

Study of the Supercritical Solution of the Stationary Negative Forced KdV Equation

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Abstract This paper considers Forced KdV equation with negative forcing term. The supercritical solitary wave solutions of the stationary Forced KdV equations are obtained. In order to obtain the solutions, the domain of the problem has been divided into three parts; the left, the middle and the right parts. The solution on the left and the right parts are obtained by an analytical method. The solution on the middle part is expressed in terms of Weierstrass elliptic function. A computer program using Mathematica is designed to produce the solutions. The complete solution is found by matching the solutions of all the three parts. It is found that there are four different solutions according to the values of the phase shift. All solutions are positive.

Keywords Stationary Forced KdV Equation; Supercritical solution.

1 Introduction

The forced KdV equation is a first -order approximation of a long nonlinear surface wave in a channel flow of an inviscid fluid of constant density over a bump or a dent. This equation is normally written in the form (Gong and Shen, 1994)

$$\eta_t + \lambda\eta_x + 2\alpha\eta\eta_x + \beta\eta_{xx} = f_x(x) \quad -\infty < x < \infty \quad t > 0 \quad (1)$$

where $\lambda > 0, \alpha < 0$ and $\beta < 0$ are constant and $f(x)$ is a given function called the forcing term which is differentiable and has a compact support i.e. it is nonzero only in a closed bounded set. Equation (1) was first derived by Akylas (1984) and in an asymptotically reduced result from Euler equations of fluid motion and the corresponding boundary conditions. The unknown $\eta(x, t)$ represents the first order elevation of the free surface of the fluid from its equilibrium level. The forcing function $f(x)$ is due to the bottom topography of the fluid domain (such as a bump or dent) or due to an external pressure on the free surface such as the wind stress on the surface of an ocean.

In the absence of the forcing function term i.e $f(x) = 0$ equation (1) becomes the familiar Korteweg de Vries (KdV) equation

$$\eta_t + \lambda\eta_x + 2\alpha\eta\eta_x + \beta\eta_{xx} = 0 \quad -\infty < x < \infty \quad t > 0. \quad (2)$$

When $\eta_t = 0$ equation (1) becomes the equation of our concern, the stationary forced (sfKdV) equation,

$$\lambda\eta_x + 2\alpha\eta\eta_x + \beta\eta_{xx} = f_x(x) \quad -\infty < x < \infty \quad (3)$$

Equation (3) was first derived by Shen (1993). The solutions of equation (3) are categorized according to the value of λ (Upstream flow velocity) as follows (Shen, 1993).

- (i) Supercritical stationary waves. They occur only when λ is positive and sufficiently large.
- (ii) Subcritical stationary waves. They occur only when λ is negative and sufficiently small.
- (iii) Unsteady periodic soliton radiation. This solution appears when $|\lambda|$ is small. Such a solution is called the transcritical solution.

2 Problem Statement

In this paper we study supercritical solitary wave solutions of equation (3) with negative forcing terms in the rectangular dent. In this work we shall take the value of λ equal to 3 and the length of rectangular dent equals 2 in order to compare with the previous results.

Solitary waves refer to any surface wave profile that dies out at infinity which means that the free surface elevation λ has the property

$$\eta \rightarrow 0, \quad \eta_x \rightarrow 0, \quad \text{and} \quad \eta_{xx} \rightarrow 0 \quad \text{as} \quad x \rightarrow \pm\infty \quad (4)$$

Integrating equation (3) once with respect to the independent variable from $-\infty$ to x , we have

$$\lambda\eta + \alpha\eta^2 + \beta\eta_{xx} = f(x), \quad -\infty < x < \infty, \quad \lambda > 0 \quad (5)$$

with

$$\eta \rightarrow 0, \quad \text{and} \quad \eta_x \rightarrow 0 \quad \text{as} \quad x \rightarrow \pm\infty \quad (6)$$

In this paper we shall study equation (5) and (6) with the following conditions

$$(i) \quad \alpha = -\frac{3}{4} < 0, \quad \beta = -\frac{1}{6} \quad \text{and} \quad \lambda = 3.$$

$$(ii) \quad f(x) = \begin{cases} -\frac{1}{2}, & -\frac{a}{2} \leq x \leq \frac{a}{2} \\ 0, & \text{otherwise} \end{cases}$$

where a is a positive constant representing the length of a rectangular dent.

