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Application of Ruscheweyh Derivative to Univalent Functions with Negative Coefficients

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Abstract The systematic investigation of a new class of analytic functions, which is defined in terms of Ruscheweyh derivative, is presented. Apart from coefficient bounds, results like distortion property, closure property and radius of starlikeness of this class of functions is obtained.

Keywords Ruscheweyh derivative, distortion theorem, coefficient bounds.

1 Introduction

Let S denote class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the open disc $U = \{z : |z| < 1\}$. The Hadamard product or convolution of two functions of S given by (1.1) is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \tag{1.2}$$

where

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \tag{1.3}$$

In [2], Ruscheweyh introduced the class K_n of functions $f(z) \in S$ satisfying the inequality

$$\mathrm{Re}\left\{\frac{z^n f(z)^{(n+1)}}{z^{n-1} f(z)^{(n)}}\right\} > \frac{n+1}{2}, \quad (n \in N_0 = N \cup \{0\}, z \in U) \tag{1.4}$$

with basic inclusion property $K_{n+1} \subset K_n$. We also note that K_0 is the class $S^*(1/2)$ of (normalized) starlike functions or order $1/2$.

Let

$$D^n f(z) = \frac{z(z^{n-1} f(z))}{n}, n \in N. \tag{1.5}$$

This symbol $D^n f(z)$ was named as Ruscheweyh derivative of $f(z)$ of order n by Al-Amiri [1]. We note that

$$D^0 f(z) = f(z), \quad D' f(z) = f'(z). \tag{1.6}$$

Using Hadamard product defined by (1.2), Ruscheweyh [2] observed that if

$$D^\alpha f(z) = \frac{z}{(1-z)^{\alpha+1}} * f(z), (\alpha \leq -1), \tag{1.7}$$

then (1.5) is equivalent to (1.7) when $\alpha = n \in N$.

Let T denote subclass of S consisting functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, (a_k \geq 0). \tag{1.8}$$

Definition

Let $0 \leq \alpha < 1, 0 < \beta \leq 1$ and $1/2 < \nu \leq 1$ if $\alpha = 0$, and $1/2 < \nu \leq 1/2\alpha$, if $\alpha \neq 0$. We define a class $Q_n(\alpha, \beta, \nu)$ as the class of all functions $f(z)$ of the form given in (1.8), if $f(z)$ satisfies the condition

$$\left| \frac{(D^n f(z))' - 1}{2\nu((D^n f(z))' - \alpha) - ((D^n f(z))' - 1)} \right| < \beta.$$

2 Main Results

Theorem 1 (Coefficient Estimates) *Let function $f(z)$ be defined by (1.8). Then $f(z)$ is in the class $Q_n(\alpha, \beta, \nu)$ if and only if*

$$\sum_{k=2}^{\infty} k(1 - \beta(1 - 2\nu))\delta(n, k) a_k \leq 2\beta\nu(1 - \alpha) \tag{2.1}$$

where

$$\delta(n, k) = \left(\begin{array}{l} n + k - 1 \\ n \end{array} \right)$$

The result (2.1) is sharp.

Proof Assume that the inequality (2.1) holds and let $|z| = 1$. Then, by hypothesis, we have

$$\begin{aligned}
&|(D^n f(z))' - 1| - \beta \left|2\nu((D^n f(z))' - \alpha) - ((D^n f(z))' - 1)\right| \\
&= \left| -\sum_{k=2}^{\infty} k\delta(n, k) a_k z^{k-1} \right| - \beta \left| 2\nu(1 - \sum_{k=2}^{\infty} k\delta(n, k) a_k z^{k-1} - \alpha) + \sum_{k=2}^{\infty} k\delta(n, k) a_k z^{k-1} \right| \\
&\leq \sum_{k=2}^{\infty} k\delta(n, k) a_k - \beta(2\nu(1 - \alpha) - 2v \sum_{k=2}^{\infty} k\delta(n, k) a_k) - \beta \sum_{k=2}^{\infty} k\delta(n, k) a_k \\
&= \sum_{k=2}^{\infty} k(1 - \beta(1 - 2v))\delta(n, k) a_k - 2\beta v(1 - \alpha) \leq 0.
\end{aligned}$$

Hence, by maximum modulus theorem $f(z) \in Q_n(\alpha, \beta, v)$.

