MATEMATIKA, 2008, Volume 24, Number 2, 211–230

Modelling of Tsunami Waves

[1]**Nazeeruddin Yaacob, Norhafizah Md Sarif & [2]Zainal Abdul Aziz**
Department of Mathematics, Faculty of Science, Universiti Teknologi Malaysia
81310 Skudai, Johor, Malaysia
e-mail: [1]ny@mel.fs.utm.my, [2]zaa@mel.fs.utm.my

Abstract The nonlinear dispersive model based on the forced Korteweg-de Vries equation (fKdV) is developed from the approximation of Boussinesq shallow water type model. This provides the possibility of observing, in particular, the process of tsunami generation by atmospheric disturbances. The fKdV is then solved numerically via an explicit finite difference method. From the simulations, the roles of nonlinearity, dispersion and forcing terms in the process of tsunami generation are shown explicitly.

Keywords Tsunami waves; forced Korteweg-de Vries equation; finite difference method.

1 Introduction

The shallow water model basically describes the governing equations for the basic hydrodynamic model of tsunami generation by atmospheric disturbances (Akylas [1], Nosov & Skachko [8]). This evolution equation is derived from the simplest theory of water waves that reasonably approximates the behaviour of the real ocean and it comes from the system of coupled partial differential equations (Pelinovsky et al. [9], Layton & van de Panne [7], Guyenne & Grilli [4]).

An alternative approach to derive the appropriate evolution equation, which asymptotically approximates the Boussinesq equation, leads to the forced Korteweg-de Vries equation (fKdV). The forcing term in the fKdV is assumed to be derivable from atmospheric disturbances. Various forms of this equation have been extensively studied (see Grimshaw et al. [3], Pelinovsky et al [9]) and numerical results show that the solution contains the set of solitary waves.

The behaviour of tsunamis on the open ocean is considerably more complicated than the solitary wave model. Although some quarters doubt the relevance of soliton theory to the modeling of tsunamis (e.g. Constantin & Johnson [2]), nonetheless, the approach involving the solitary wave model gives an idea of the impressive scales involved in waves in deep water (Sattinger & Li [10]). Therefore, a numerical study of this equation is done in order to examine the phenomena of solitary wave in shallow water. In many cases, the tsunami source moves with variable speed and direction. The fKdV equation has been shown to describe the resonant mechanism of tsunami wave generation by atmospheric disturbances moving with near-critical speed (Shen, [13]). In a series of papers, Shen [11] & [12] summarized that there exist two supercritical solitary waves, and one subcritical downstream cnoidal wave.

This paper aims to demonstrate the roles of nonlinearity, dispersion, and forcing terms in the process of tsunami generation and also to develop a mathematical model in order to study the characteristics of the tsunami mechanism. Predominantly this study aims to portray the fKdV as a basic mathematical model of tsunami generation. The paper is organized as follows. Section 1 is the introduction, section 2 describes the form of the forced nonlinear dispersive equation, section 3 discusses the results and finally section 4 is the concluding remark.

2 Nonlinear Dispersive Equation

Taking into account the resonant character of tsunami generation, the dynamical system can be simplified. Here it is shown that by making further simplifying assumptions it is possible to derive simpler nonlinear, dispersive wave models which can be used for our possible initial numerical investigation. If it is assumed that waves travel in only the positive x-direction then it can be shown that to a first approximation the waves have a steady form, i.e. the nondimensional transformation,

$$\tilde{X} = \tilde{x} - \tilde{t} \tag{1}$$

can be introduced. Assuming that there is a slight variation with time or distance travelled, the appropriate scaling is,

$$\hat{X} = \sigma \tilde{X}, \tag{2}$$

$$\hat{T} = \sigma^3 \tilde{T}. \tag{3}$$

The free surface and horizontal velocity are scaled as before, according to the definitions $\hat{x} = \sigma\, \tilde{x}, \quad \hat{t} = \sigma^3\, \tilde{t}, \quad \hat{\eta} = \frac{\tilde{\eta}}{\varepsilon}$ (Ursell numbers) and $\hat{\bar{u}} = \frac{\tilde{u}}{\varepsilon}$, respectively.

Substituting into a Boussinesq equation system (refer to Pelinovsky et al. [9]), leads to

$$\sigma^2 \frac{\partial \hat{\bar{u}}}{\partial \hat{T}} - \frac{\partial \hat{\bar{u}}}{\partial \hat{X}} + \varepsilon\, \hat{\bar{u}} \frac{\partial \hat{\bar{u}}}{\partial \hat{X}} + \frac{\partial \hat{\eta}}{\partial \hat{X}} + \frac{\sigma^2}{3} \frac{\partial^3 \hat{\bar{u}}}{\partial \hat{X}^3} - \frac{\sigma^4}{3} \frac{\partial^3 \hat{\bar{u}}}{\partial \hat{X}^2 \partial \hat{T}} + \frac{1}{\varepsilon\rho} \frac{\partial p_{atm}}{\partial \hat{X}} = 0, \tag{4}$$

$$\sigma^2 \frac{\partial \hat{\eta}}{\partial \hat{T}} - \frac{\partial \hat{\eta}}{\partial \hat{X}} + \frac{\partial}{\partial \hat{X}} \left((1 + \varepsilon \hat{\eta})\ \hat{\bar{u}} \right) = 0, \tag{5}$$

where $p_{atm}$is the pressure due to atmospheric disturbances.

To the lowest order of approximation both equations reduce to,

$$\frac{\partial \hat{\eta}}{\partial \hat{X}} = \frac{\partial \hat{\bar{u}}}{\partial \hat{X}} + O\left(\varepsilon, \sigma^2\right) \tag{6}$$

under the assumption that the initial conditions are consistent with this statement. Hence by integrating, assuming that the order stays the same,

$$\hat{\eta} = \hat{\bar{u}} + O\left(\varepsilon, \sigma^2\right). \tag{7}$$

Adding equations (4) and (5) and using the relation (7), we have

$$2\sigma^2 \frac{\partial \hat{\eta}}{\partial \hat{T}} + 3\varepsilon \hat{\eta} \frac{\partial \hat{\eta}}{\partial \hat{X}} + \frac{\sigma^2}{3} \frac{\partial^3 \hat{\eta}}{\partial \hat{X}^3} + \frac{1}{\varepsilon\rho} \frac{\partial p_{atm}}{\partial \hat{X}} = O\left(\sigma^4\right), \tag{8}$$

where it is assumed from the definition of the Ursell number that $\varepsilon = O\left(\sigma^2\right)$. In dimensionless variables this reads,

$$\frac{\partial \eta}{\partial t} + c\frac{\partial \eta}{\partial x} + \alpha\overline{\eta}\frac{\partial \eta}{\partial x} + \beta\frac{\partial^3 \eta}{\partial x^3} = \frac{\partial f}{\partial x}, \tag{9}$$

where

$$\alpha = 3c/2h, \qquad \beta = ch^2/6, \qquad f = -hp_{atm}/2c\rho. \tag{10}$$

Equation (9) is the forced version of the Korteweg-de Vries equation and currently it is considered as one of the basic nonlinear equations of mathematical physics. As mentioned before, $\eta\left(x,t\right)$ describes the free surface profile, $f'\left(x\right)$ is the forcing function, x is the spatial coordinate along the channel, and t is time. Its solution contains the set of solitary waves and dispersive tails (variable wave packets, refer Shen [13]).

3 Results and Discussions

In this section, a numerical scheme is developed for the simulation of the fKdV equation. By using an explicit finite difference method, this equation is solved by using a numerical scheme in Matlab 6.1 (for example Hanselman & Littlefield [5]). The results of the computations are presented for various initial conditions and time.

3.1 Finite Difference Method

The finite difference method is developed by first partitioning the spatial domain into a set of non-coincident points or nodes here, and for simplicity of presentation, the discussion is limited to a constant spacing between adjacent nodes,

$$x_i = x_1 + \left(i-1\right)\Delta x,\ i = 1, 2, \ldots, N, \tag{11}$$

$$\Delta x = \frac{x_{\text{upper}} - x_{\text{lower}}}{N}, \tag{12}$$

and Δx is termed the grid size or grid spacing.