In order to obtain the solution of equation (5) and (6) the domain of the problem has been divided into three parts. The left $\left(x \leq -\frac{a}{2}\right)$, the middle $\left(-\frac{a}{2} < x < \frac{a}{2}\right)$ and the right parts $\left(x \geq \frac{a}{2}\right)$. The complete solution was found by matching the solutions of all parts. Section 3 and Section 4 describes the method of finding solution on the left and the right part respectively. Section 5 describes the solution in the middle part. In Section 7 we combine the solutions obtained in Section 4 and Section 5.

3 Left-Side Solution

In this Section we need to solve equations (5) and (6) in the region of zero forcing term by using analytical method. Meanwhile, the solution in the region of nonzero forcing term (middle part) will be discussed later in Section 5.

Outside the region of the forcing term, equations (5) and (6) become

$$\lambda\eta + \alpha\eta^2 + \beta\eta_{xx} = 0 \quad \frac{a}{2} \leq x \leq -\frac{a}{2}, \quad \lambda > 0 \quad (7)$$

$$\eta \rightarrow 0, \quad \text{and} \quad \eta_x \rightarrow 0 \quad \text{as} \quad x \rightarrow \pm\infty. \quad (8)$$

We need to find the solution in the negative region $\left(x \leq -\frac{a}{2}\right)$ outside the rectangular bump. In this region equations (7) and (8) are written as

$$\lambda\eta + \alpha\eta^2 + \beta\eta'' = 0 \quad -\infty < x < -\frac{a}{2}, \quad \lambda > 0 \quad (9)$$

$$\eta \rightarrow 0, \quad \text{and} \quad \eta_x \rightarrow 0 \quad \text{as} \quad x \rightarrow -\infty. \quad (10)$$

where equation (10) is the half negative solitary wave conditions. We start by rearranging equation (9) in the form

$$\beta\eta'' = -\lambda\eta - \alpha\eta^2 \quad -\infty < x \leq -\frac{a}{2}, \quad \lambda > 0 \quad (11)$$

where $\eta' = \frac{d\eta}{dx}$.

The procedure we adopt here in order to integrate equation (11) is that we write η'' as

$$\eta'' = \eta' \frac{d\eta'}{d\eta} \quad (12)$$

Substituting equation (12) into equation (11) we obtain

$$\beta\eta' d\eta' = (-\lambda\eta - \alpha\eta^2) d\eta \quad (13)$$

Equation (13) is simply a first order separable equation which can be integrated easily to get

$$\frac{\beta}{2} (\eta')^2 = -\frac{\lambda}{2} \eta^2 - \frac{\alpha}{3} \eta^3 + C. \quad (14)$$

Multiplying both sides of equation(14) by $\frac{3}{\alpha}$ we then obtain

$$\frac{3\beta}{2\alpha} (\eta')^2 = -\frac{3\lambda}{2\alpha} \eta^2 - \eta^3 + \frac{\alpha}{3} C \quad (15)$$

where $\frac{\beta}{\alpha}$ is positive constant and C is the integration constant.

