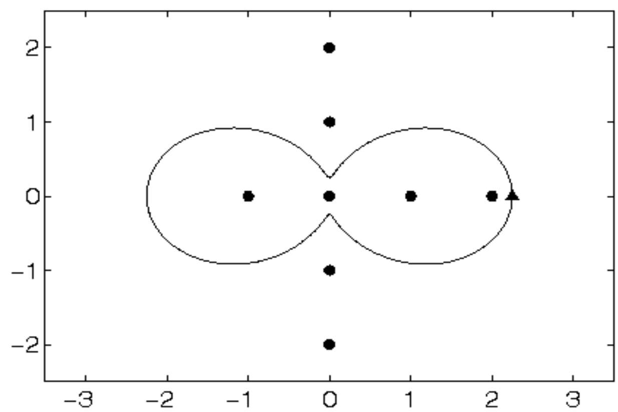

Figure 2: The Curve Γ_2, the Interior and Exterior Test Points.

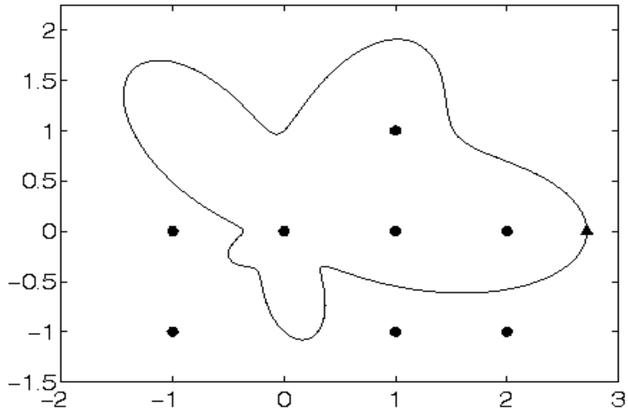

Figure 3: The Curve Γ_3, the Interior and Exterior Test Points.

The interior of amoeba boundary shown in Figure 2 is a nonconvex and complicated region. It was used in [2] to illustrate that the programs given there can handle solutions on such unusual boundaries. In this paper, we use this boundary to show that our method can also work efficiently for such regions.

For the test problems, we use the following Neumann problems. These problems had been used in [2]. The first problem is the interior Neumann problem with the condition $u(0) = 0$, has the unique solution

$$u(z) = \mathrm{e}^x \cos y - 1, \quad z = x + \mathrm{i}y \in \Omega.$$

The second problem is the exterior Neumann problem which has the unique solution

$$u(z) = \frac{x}{x^2 + y^2}, \quad z = x + \mathrm{i}y \in \Omega^-.$$

Let $T(\eta)$ denotes the unit tangent of Γ at the point $\eta(t) \in \Gamma$, then $T(\eta) = \dot{\eta}(t)/|\dot{\eta}(t)|$. Hence, the unit normal vector $\mathbf{n}$ to Γ at the point $\eta(t) \in \Gamma$ is given by

$$\mathbf{n} = (\mathbf{n}_x, \mathbf{n}_y)$$

where $\mathbf{n}_x = \mathrm{Re}(-\mathrm{i}T(\eta(t)))$ and $\mathbf{n}_y = \mathrm{Im}(-\mathrm{i}T(\eta(t)))$. Hence, for the above interior and exterior Neumann problems, the function $\gamma(t)$ is given by

$$\gamma(t) = \left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\eta(t)} = \left.\left(\frac{\partial u}{\partial x}\mathbf{n}_x + \frac{\partial u}{\partial y}\mathbf{n}_y\right)\right|_{\eta(t)}.$$

The maximum error norm $\|u - u_n\|_\infty$ between the exact boundary values u and the approximate boundary values u_n at the node points is presented in Table 1 for the interior Neumann problem and in Table 5 for the exterior Neumann problem. The absolute error $|u(z) - u_n(z)|$ at four test points z inside Γ for the interior Neumann problem is listed in Tables 2–4. The error $|u(z)-u_n(z)|$ at four test points z outside Γ for the exterior Neumann problem is listed in Tables 6–8. The numerical results are presented for various values of n and m where n is the number of node points given in (56) and $m+1$ is the number of Gaussian abscissa used in (60).

Table 1: The Error $\|u - u_n\|_\infty$ for the Interior Neumann Problem on the Boundaries Γ_1, Γ_2 and Γ_3.

m	n	Γ_1	Γ_2	Γ_3
64	32	1.43(−05)	7.56(−03)	2.54(−01)
	64	3.20(−11)	1.58(−04)	7.13(−03)
	128	3.56(−14)	9.21(−08)	5.00(−05)
	256	———	6.84(−14)	1.26(−07)
	512	———	———	1.31(−07)
128	32	1.43(−05)	7.56(−03)	2.54(−01)
	64	3.20(−11)	1.58(−04)	7.13(−03)
	128	1.44(−14)	9.21(−08)	5.00(−05)
	256	———	9.59(−14)	3.58(−09)
	512	———	———	2.91(−13)

Table 2: The Error $|u(z) - u_n(z)|$ for the Interior Neumann Problem on the Boundary Γ_1.