The fKdV is solved numerically by this explicit finite difference method, where the basic form being suggested by Vliegenthart [16]. The following notations are used.

$$\frac{\partial \eta}{\partial t} = \frac{1}{\Delta t}\left(\eta_{i,j+1} - \eta_{i,j}\right) - \frac{1}{6}\Delta x^2\frac{d^3\eta}{dt^3} + o\left(\Delta x^4\right), \tag{13}$$

$$\frac{\partial \eta}{\partial x} = \frac{1}{2\Delta x}\left(\eta_{i+1,j} - \eta_{i-1,j}\right) - \frac{1}{6}\Delta x^2\frac{d^3\eta}{dx^3} + o\left(\Delta x^4\right), \tag{14}$$

$$\begin{aligned}\tfrac{\partial^3 \eta}{\partial x^3} = \tfrac{1}{2(\Delta x)^3}&\left(\eta_{i+2,j} - 2\eta_{i+1,j} + 2\eta_{i-1,j} - \eta_{i-2,j}\right)\\ &+\tfrac{7}{60}\Delta x^2\frac{d^5\eta}{dx^5} + o\left(\Delta x^4\right),\end{aligned} \tag{15}$$

$$\bar{\eta} = \frac{\eta_{i+1,j} + \eta_{i,j} + \eta_{i-1,j}}{3}. \tag{16}$$

Discretisation of the differential equation turns it to a difference equation. Substituting (13)-(16) into fKdV equation (9) and only taking the first order term, we obtain

$$\begin{aligned}&\frac{\eta_{i,j+1}-\eta_{i,j}}{\Delta t}+c\frac{\eta_{i+1,j}-\eta_{i-1,j}}{2\Delta x}\\&+\frac{3c}{2h}\frac{\eta_{i+1,j}-\eta_{i-1,j}}{2\Delta x}\left(\frac{\eta_{i+1,j}+\eta_{i,j}+\eta_{i-1,j}}{3}\right)\\&+\frac{ch^2}{6}\left(\frac{\eta_{i+2,j}-2\eta_{i+1,j}+2\eta_{i-1,j}-\eta_{i-2,j}}{2\left(\Delta x\right)^3}\right)=f'.\end{aligned}\tag{17}$$

Rearranging equation (17), we have

$$\begin{aligned}\eta_{i,j+1}=\eta_{i,j}&-\frac{c}{2}\frac{\Delta t}{\Delta x}\left(\eta_{i+1,j}-\eta_{i-1,j}\right)-\frac{c}{4h}\frac{\Delta t}{\Delta x}\left(\eta_{i+1,j}-\eta_{i-1,j}\right)\left(\eta_{i+1,j}+\eta_{i,j}+\eta_{i-1,j}\right)\\&-\frac{ch^2}{12}\frac{\Delta t}{\left(\Delta x\right)^3}\left(\eta_{i+2,j}-2\eta_{i+1,j}+2\eta_{i-1,j}-\eta_{i-2,j}\right)+f'.\end{aligned}\tag{18}$$

This scheme is consistent with equation (9) and the truncation error is

$$o\left[\left(\Delta t\right)^3\right]+o\left[\Delta t\left(\Delta x\right)^2\right].$$

We then simulate the difference equation (18) with an initial condition of form

$$\eta_{i,0}=12\sec\mathrm{h}^2\left(i\Delta x-x_0\right).\tag{19}$$

Another initial condition, for the development of undular bore in shallow water, can also be set for equation (18) as

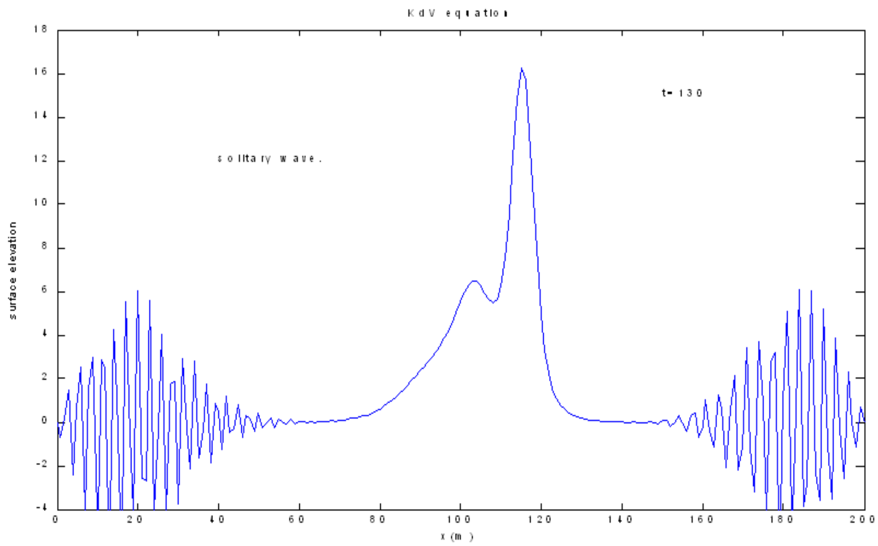
$$\eta_{i,0}=1/2\left[1-\tanh(x_i-25)/5\right].\tag{20}$$

3.2 Stability of the Numerical Scheme

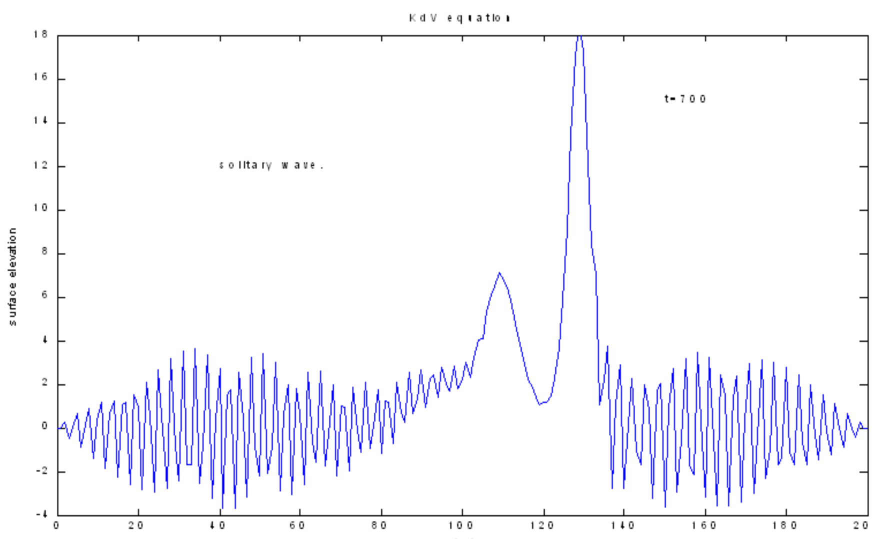
Explicit finite difference method is simpler to formulate, however it is less stable compared to the other schemes of finite difference method such as the implicit method. A numerical scheme is said to be stable if the graph of the solution does not change dramatically in a short time span, otherwise it is said to be unstable (Tiong & Zainal, [15]). As shown by Vliegenthart [16], the numerical scheme is only conditionally stable and expected to be stable for $\Delta t \leq \Delta x^3$.

The accuracy of the numerical calculations was checked in various ways; first, the known exact linear solution was compared with the corresponding numerical solution. Secondly, the numerical scheme was successfully tested against known soliton solutions of the KdV equation. Finally, the use of refined integration steps $\Delta t,\ \Delta x$ and a larger computational domain for x did not alter the results within the accuracy reported here, especially in the domain x.

In order to investigate influences of mesh size on the numerical simulation for the KdV equation, two different meshes ($\Delta t = 0.0005,\ \ \Delta t = 0.001$) are considered. However, using these values to our numerical scheme, we found that the numerical scheme is unstable (see Figure 1 (a) and Figure 1(b)). From these figures, these show the influences of different mesh sizes on the magnitude and distribution of simulated wave crests and trough. It is observed that the smaller mesh size gives better results.