By the half negative solitary wave conditions (10), equation (15) has a real double root, r_0 and r_0 is smaller than the third real root r . Then equation (15) can be written as

$$\frac{3\beta}{2\alpha} (\eta')^2 = (r - \eta)(\eta - r_0), \quad r_0 < r. \quad (16)$$

If we now let

$$v = \eta - r_0 \quad \zeta = \sqrt{\frac{2\alpha}{9\beta}} x,$$

then equation (16) can be written as

$$\frac{1}{3} \left(\frac{dv}{d\zeta} \right) = v^2 (s_1 - v), \quad \text{where} \quad s_1 = r - r_0 > 0. \quad (17)$$

Equation (17) can be integrated to yield

$$v = s_1 \sec h^2 \sqrt{\frac{3s_1}{4}} \zeta. \quad (18)$$

Hence, the actual solution on the left side is

$$\eta(x) = -\frac{3\lambda}{2\alpha} \sec h^2 \sqrt{\frac{-\lambda}{4\beta}} (x - L_0), \quad (19)$$

where L_0 is the phase shift to be determined in Section 6. All graphs of these solutions are shown in Figure 1. In Section 6 we shall show how to calculate L_0 . For the purpose of illustration we now use the values of L_0 obtained in Section 6 to produce the appropriate solution of the sfKdV on the left side.

The graphs shown in Figure 1 are the solution of the sfKdV equation on the left side when $\lambda = 3$ and $a = 2$. The values of L_0 used for this purpose are 0.16614, -0.153332, -0.795703 and -1.8466578 for graph (a), (b), (c) and (d) respectively.

4 The Right-Side Solution

In this section we determine the solution on the right side $\left(x \geq \frac{a}{2}\right)$ that lies outside the rectangular dent. In this region equations (7) and (8) become

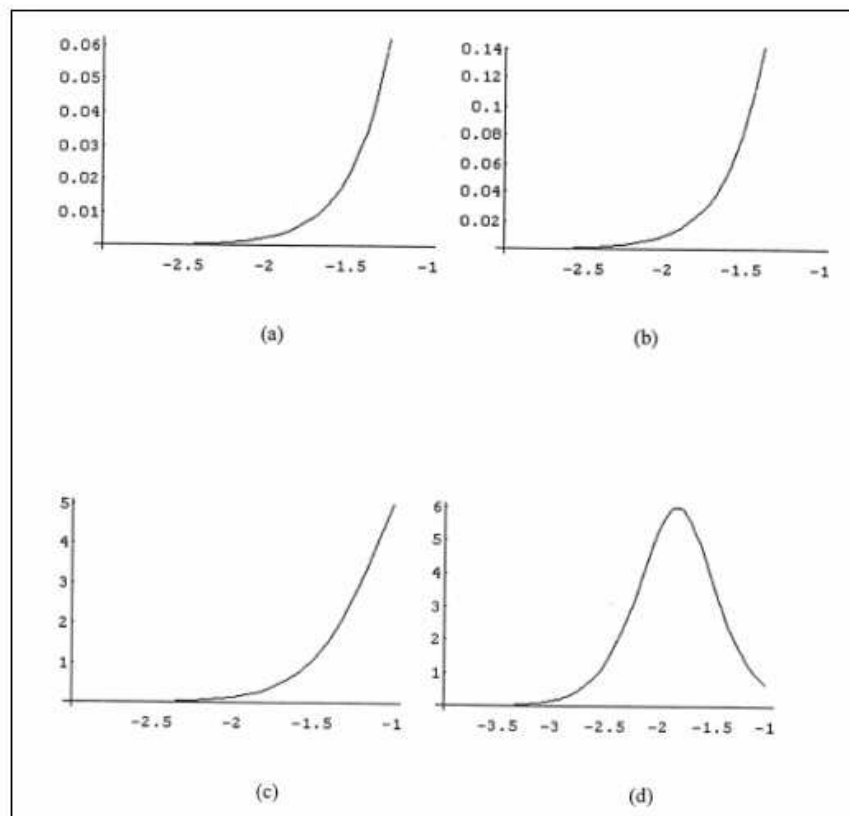
$$\lambda\eta + \alpha\eta^2 + \beta\eta'' = 0, \quad \frac{a}{2} \leq x < \infty, \quad \lambda > 0 \quad (20)$$

$$\eta \rightarrow 0 \quad \eta_x \rightarrow 0 \quad \text{as} \quad x \rightarrow +\infty \quad (21)$$

where equation (21) is normally termed as the positive half solitary wave conditions.