m	n	$z = 0$	$z = i$	$z = 2i$	$z = 3i$
64	16	1.28(−01)	4.93(−02)	5.64(−02)	1.15(−02)
	32	5.24(−03)	9.64(−04)	5.43(−04)	2.19(−03)
	64	7.97(−06)	1.27(−06)	2.74(−07)	1.25(−06)
	128	1.85(−11)	1.35(−12)	8.93(−12)	1.13(−11)
	256	1.37(−15)	2.00(−15)	5.33(−15)	8.44(−15)

Table 3: The Error $|u(z) - u_n(z)|$ for the Interior Neumann Problem on the Boundary Γ_2.

m	n	$z = -1$	$z = 0$	$z = 1$	$z = 2$
64	16	5.25(−02)	1.13(−03)	4.05(−02)	3.79(−01)
	32	2.64(−03)	2.11(−07)	2.44(−03)	2.73(−02)
	64	3.54(−05)	9.45(−13)	3.29(−05)	3.73(−04)
	128	1.06(−08)	2.32(−16)	9.82(−09)	6.85(−08)
	256	5.55(−16)	———	4.44(−16)	3.55(−15)

Table 4: The Error $|u(z) - u_n(z)|$ for the Interior Neumann Problem on the Boundary Γ_3.

m	n	$z = 0$	$z = 1$	$z = 2$	$z = 1 + i$
128	32	5.98(−04)	1.86(−02)	5.69(−02)	9.48(−04)
	64	8.68(−08)	4.35(−04)	9.56(−04)	4.05(−04)
	128	9.43(−13)	2.14(−06)	2.02(−06)	1.53(−06)
	256	1.25(−16)	2.91(−11)	2.74(−11)	2.04(−11)
	512	———	4.66(−15)	1.51(−14)	5.55(−16)

Table 5: The Error $\|u - u_n\|_\infty$ for the Exterior Neumann Problem on the Boundaries Γ_1, Γ_2 and Γ_3.

m	n	Γ_1	Γ_2	Γ_3
128	32	7.50(−02)	4.61(−02)	6.14(−01)
	64	3.04(−03)	1.60(−03)	2.86(−02)
	128	5.52(−06)	1.19(−06)	1.48(−04)
	256	4.50(−06)	5.08(−12)	1.20(−08)
	512	4.68(−06)	5.16(−12)	4.82(−10)
256	32	7.50(−02)	4.61(−02)	6.14(−01)
	64	3.04(−03)	1.60(−03)	2.86(−02)
	128	4.64(−06)	1.19(−06)	1.48(−04)
	256	1.09(−11)	4.23(−13)	1.20(−08)
	512	6.05(−13)	6.50(−14)	4.53(−14)

Table 6: The Error $|u(z) - u_n(z)|$ for the Exterior Neumann Problem on the Boundary Γ_1.

m	n	$z = -4$	$z = -2$	$z = 2$	$z = 4$
256	16	1.02(−02)	7.51(−02)	7.51(−02)	1.02(−02)
	32	5.10(−04)	6.23(−03)	6.23(−03)	5.10(−04)
	64	7.85(−07)	1.94(−05)	1.94(−05)	7.85(−07)
	128	1.82(−12)	1.00(−10)	1.00(−10)	1.82(−12)
	256	2.78(−16)	7.77(−16)	1.11(−16)	4.44(−16)

Table 7: The Error $|u(z) - u_n(z)|$ for the Exterior Neumann Problem on the Boundary Γ_2.

m	n	$z = -2\mathrm{i}$	$z = -\mathrm{i}$	$z = \mathrm{i}$	$z = 2\mathrm{i}$
128	16	2.51(−04)	2.56(−02)	2.56(−02)	2.51(−04)
	32	3.57(−06)	6.25(−04)	6.25(−04)	3.57(−06)
	64	6.88(−12)	2.86(−07)	2.86(−07)	6.86(−12)
	128	1.62(−14)	4.36(−13)	4.37(−13)	1.81(−14)
	256	2.10(−14)	2.37(−14)	2.37(−14)	2.18(−14)

Table 8: The Error $|u(z) - u_n(z)|$ for the Exterior Neumann Problem on the Boundary Γ_3.

m	n	$z = -1$	$z = -1 - \mathrm{i}$	$z = 1 - \mathrm{i}$	$z = 2 - \mathrm{i}$
256	32	1.79(−02)	5.62(−03)	6.29(−02)	2.60(−02)
	64	5.88(−04)	2.13(−07)	1.09(−03)	9.83(−04)
	128	1.55(−07)	7.76(−07)	4.21(−06)	2.31(−06)
	256	8.72(−11)	6.52(−11)	1.29(−10)	1.91(−10)
	512	2.22(−16)	1.22(−15)	2.22(−16)	3.89(−16)

8 Conclusions

Two uniquely solvable boundary integral equations were derived in this paper for the interior and the exterior Neumann problems. Unlike the classical boundary integral equations for the Neumann problems [1, 2, 5, 6], the derived boundary integral equations are uniquely solvable and yields directly the boundary value of the solutions of the Neumann problems.

An interior and an exterior Neumann problems were solved numerically in three test regions using the proposed method. The numerical examples illustrate that the proposed method yields approximations of high accuracy.

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