(a)



(b)

Figure 1: Numerical Demonstration of the Instability of a Solitary Wave of the KdV Equation (a) $\Delta t = 0.0005$, (b) $\Delta t = 0.001$

3.3 Numerical Results

The numerical procedure below is carried out with $\Delta x = 0.1$. For the present problem, $\Delta x = 0.1,\ \Delta t = 0.25 \times 10^{-3}$ are sufficient for stability, in accordance with Vliegenhart [16].

3.3.1 Simulation on the Forced KdV

We now proceed to simulate the solitary wave evolution for KdV equation with the forcing in motion. To demonstrate the roles of solitons in the forced dynamics, the assumption in the scheme has been made that the forcing term being introduced is sufficiently weak (Pelinovsky et al. [9]), so that the soliton retains its form in the process of interaction.

We consider for the cases $h = 1.0$ m and $h = 0.5$ m. Until up to $t = 300$, it is found that the solitary wave does not change its shape. However, at (c) $t = 1000$, the soliton interaction begins and from Figure 2(c) the non-linear steepening dominates. Now, it is the dispersion term that is small and we can see the effect of the nonlinear 'convection' term. Where η is large the 'convection' is rapid, but out in front where η is small the 'convection' is slower. This allows the fast peak to catch up with the slow front end, causing wave steepening. (An effect like this causes ocean waves to steepen and then break at the beach). In the case of three solitons, the wave packet has very small amplitude. It is readily seen that the peaks of the soliton lie approximately on a straight line. This is due to the fact that the velocity of the soliton is proportional to its amplitude, and so that the distances covered by them will be proportional to their amplitude.

The phenomena of a solitary wave moving into a region of decreasing depth and producing solitons are shown in Figures 2(a)-2(c), 3(d)-3(f), 4(g) and 5(a)-5(f). From the above result, it shows many similarities with the standard theory of the KdV equations. Standard KdV theory then predicts that exactly n solitons will eventually appear on the shelf if

$$h = \left[\frac{1}{2} n\left(n+1\right)\right]^{-4/9} \tag{21}$$

where the soliton amplitude is $2Am^2/n\left(n+1\right),\ m = 1, 2, \ldots, n$, where A is the amplitude of the initial solitary wave. This prediction is made by Johnson [6].

In our simulation of the fKdV equation, we have found that for Case 1 at $h = 1.0\,m$, 3 solitons were produced with an oscillatory tail (Figure 3(f)). In this case, according to equation (21), this corresponds to $1 < n < 2$, and 2 solitons would be produced. For Case 2 at $h = 0.5$ m, the case where the depth is a bit shallower than Case 1, 6 solitons were produced with an oscillatory tail behind them (Figure 5(e)). According to equation (21), this corresponds to $2 < n < 3$, i.e. 3 solitons would be produced. It is seen that in the simulations, equation (21), which is derived for the variable coefficient KdV equation is not compatible for the above cases (also see Tiong & Zainal, [15]). This certainly requires further examination.

Now, we focus on the effect of forcing in soliton. Due to the effect of forcing, the solitary wave amplitude and position will change. Numerical simulations of this equation show that the imposition of forcing generates soliton, as well as nonlinear wave packet. In our simulation of the forced Korteweg-de Vries equation, it seems that the soliton amplitude grows at the moment of interaction and recovers after interaction. Figures 2, Figures 3 and Figures 5 (a)-(f) show the evolution of the wave disturbance at different values of h. Solitons are successively generated at $N < 150$ and propagate in front of the pressure distribution: after each soliton reaches certain equilibrium amplitude, a new soliton of slightly smaller equilibrium amplitude is released. Immediately, behind the pressure excitation, a trough appears (to balance more or less the amount of water used to form a soliton), and at larger distances from pressure, the wave disturbance is highly oscillatory with a larger amplitude.

As shown by the numerical solution of the fKdV equation, the simplified model given here yields a physical representation of the interaction between the soliton and forcing in motion. In reality, the number of waves interacting with the atmospheric disturbances can be high and they may interact between them, sometimes forming large amplitude waves.

3.3.2 Simulation of Undular Bore

If we change slightly the initial condition in the fKdV equation i.e. using (20) instead, then another simulation will be formed, which is known as the undular bore. The term bore is used for a moving discontinuity. The bore or hydraulic jump has exactly the same dynamic function as shock wave that is it converts the kinetic energy in the incident flow into some other forms.

In shock wave, the energy goes into random motion of the molecules, and generates heat. In the bore, the energy then becomes transformed into a random motion of the water on a small scale, and produces turbulence. Nonetheless, in some circumstances part or all of the energy goes into generating surface waves, giving an undular bore situation. Figures 6 (a-f) show these possibilities.

For some initial conditions it is true that an initial wave of elevation will steepen and eventually form a hydraulic jump or bore, as mentioned before, and this will happen when the nonlinearity is quite strong, as is shown in Figure 6 (c). But if the initial wave of elevation is not very marked, then other terms in the full equation have an effect and a moving equilibrium state may be reached which propagates unperturbed shape. Another reason why a bore may not form in certain circumstances is that friction (due for example, to turbulence or obstructions) may be relatively important, and this can remove energy at such a rate that no discontinuity needs to be formed.

Tsunami can thus be considered as a big solitary wave, i.e. soliton. Hence, tsunami may possesses similar properties of soliton such as representing a wave of permanent form, localized, and interact strongly with other solitons. In our simulations, we observe that for large t, the graph consists of a finite number of solitons depending on the depth and initial condition. When a soliton propagates into a different depth, as a tsunami leaves the deep water and moves to shallower water, it will deform. This situation is certainly due to the nonlinear and dispersive effects, which depend on the water depth, and will no longer balance each other anymore as in the case of constant depth. The larger the nonlinearity, as observable from the simulations, the bigger is the amplitude produced. This case also explains the freak wave formation on shallow water waves. Freak wave is always a wave with small ratio of nonlinearity to dispersion, so it is almost a linear dispersive wave (e.g. Tan [14])

As obtained in section 3.3.1, the influence of nonlinear effects on the large amplitude appearance is demonstrated and its probability of occurrences increases as nonlinearity becomes significant. Consequently, the results obtained thus far would allow the study of unidirectional waves in shallow water within the precepts of KdV model, and which can be related to tsunami waves. Such numerical experiments demonstrated dependence of wave statistics and ratio of nonlinear effect to dispersion, and what leads to the growing of the asymmetry in wave field with time increasing of the contribution of the large waves in total distribution.

Since the velocities of the solitons are proportional to their amplitude, solitons are arranged in order of increasing amplitude, i.e. the largest are in front. The rapidly wave packet (or tail) is seen to remain behind in accordance to the dispersion equation. It can be shown that from our simulation, if we begin from an initial profile, such as of Figure 2(a) and Figure 5(a), but with the taller wave somewhat to the left of the shorter, then the development observed is as depicted in Figure 5(c). In this case, the taller wave catches up and becomes intact and undistorted.

4 Conclusions

We have discussed the basic modelling of tsunami waves. The characteristics of tsunami have been compared and the problem of tsunami is formulated into a mathematical model, with a few physical considerations taken into account. Shallow water theory is seen as a viable theoretical model to describe the properties of tsunami waves. Nonetheless from the results above, we can conclude that

(a) Tsunami wave can be shown to act as a soliton and solitary wave and to possess similar properties of solitons.

(b) As the depth of sea decreases, the number of solitons increases as the nonlinear effects become larger than the dispersive effects.

(c) The larger the nonlinearity, the bigger amplitudes of the waves are produced.

(d) Due to the effect of forcing, the solitary wave and its position will change. A trough will appear (to balance more or less the amount of water used to form a soliton) when the soliton interact with the weak forcing.