Conversely, suppose that $f(z) \in Q_n(\alpha, \beta, v)$, then

$$\begin{aligned}
&\left| \frac{(D^n f(z))' - 1}{2v((D^n f(z))' - \alpha) - ((D^n f(z))' - 1)} \right| \\
&= \left| \frac{-\sum\limits_{k=2}^{\infty} k\delta(n,k) a_k z^{k-1}}{2v(1-\alpha) + \sum\limits_{k=2}^{\infty} k(1-2v)\delta(n,k) a_k z^{k-1}} \right| \qquad (2.2)\\
&< \beta, \ z \in U.
\end{aligned}$$

Since $|Re(z)| < |z|$ for all z, we have

$$\operatorname{Re}\left\{ \frac{-\sum\limits_{k=2}^{\infty} k\delta(n,k) a_k z^{k-1}}{2v(1-\alpha) + \sum\limits_{k=2}^{\infty} k(1-2v)\delta(n,k) a_k z^{k-1}} \right\} < \beta. \qquad (2.3)$$

Choose value of z on real axes so that $(D^n f(z))'$ is real. Upon clearing the denominator in (2.3) and letting $z \to 1$ through real values, we obtain

$$\sum_{k=2}^{\infty} k\delta(n,k) a_k \leq \beta(2\nu(1-\alpha) + \sum_{k=2}^{\infty} k(1-2\nu)\delta(n,k) a_k).$$

This gives required result. Finally, for the function,

$$f(z) = z - \frac{2\beta\nu(1-\alpha)}{k(1-\beta(1-2\nu))\delta(n,k)} z^k \qquad (2.4)$$

the result is sharp.

Corollary 1 *Let the function $f(z)$ defined by (1.8) be in the class $Q_n(\alpha, \beta, \nu)$, then*

$$a_k \leq \frac{2\beta\nu(1-\alpha)}{k(1-\beta(1-2\nu))\delta(n,k)}, (k \geq 2).$$

The result is sharp for the function given by (2.4).

Now we prove distortion property.

Theorem 2 *Let $f(z)$ defined by (1.8) in the class $Q_n(\alpha, \beta, \nu)$, then we have*

$$r - \frac{\beta\nu(1-\alpha)}{(1-\beta(1-2\nu))\delta(n,k)} r^2 \leq |f(z)| \leq r + \frac{\beta\nu(1-\alpha)}{(1-\beta(1-2\nu))\delta(n,k)}. \qquad (2.5)$$

The result (2.5) is sharp.

Proof Since $f(z) \in Q_n(\alpha, \beta, \nu)$ and in view of inequality (2.1) of Theorem 1, we obtain,

$$2(1-\beta(1-2v))\delta(n,2) \sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} k(1-\beta(1-2v))\delta(n,k) a_k \leq 2\beta\nu(1-\alpha) \qquad (2.6)$$

which implies

$$\sum_{k=2}^{\infty} a_k \leq \frac{\beta\nu(1-\alpha)}{(1-\beta(1-2\nu))\delta(n,2)}.$$

Therefore we can show that

$$|f(z)| \geq r - r^2 \sum_{k=2}^{\infty} a_k \geq r - r^2 \frac{\beta\nu(1-\alpha)}{(1-\beta(1-2\nu))\delta(n,2)}$$

and

$$|f(z)| \leq r + r^2 \sum_{k=2}^{\infty} a_k \leq r + r^2 \frac{\beta\nu(1-\alpha)}{(1-\beta(1-2\nu))\delta(n,2)}.$$

This completes the proof. Finally, by taking function,

$$f(z) = z - \frac{\beta\nu(1-\alpha)}{(1-\beta(1-2\nu))\delta(n,2)} z^2 \tag{2.7}$$

we can show the result is sharp.

Now we prove the closure properties of aforementioned class $Q_n(\alpha, \beta, \nu)$. Let the function, for $i = 1, 2, 3, \ldots, m$, be defined by

$$f_i(z) = z - \sum_{k=2}^{\infty} a_{k,i} z^k, \ \ a_k \geq 0, \ \ z \in U. \tag{2.8}$$

Theorem 3 *Let the function $f_i(z)$ given by (2.8) be in the class $Q_n(\alpha, \beta, \nu)$ for every $i = 1, 2 \ldots, m$. Then the function $h(z)$ defined by*

$$h(z) = \sum_{i=2}^{m} c_i f_i(z) \tag{2.9}$$

where

$$c_i \geq 0 \quad \text{and} \quad \sum_{i=2}^{m} c_i = 1 \tag{2.10}$$

is also in the class $Q_n(\alpha, \beta, \nu)$.