The difference between the left and the right region is simply on the solitary wave conditions. The same procedure as used for the left side can be applied to find the solution in the right and we thus obtain

$$\eta(x) = -\frac{3\lambda}{2\alpha} \sec h^2 \sqrt{\frac{-\lambda}{4\beta}} (x - L_1), \quad (22)$$

**Figure 1:** Left-Side Solution

where the phase shift $L_1 = -L_0$. All graphs of these solutions are shown in

The graphs shown in Figure 2 are the solutions of the sfKdV equation on the right side when $\lambda = 3$ and $a = 2$. The values of L_1 used for this purpose are -0.116614, 0.1553332, 0.7957503 and 1.8466578 for graph (a), (b), (c) and (d) respectively.

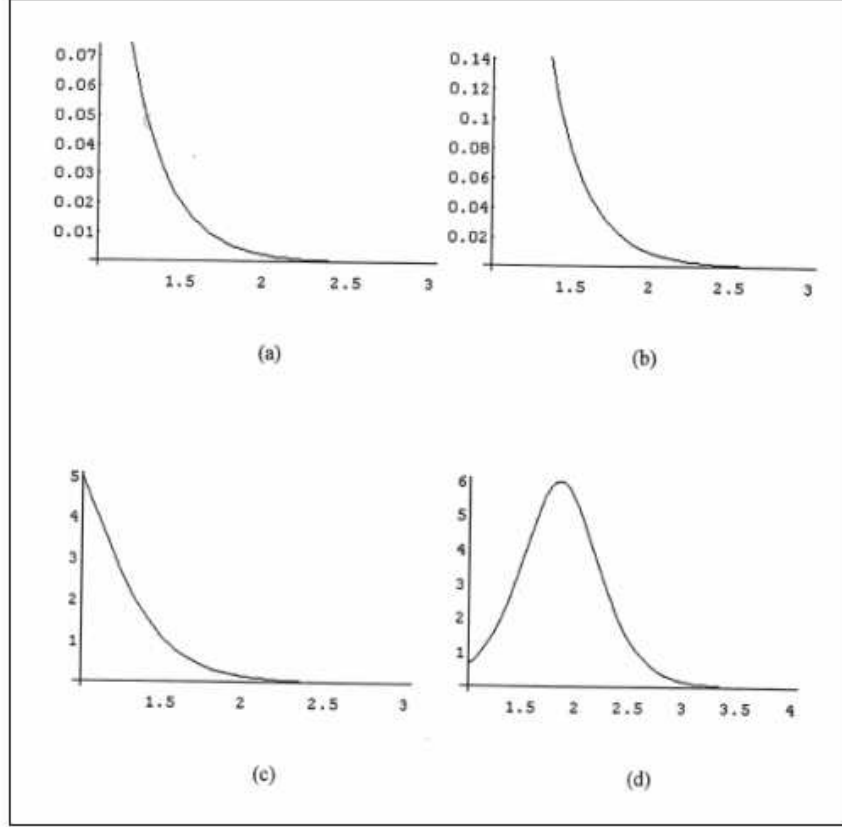


Figure 2: Right-Side Solution

5 The Middle Solution

At this stage, we have obtained the solutions at both sides outside the region of the forcing terms discussed in Section 3 and Section 4, respectively. What is left now is to find the solutions in the middle. This means that we need to solve equations (5) and (6) in the region of the forcing terms. We will show that the solution in this region can be expressed in terms of Weierstrass elliptic functions. The solutions we obtained in Section 3, 4, and 5 will be combined to give the full solutions. This will be presented in Section 7.

In this region, $|x| \leq \frac{a}{2}$, the solution must satisfy

$$\lambda\eta + \alpha\eta^2 + \beta\eta_{xx} = 1, \quad \lambda > 0. \quad (23)$$

We shall be dealing with equation (23). In order to solve this equation we need to deduce some conditions. These conditions come from continuity of η and η' at $x = -\frac{a}{2}$ and $x = \frac{a}{2}$.