(e) The velocities of the solitons are proportional to their amplitudes, and solitons are arranged in order of increasing amplitude, i.e. the largest are in front.

Acknowledgment

This research was partially supported by MOHE Fundamental Research Grant, Vot. No.78082.

References

[1] T.R. Akylas, *On The Excitation of Long Nonlinear Water Waves By A Moving Pressure Distribution,* Journal Fluid Mechanics, 141(1984), 455-466

[2] A. Constantin & R.S. Johnson, *Modelling Tsunamis,* J. Phys. A: Math. Gen, 39(2006), L215-L217.

[3] R. Grimshaw, E. Pelinovsky & X. Tian, *Interaction of a Solitary Wave with an External Force,* Physica *D*, 77(1994) 405-433.

[4] P. Guyenne & S.T. Grilli, *Computation of Three-Dimensional Overturning Waves in Shallow Water, Dynamics and Kinematics,* in Proc. 13^{th} Int. Offshore and Polar Eng. Conf. Honolulu, Hawaii, 2003.

[5] D. Hanselman & B. Littlefield, *Mastering Matlab®* 6, A Comprehensive Tutorial and Reference, Prentice Hall, Upper Saddle River, N.J., 2001

[6] R.S. Johnson, *The Korteweg-de Vries Equation with Variable Coefficients,* Lecture in Applied Mathematics, American Mathematical Society, 15, 1974.

[7] A.T. Layton & van de Panne, *A Numerically Efficient and Stable Algorithm for Animating Water Waves,* Visual Computer, 18(2002), 41-53

[8] M.A. Nosov & S.N. Skachko, *Nonlinear Tsunami Generation Mechanism,* Natural Hazard and Earth Sciences, 1(2001), 251-253.

[9] E. Pelinovsky, T. Talipova, A. Kurkin & C. Kharif, *Nonlinear Mechanism of Tsunami Wave Generation by Atmospheric Disturbances,* Natural Hazard and Earth Sciences, 1(2001), 243-250.

[10] D.H. Sattinger & Li, Yi, *Matlab Codes for Nonlinear Dispersive Wave Equations,* Technical Report, at http:// www.math.usu.edu/∼dhs/, 1998.

[11] S.S. Shen, *Locally Forced Critical Surface Waves in Channels of Arbitrary Cross Section,* Journal of Fluid Mechanics, 42(1991), 147-188.

[12] S.S. Shen, *Forced Solitary Waves and Hydraulic Falls in Two Layer Flows,* Journal of Fluid Mechanics, 234(1992), 583-612.

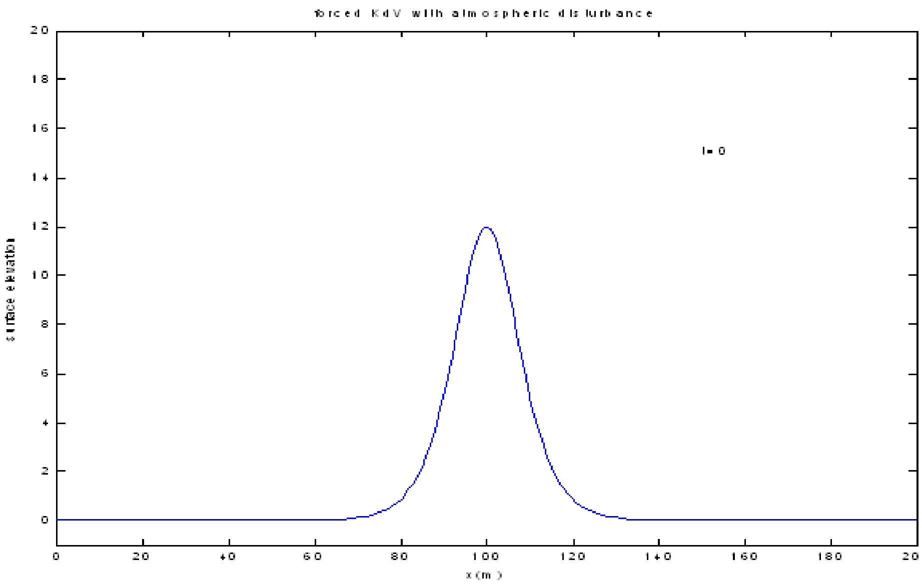
[13] S.S Shen, *A Course on Nonlinear Waves*, Kluwer Academic Publisher, (1993) 53-70.

[14] W.N. Tan, *Theory of Bichromatic Wave Groups Amplitude Amplification using Implicit Variational Method*, PhD Thesis, Universiti Teknologi Malaysia, Skudai, 2006.

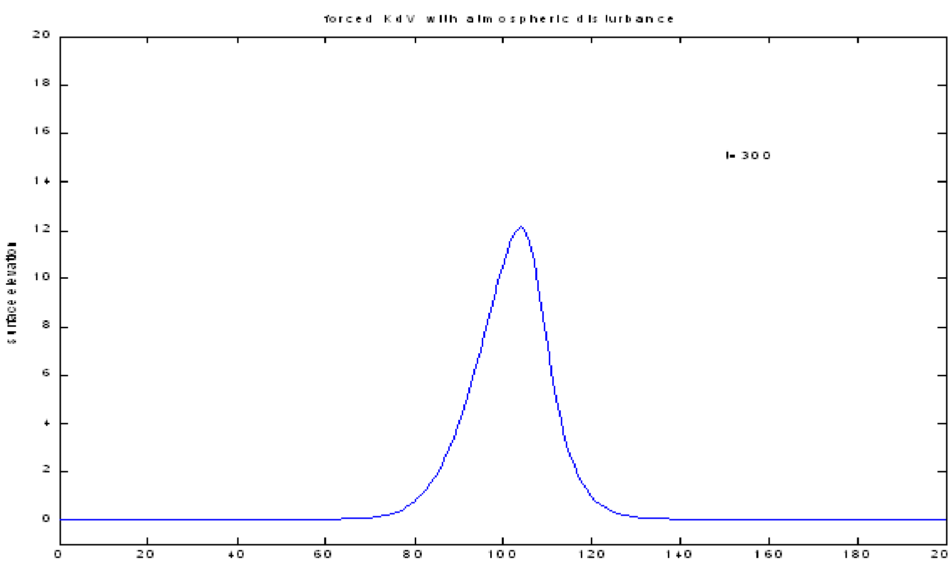
[15] K.M. Tiong & A.A. Zainal, *On The Splitting of A Solitary Wave Propagating Over a Slowly Varying Topography,* in *Prosiding Simposium Kebangsaan Sains Matematik XI,* Amran Ahmed et al (eds.), (2003) 149-157, UMS, Kota Kinabalu.

[16] A.C. Vliegenthart, *On Finite Difference Methods for the Korteweg de Vries Equation,* Journal of Engineering Mathematics, 5(1970), 137-155.

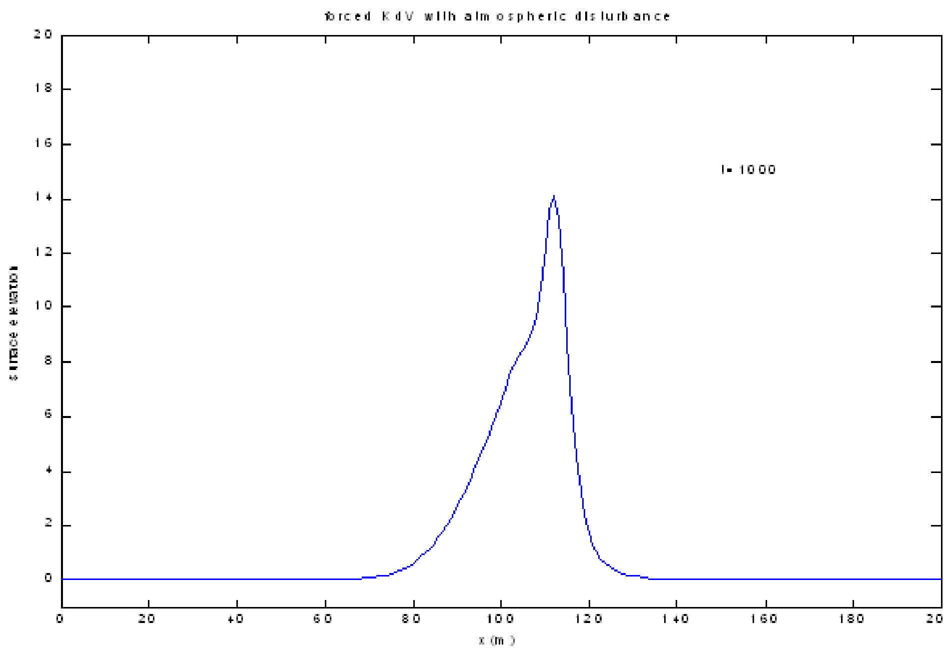
Case 1: $h = 1.0$ m



(a)