Proof Observing definition of $h(z)$ in (2.9), we get

$$h(z) = z - \sum_{k=2}^{\infty} (\sum_{i=1}^{m} c_i a_{k,i}) z^k. \tag{2.11}$$

Further, since $f_i(z)$ are in $Q_n(\alpha, \beta, \nu)$ for every $i = 1, 2, \ldots, m$, we get

$$\sum_{k=2}^{\infty} k(1-\beta(1-2\nu))\delta(n,k) a_{k,i} \leq 2\beta\nu(1-\alpha). \tag{2.12}$$

Hence, in view of (2.12), we see that

$$\begin{aligned}
&\sum_{k=2}^{\infty} k(1-\beta(1-2\nu))\delta(n,k)(\sum_{i=1}^{m} c_i a_{k,i}) \\
&= \sum_{i=1}^{m} c_i(\sum_{k=2}^{\infty} k(1-\beta(1-2\nu))\delta(n,k)a_{k,i}) \\
&\leq \sum_{i=2}^{m} c_i(2\beta\nu(1-\alpha)) \leq 2\beta\nu(1-\alpha).
\end{aligned} \tag{2.13}$$

This immediately implies that $h(z)$ belongs to $Q_n(\alpha,\beta,\nu)$. This completes proof of Theorem 3.

Theorem 4 *Let*

$$f_1(z) = z \text{ and } f_k(z) = z - \frac{2\beta\nu(1-\alpha)}{k(1-\beta(1-2\nu))\delta(n,k)} z^k, (k \geq 2) \tag{2.14}$$

for $0 \leq \alpha < 1, 0 < \beta \leq 1, 1/2 < v \leq 1$ *if* $\alpha = 0$ *or* $1/2 < v \leq 1/2\alpha,$ *if* $\alpha \neq 0$.

Then $f(z)$ *is in the class* $Q_n(\alpha,\beta,\nu)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z), \tag{2.15}$$

where $\lambda_k \geq 0, k \geq 1$ *and* $\sum\limits_{k=0}^{\infty} \lambda_k = 1$.

Proof Suppose that

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) = z - \sum_{k=1}^{\infty} \frac{2\beta\nu(1-\alpha)}{k(1-\beta(1-2v))\delta(n,k)} \lambda_k z^k \tag{2.16}$$

then it follows that

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{2\beta\nu(1-\alpha)}{k(1-\beta(1-2v))\delta(n,k)} \lambda_k . k(1-\beta(1-2v))\delta(n,k) \\
&= 2\beta\nu(1-\alpha)\sum_{k=2}^{\infty} \lambda_k = 2\beta\nu(1-\alpha)(1-\lambda_1) \\
&\leq 2\beta\nu(1-\alpha).
\end{aligned} \tag{2.17}$$

Hence, in view of inequality (2.1) of Theorem 1, $f(z) \in Q_n(\alpha,\beta,\nu)$.

Conversely, assume that $f(z)$ defined by (1.8) belongs to class $Q_n(\alpha,\beta,\nu)$ then

$$a_k \leq \frac{2\beta\nu(1-\alpha)}{k(1-\beta(1-2v))\delta(n,k)}, k \geq 2.$$

Setting

$$\lambda_k = \frac{k(1-\beta(1-2v))\delta(n,k)}{2\beta\nu(1-\alpha)} a_k$$

and $\lambda_1 = 1 - \sum\limits_{k=2}^{\infty} \lambda_k$, we can see that $f(z)$ can be expressed in the form (2.15). This completes the proof.

Corollary 2 *The extreme points of class $Q_n(\alpha, \beta, \nu)$ are the functions $f_k(z)(k \geq 1)$ given by (2.14) .*

Lastly we obtain radius of convexity.

Theorem 5 *Let $f(z)$ defined by (1.8) be in $Q_n(\alpha, \beta, \nu)$ then $f(z)$ is starlike of order $\rho(0 \leq \rho < 1)$ in disc*

$$|z| < r(\alpha, \beta, \nu, \rho) = \inf_k \left\{ \frac{(1-\rho)k(1-\beta(1-2v))\delta(n,k)}{(k-\rho)(2\beta\nu(1-\alpha))} \right\}^{1/k-1}. \tag{2.18}$$

The result is sharp.

Proof It is suffices to prove

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \rho, \tag{2.19}$$

that is to prove,

$$\frac{\sum\limits_{k=2}^{\infty} (k-1)a_k z^k}{1 - \sum\limits_{k=2}^{\infty} a_k z^k} \leq 1 - \rho$$

which is equivalent to

$$\sum_{k=2}^{\infty} (k-\rho)a_k z^{k-1} \leq (1-\rho)$$

which gives

$$\sum_{k=2}^{\infty} \frac{(k-\rho)}{1-\rho} a_k z^{k-1} \leq 1.$$

In view of Theorem 1, it is possible only when

$$\sum_{k=2}^{\infty} \frac{(k-\rho)}{1-\rho} a_k z^{k-1} \leq \sum_{k=2}^{\infty} \frac{k(1-\beta(1-2v))\delta(n,k)}{2\beta\nu(1-\alpha)} a_k. \tag{2.20}$$

After simplification we get required result. This completes proof of Theorem 5.

3 Conclusion

The work presented here is generalization of work done by earlier researchers. Further the research can be done by using fractional calculus operators for this class.

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