Now at $x = -\frac{a}{2}$, we have

$$\eta\left(-\frac{a}{2}\right) \equiv \eta_0 \quad \text{and} \quad \eta'\left(-\frac{a}{2}\right) \equiv \eta_1 \quad (24)$$

where

$$\eta_0 = -\frac{3\lambda}{2\alpha} \operatorname{sech}^2 \sqrt{\frac{-\lambda}{4\beta}} (x + L_0) \quad (25)$$

$$\eta_1 = \sqrt{-\frac{\lambda}{\beta}} \eta_0 \tanh \sqrt{\frac{-\lambda}{4\beta}} \left(\frac{a}{2} + L_0\right) \quad (26)$$

At $x = \frac{a}{2}$ we have

$$\begin{aligned} \eta\left(-\frac{a}{2}\right) &= \eta\left(\frac{a}{2}\right), \\ \eta'\left(-\frac{a}{2}\right) &= \eta'\left(\frac{a}{2}\right) \quad \text{or} \quad \eta'\left(-\frac{a}{2}\right) = -\eta'\left(\frac{a}{2}\right) \end{aligned} \quad (26^*)$$

Integrating equation (23) from $-\frac{a}{2}$ to x ($< \frac{a}{2}$), we find

$$\frac{\beta}{2} (\eta')^2 = \eta - \frac{\lambda}{2} \eta^2 - \frac{\alpha}{3} \eta^3 + \eta_0 \quad (27)$$

Equation (27) can be written as

$$(\eta')^2 = b_1 \eta^3 + b_2 \eta^2 + b_3 \eta + b_4 \quad (28)$$

where

$$b_1 = -\frac{2\alpha}{3\beta}, \quad b_2 = -\frac{\lambda}{\beta}, \quad b_3 = -\frac{1}{\beta} \quad \text{and} \quad b_4 = -2b_3\eta_0.$$

By making the transformation

$$\eta = c_1 u + c_2$$

Equation (28) is converted into

$$c_1^2 u'^2 = b_1 (c_1 u + c_2)^3 + b_2 (c_1 u + c_2)^2 + b_3 (c_1 u + c_2) + b_4 \quad (29)$$

Dividing both sides of equation (29) by $(c_1)^2$ we obtain

$$\begin{aligned} (u')^2 &= (b_1 c_1) u^3 \\ &+ \left[3b_1 \left(-\frac{b_2}{3b_1} \right) + b_2 \right] u^2 \\ &+ \left[3b_1 \left(\frac{b_2^2}{9b_1^2} \right) + 2b_2 \left(-\frac{b_2}{3b_1} \right) + b_3 \right] \frac{u}{c_1} \\ &+ (b_1 c_2^3 + b_2 c_2^2 + b_3 c_2 + b_4) \frac{1}{c_1} \end{aligned} \quad (30)$$

Equation (30) is now written in a simplified form as

$$u'^2 = 4u^3 - g_2u - g_3 \quad (31)$$

where

$$c_1 = \frac{4}{b_1}, c_2 = -\frac{b_2}{3_1}, g_2 = -\frac{(c_2c_2)}{c_1}, \text{ and } g_3 = -\frac{(b_1c_2^2 + b_3c_2 + b_4)}{c_1^2}$$

In equation (31), g_2 is constant and g_3 is a function of L_0 for given λ, α, β and a . The general solution of equation (31) can be expressed in term of Weierstrass elliptic function (Gong, 1994)

$$u(x) = \wp(x + T, g_2, g_3)$$

where T is a constant.