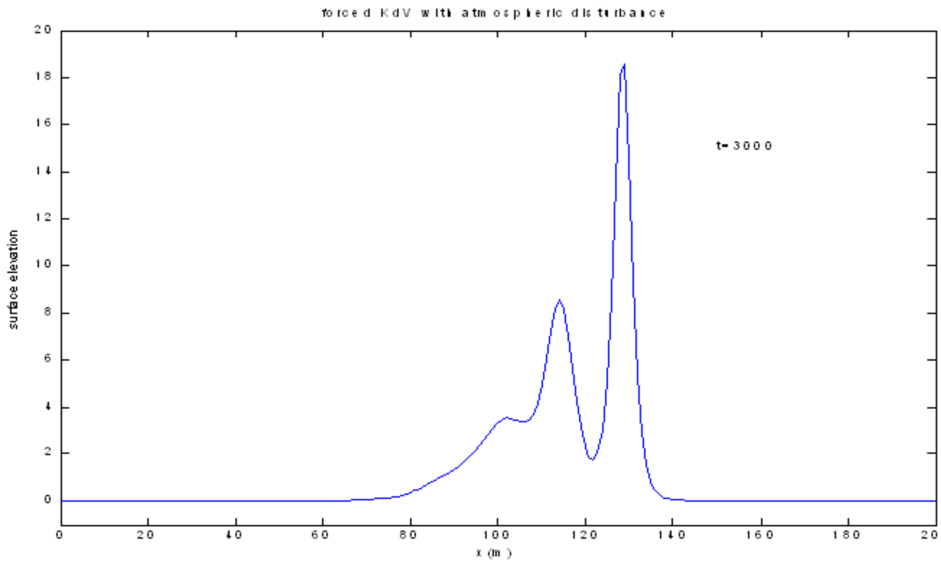


(b)

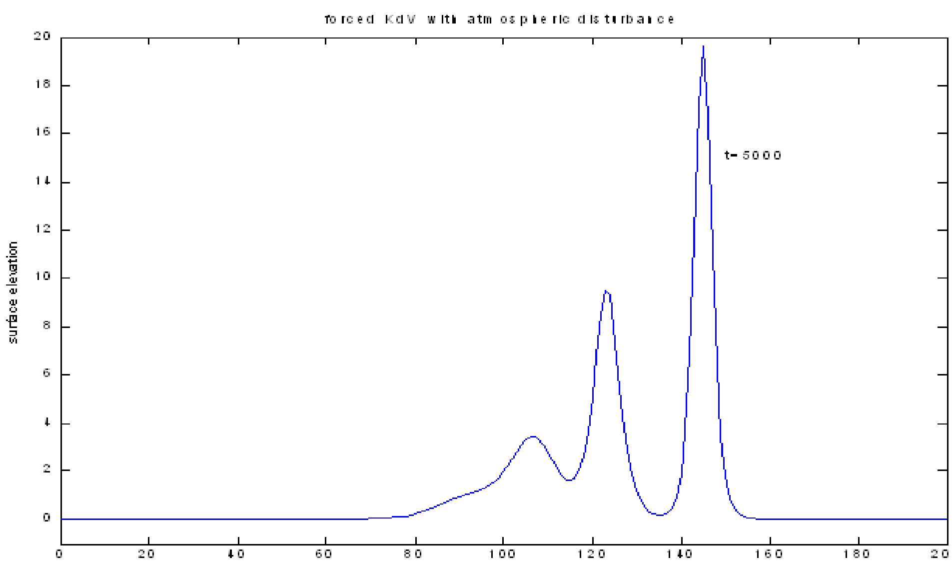


(c)

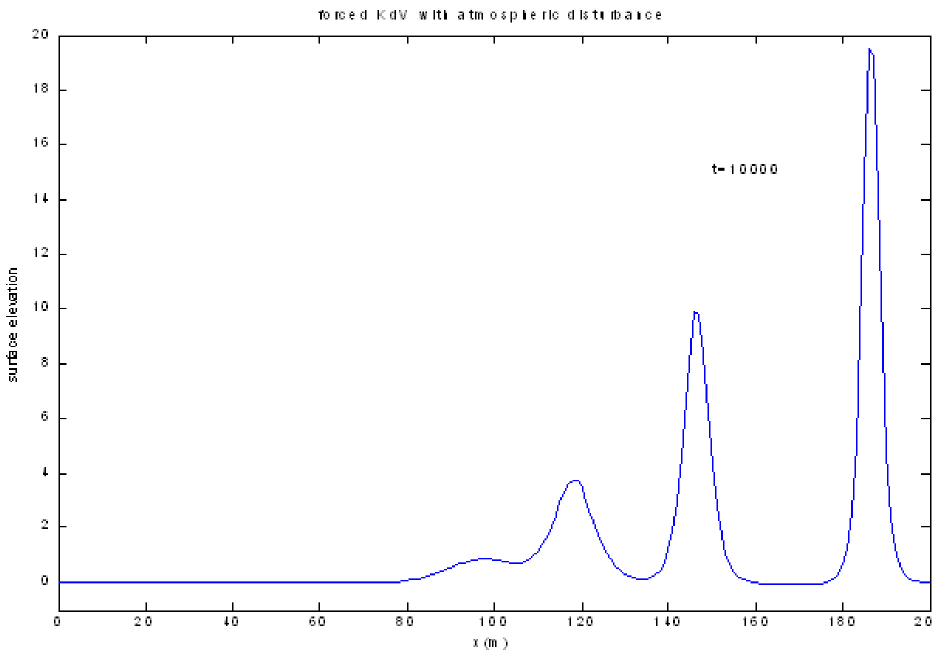
Figure 2: (a-c) Show an Evolution of Solitary Wave at Shallow Water


forced KdV with atmospheric disturbance
t=3000
surface elevation
x (m)
0
2
4
6
8
10
12
14
16
18
20
0
20
40
60
80
100
120
140
160
180
200

(d)

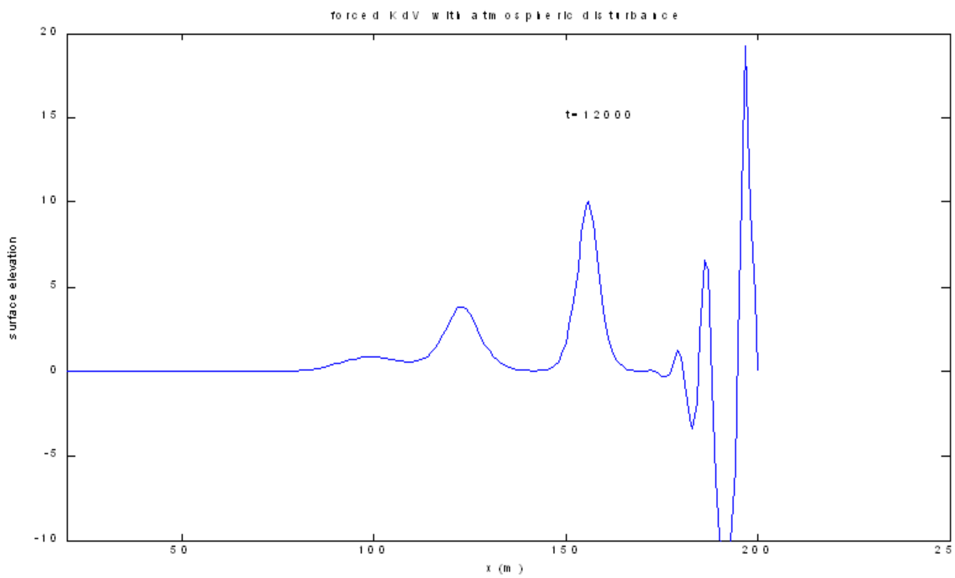

forced KdV with atmospheric disturbance
t=5000
surface elevation
x (m)
0
2
4
6
8
10
12
14
16
18
20
0
20
40
60
80
100
120
140
160
180
200

(e)



(f)

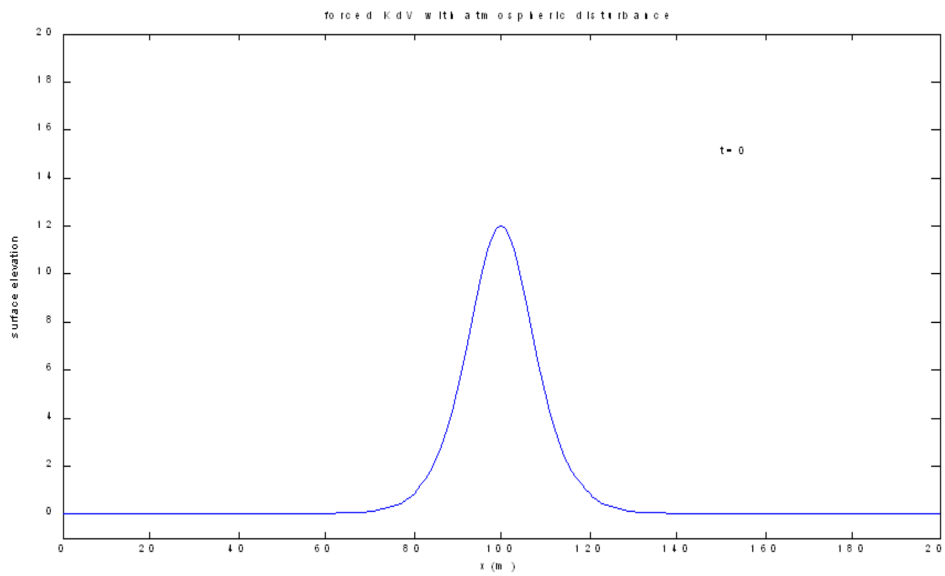
Figure 3: (d)-(f) Show Collision of Three KdV Soliton; the Evolution of the Initial Profile $u\,(x,0) = 12\,\mathrm{sech}^2 x,$ at (d) $t = 3000$, (e) $t = 5000$, and (f) $t = 10000$ (continued)



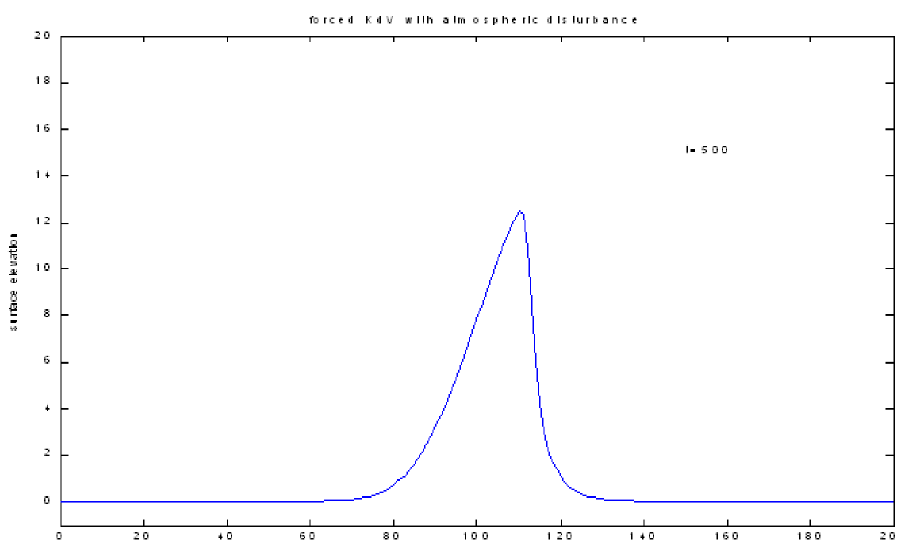
(g)

Figure 4: The Evolution of Forced KdV at (g) $t = 15000$

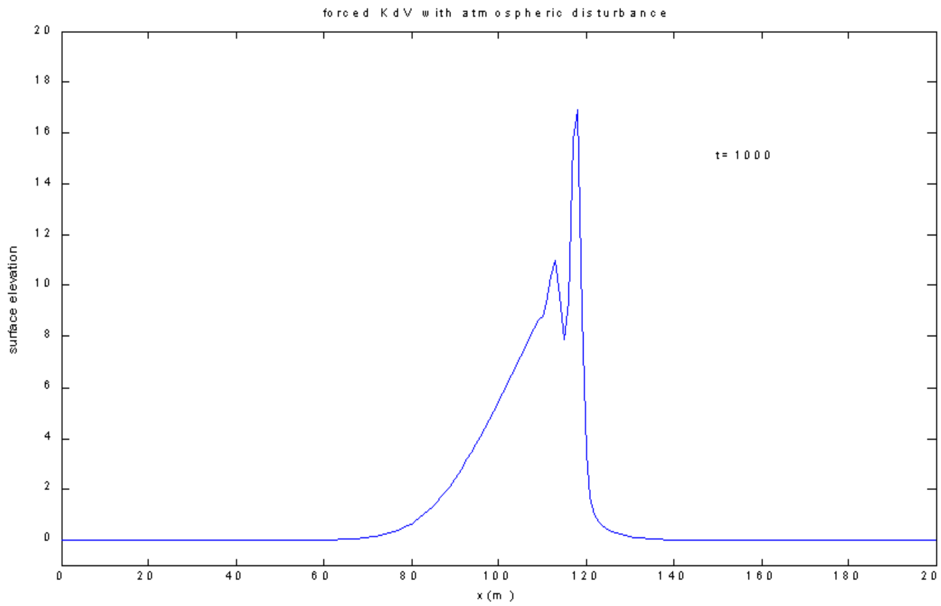
Case 2: $h = 0.5$ m



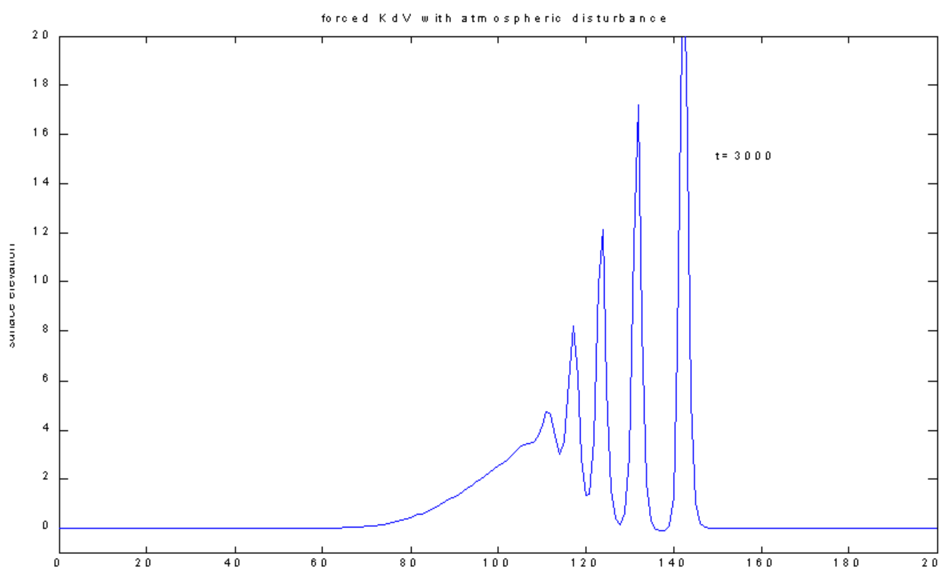
(a)