Thus, in the region of the non-zero forcing term $\left(|x| \leq \frac{a}{2}\right)$ the supercritical solitary wave solution can be written as

$$\eta(x) = c_1\wp(x + T, g_2, g_3) + c_2. \quad (32)$$

Equations (25) and (26) are now respectively reduced to

$$\eta_0 = c_1\wp\left(-\frac{a}{2} + T, g_2, g_3\right) + c_2 \quad (33)$$

and

$$\eta_1 = c_1\wp\left(-\frac{a}{2} + T, g_2, g_3\right) \quad (34)$$

From Wang and Guo(1989), we find the identity

$$(x + T, g_2, g_3) = \frac{1}{4} \left\{ \frac{\wp'(x, g_2, g_3) - \wp'(T, g_2, g_3)}{\wp(x, g_2, g_3) - \wp(T, g_2, g_3)} \right\}^2 - \wp(x, g_2, g_3) - \wp(T, g_2, g_3) \quad (35)$$

Substituting equations (33) and (34) into equation (35), we then obtain

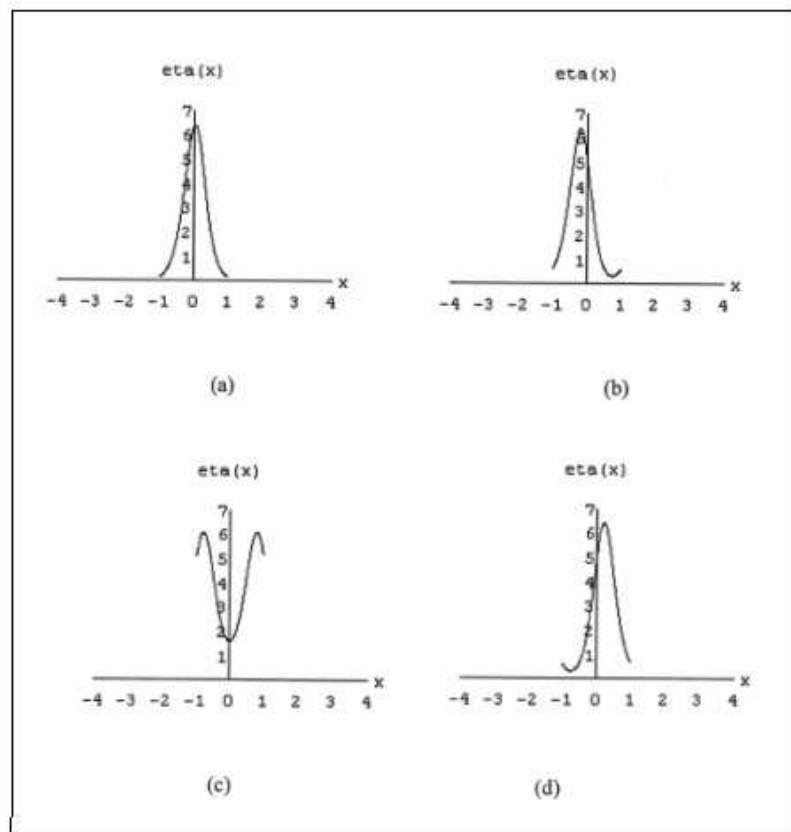
$$(x + T, g_2, g_3) = \frac{1}{4} \left\{ \frac{\wp'(x, g_2, g_3) - \frac{\eta_1}{c_1}}{\wp(x, g_2, g_3) - \left(\frac{\eta_0 - c_2}{c_1}\right)} \right\}^2 - \wp(x, g_2, g_3) - \frac{\eta_0 - c_2}{c_1} \quad (36)$$

We now observe that in equation (36), the right hand side is independent of constant T . We then have

$$(x + T, g_2, g_3) = \frac{1}{4} \left\{ \frac{c_1\wp'(x, g_2, g_3) - \eta_1}{\wp(x, g_2, g_3) - \eta_0 + c_2} \right\}^2 - \wp(x, g_2, g_3) - \frac{\eta_0 - c_2}{c_1} \quad (37)$$

Using Mathematica, substituting equation (37) into equation (32), we get the solution on the middle side. All graphs of these solutions are shown in Figure 3.

Figure 3 shows the solutions of the sfKdV in the middle, when $\lambda = 3$ and $a = 2$. The values of L_0 are -0.116614, 0.1553332, 0.7957503 and 1.8466578 for graph (a), (b), (c) and (d) respectively.