(b)

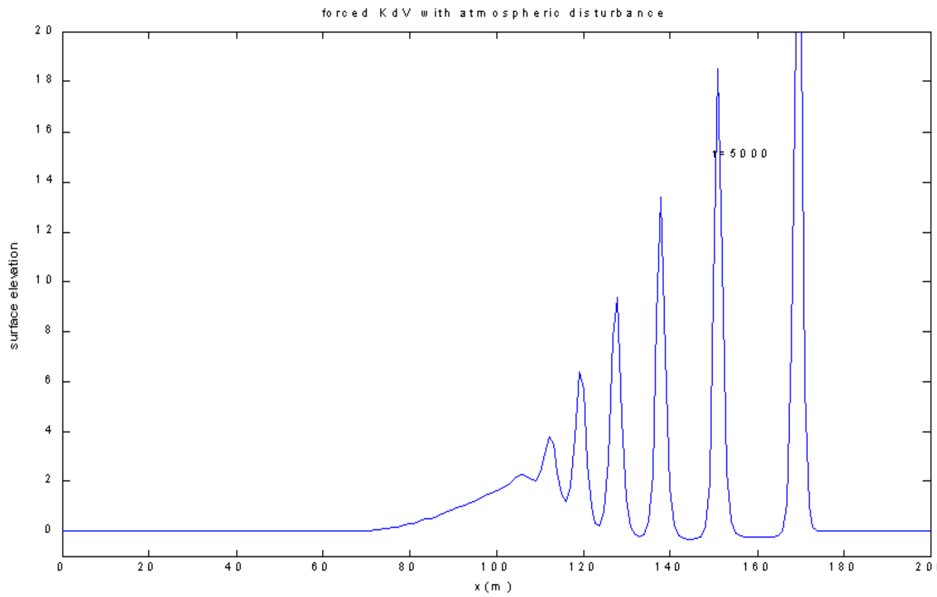


(c)

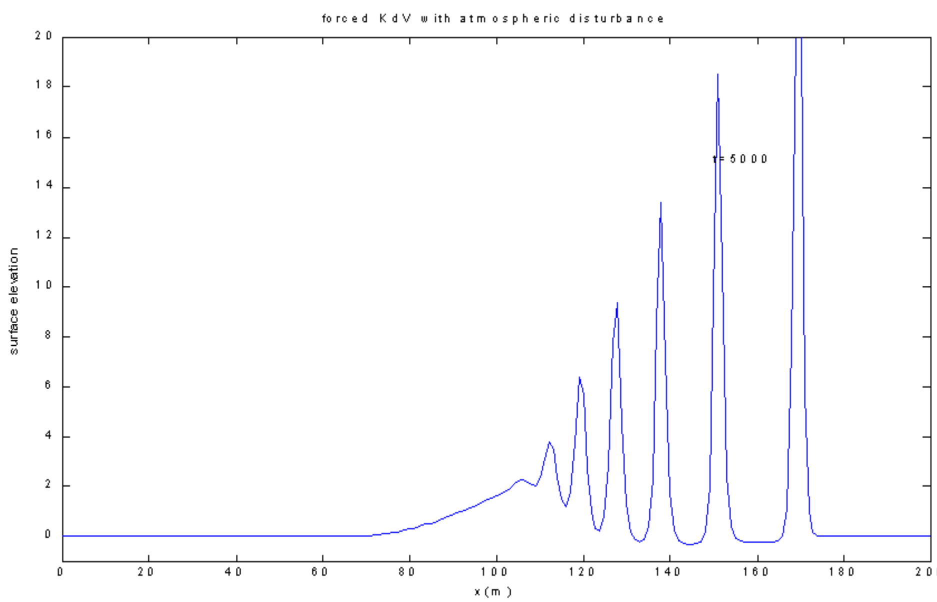


(d)

The wave starts to steepen earlier when the depth is decreasing.

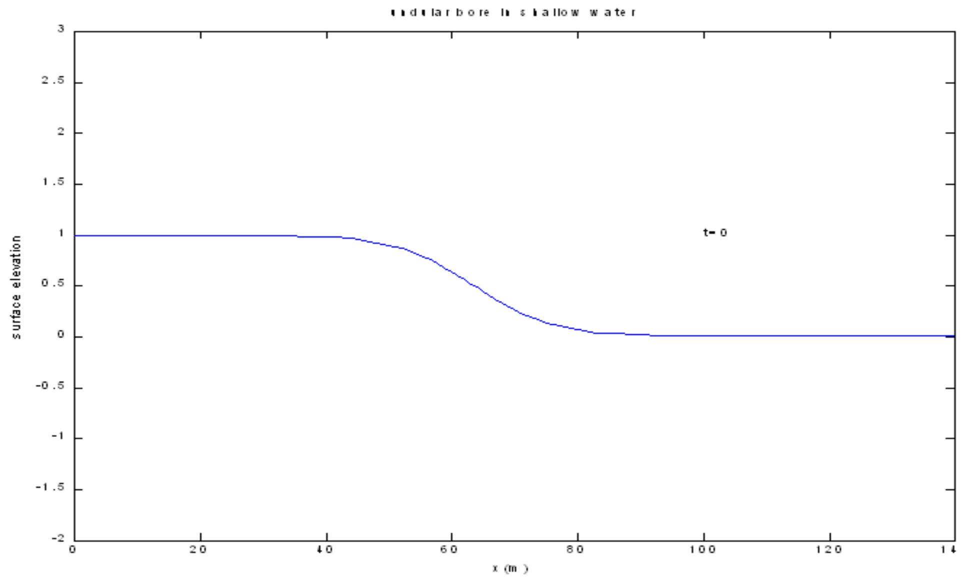


(e)

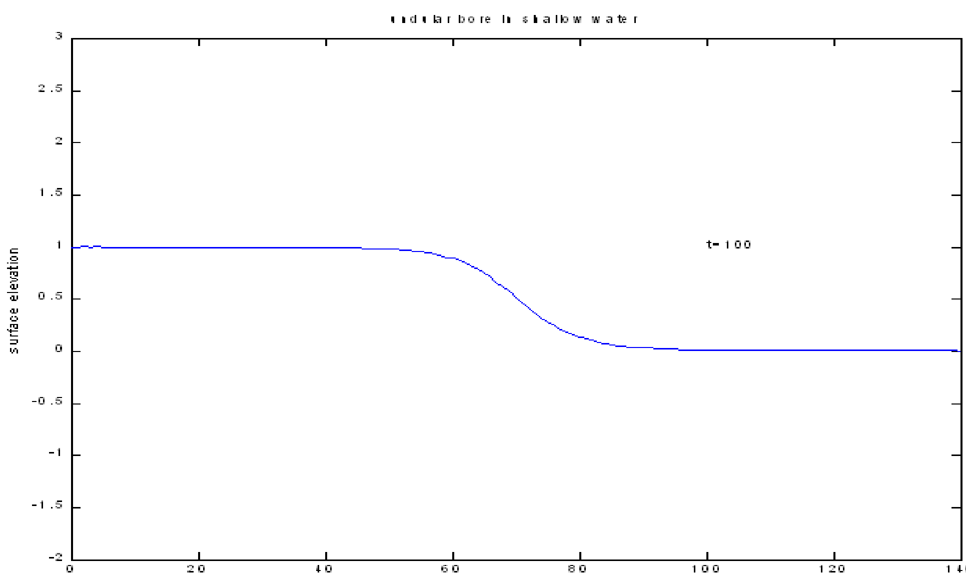


(f)