**Figure 3:** The Middle Solution

6 The Phase Shift L_0

In this section we shall determine the phase shift L_0 which depends on the forcing terms and the continuity conditions between each adjacent part.

Substituting $x = \frac{a}{2}$ and $x = -\frac{a}{2}$ into equation (32) we get

$$\eta\left(\frac{a}{2}\right) = c_1 \wp\left(\frac{a}{2} + T, g_2, g_3\right) \quad \text{and} \quad \eta\left(-\frac{a}{2} + T, g_2 + g_3\right) + c_2.$$

Substituting the above expression in equation (26*) we obtain

$$\wp\left(\frac{a}{2} + T, g_2, g_3\right) - \wp\left(-\frac{a}{2} + T, g_2 + g_3\right) = 0 \quad (38)$$

From equations (33), (34), (37) and (38) we can obtain

$$B(\lambda, L_0) = \frac{c_1}{4} \left\{ \frac{c_1 \wp'(a, g_2, g_3) - \eta_1}{c_1 \wp(a, g_2, g_3) - \eta_0 + c_2} \right\}^2 - c_1 \wp(a, g_2, g_3) + 2c_2 - 2\eta_0 = 0 \quad (39)$$

Equation (39) determines the phase shift L_0 .

Equation (39) have a solitary wave solution only when $B(\lambda, L_0) = 0$ (Gong, 1994). Hence, $B(\lambda, L_0) = 0$ is the condition used to calculate the phase shift L_0 . Using Mathematica, the function $B(\lambda, L_0)$ was plotted versus trial L_0 . The intersection point with L_0 -axis was found as the values of L_0 . (see Figure 4)

This curve has four intersection point with L_0 -axis, i.e. $B(\lambda, L_0)$ has four zero, $L_0 = 0.16614, -0.15332, -0.795703$ and -1.8466578 that means we have four supercritical solitary wave solutions.

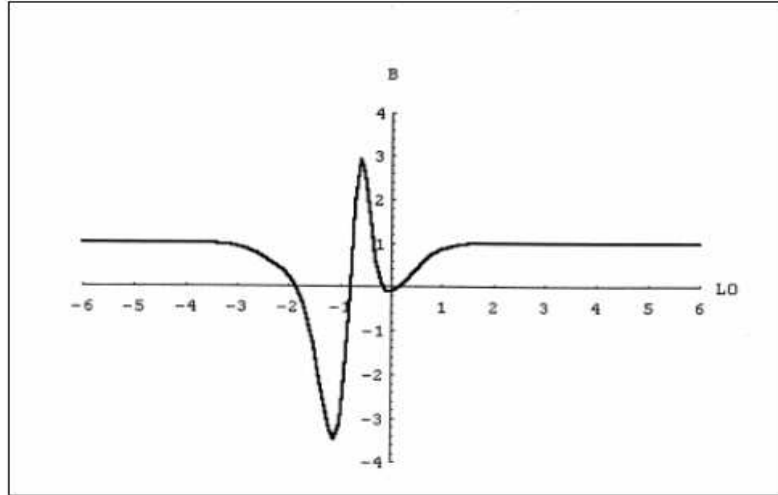


Figure 4: The Curve of $B(\lambda, L_0)$ versus L_0

7 Complete Solutions

We have obtained the solutions of the sfKdV equation at both sides and also in the middle as discussed in Section 3, 4, and 5, respectively. In this Section we shall give the complete solutions of the sfKdV equation. This means that we need to combine the solutions on both side with the middle ones.

We rewrite the solutions outlined in Section 3 as

$$\eta(x) = -\frac{3\lambda}{2\alpha} \sec h^2 \sqrt{\frac{-\lambda}{4\beta}} (x - L_0) \quad (40)$$

for the left side, and

$$\eta(x) = -\frac{3\lambda}{2\alpha} \sec h^2 \sqrt{\frac{-\lambda}{4\beta}} (x - L_1) \quad (41)$$

for the right side. The solution in the middle parts

$$\eta(x) = c_1 \wp(x + T, g_2, g_3) + c_2 \quad (42)$$

By matching equations (40), (41) and (42) we get

$$\eta(x) = \begin{cases} -\frac{3\lambda}{2\alpha} \sec h^2 \sqrt{\frac{-\lambda}{4\beta}} (x - L_0), & -\infty < x \leq -\frac{a}{2} \\ c_1 \wp(x + T, g_2, g_3) + c_2, & -\frac{a}{2} < x < \frac{a}{2} \\ -\frac{3\lambda}{2\alpha} \sec h^2 \sqrt{\frac{-\lambda}{4\beta}} (x - L_1), & \frac{a}{2} \leq x < \infty \end{cases} \quad (43)$$

Equation (43) is the complete solution of sfKdV equation.

The graphs shown in Figure 5 are the complete solutions of the sKdV equation when $\lambda = 3$ and $a = 2$. The values of L_0 are -0.116614 , 0.1553332 , 0.7957503 and 1.8466578 for graphs (a), (b), (c) and (d) respectively

8 Conclusion

In this paper we have studied the solutions of the stationary Korteweg-deVries (sfKdV) equation. In order to obtain the solution of the sfKdV equation, the domain of the problem has been divided into three parts, the left, the middle and the right parts. The solutions on the left and right parts were obtained in Section 3 and Section 4 by using analytical method. The solution on the middle was expressed in the terms of Weierstrass elliptic function. In both cases we have constructed two computer programs using Mathematica to produce the solutions. The complete solution of the sfKdV equation is given in Section 7, by matching the solutions of the three parts i.e. the left, the middle and the right parts. We have found that the solutions of the stationary Korteweg-deVries (sfKdV) equation produce four different solutions according to the value of the phase shift L_0 . All solutions are positive. Further research can be carried out for large negative forcing term.

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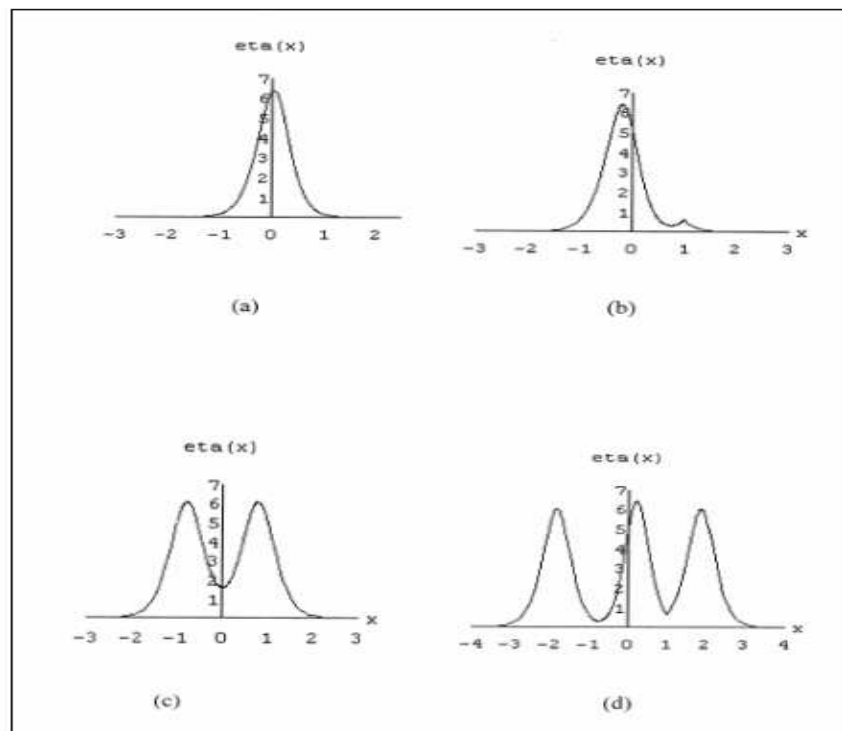


Figure 5: The Complete Solution

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