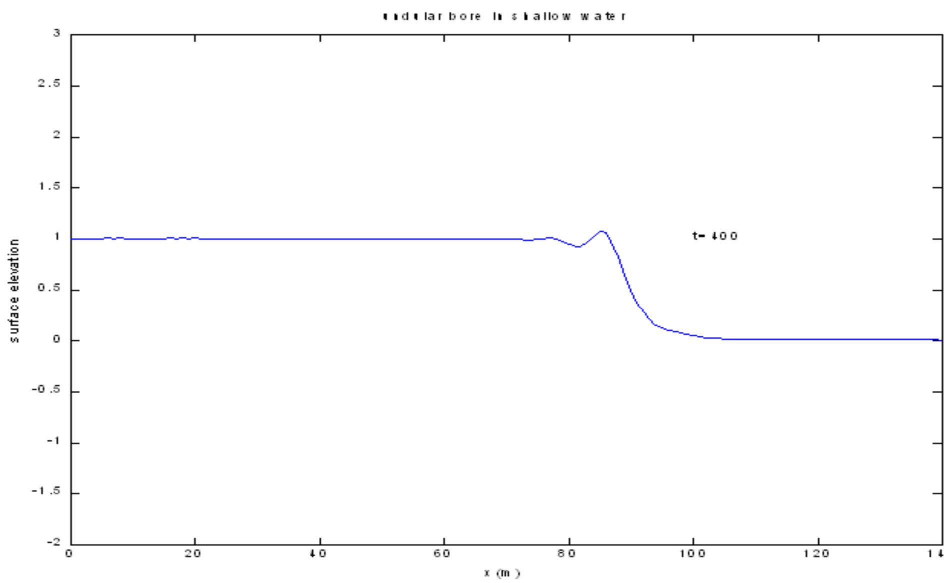
Figure 5: (a)-(f) Show the Evolution of Forced KdV at (a) $t = 0$, (b) $t = 500$, (c) $t = 1000$, (d) $t = 30000$, (e) $t = 5000$, (f) $t = 10000$


undular bore in shallow water
surface elevation
x (m)
t=0

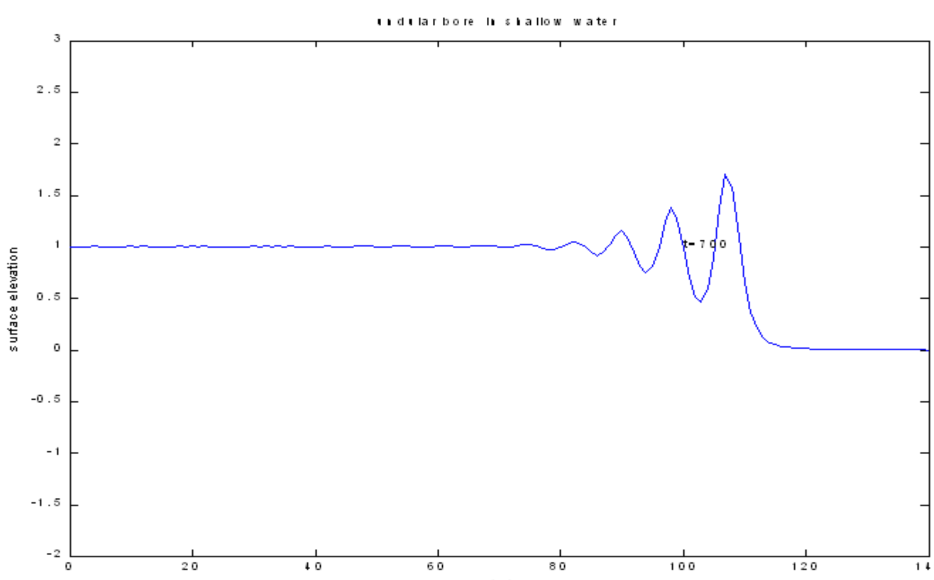
(a)


undular bore in shallow water
surface elevation
x (m)
t=100

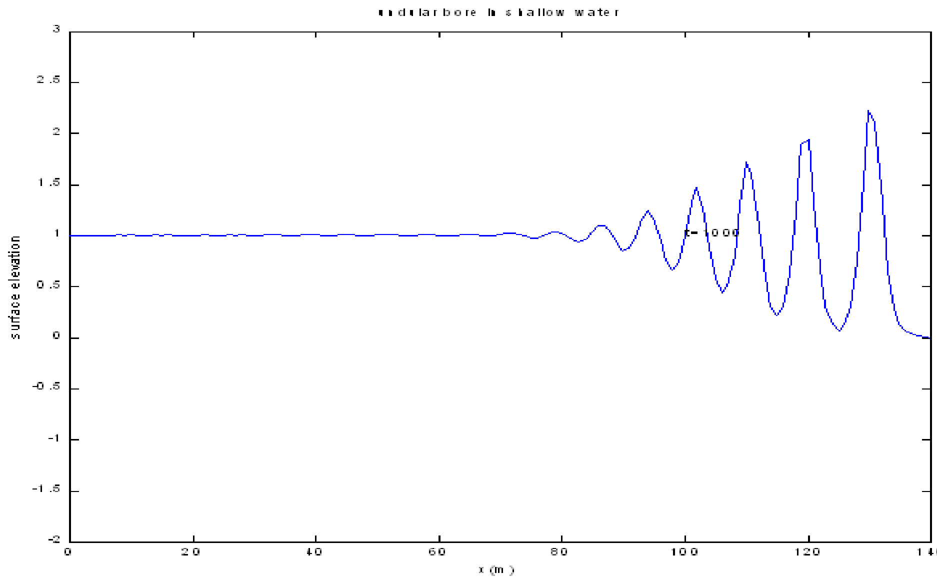
(b)


undular bore in shallow water
surface elevation
x (m)
t= 400
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
20
40
60
80
100
120

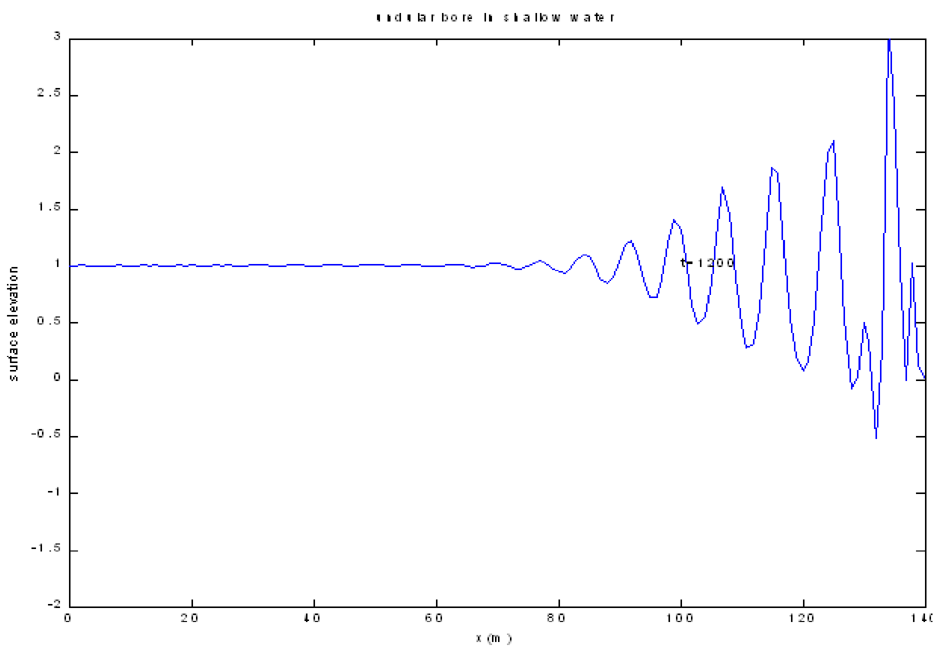
(c)


undular bore in shallow water
surface elevation
x (m)
t= 700
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
20
40
60
80
100
120

(d)



(e)



(f)

Figure 6: (a)-(f) Show the Evolution of Undular Bore at Shallow Water