Some Bounds of Radius of Convexity of a Class of Analytic Functions

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Abstract Let $G(\alpha, \delta)$ denote the class of function $f$, $f(0) = f'(0) - 1 = 0$ for which $\Re \{e^{i\alpha} f'(z)\} > \delta$ in $D = \{z : |z| < 1\}$ where $|\alpha| \leq \pi$ and $\cos \alpha > \delta$. We obtain some sharp results related to its radius of convexity.

Keywords Analytic functions; convex; and radius of convexity.

1 Introduction

We denote $G(\alpha, \delta)$ the class of normalised analytic functions $f$ in the unit disc $D$ where

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

satisfying $\Re \{e^{i\alpha} f'(z)\} > \delta$ where $|\alpha| \leq \pi$ and $\cos \alpha > \delta$. This class of functions has been studied by many researches among others MacGregor [5] who extensively looked into its basic properties for $G(0, 0)$, Goel and Mehrok [3] for $G(0, \delta)$ ($\delta \geq 0$) and Silverman and Silvia [6] for $G(\alpha, 0)$. Daud [1] obtained some basic properties for $G(\alpha, \delta)$ including its representation theorem, extremals and argument. In this paper we have extended the later work by looking at, in particular, the radius of convexity of the class.

Let $S$ be the class of normalized univalent functions analytic in the unit disc. Let $K(\beta)$ denote the subclass of $S$ consisting of functions $g(z)$ for which

$$\Re \left\{ 1 + \frac{g''(z)}{g'(z)} \right\} \geq \beta \quad (0 \leq \beta \leq 1).$$

This class is called convex of order $\beta$. In particular $f(z)$ is in the class $G(\alpha, \delta)$ if there exists a function $g(z) \in K(\beta)$ such that

$$\Re \left\{ e^{i\alpha} \frac{f'(z)}{g'(z)} \right\} > \delta \quad (0 \leq \delta \leq 1, |z| < 1).$$

Define a function $p$ is in the class $P$ if and only if

$$p(z) = \int_{|z|=1} \frac{1+z\overline{x}}{1-z\overline{x}} d\mu(x),$$

for some probability measure $\mu$. Denote by $P_\delta$ the functions $p(z)$ that are analytic in $|z| < 1$ and satisfy the conditions $p(0) = 1$ and $\Re p(z) > \delta$, and set $P_0 = p$. It is well known that a function $q(z)$ is in $P_\delta$ if and only if there exists a function $p(z) \in P$ such that

$$q(z) = (1-\delta)p(z) + \delta = \frac{p(z) + h}{1 + h},$$

for some $h > 0$. The radius of convexity of $q(z)$ is given by $R_q = \max\{z : |z| < 1, \Re q(z) > \delta\}$. In the next section, we shall obtain some sharp results related to the radius of convexity of $q(z)$.
where

\[ h = \frac{\delta}{1 - \delta}. \]  

Thus if \( f(z) \in C(\beta, \delta) \), then we may write

\[ e^{\imath \alpha} \frac{f'(z)}{g'(z)} = \frac{p(z) + h}{1 + h} \]

so that

\[ f'(z) = e^{-\imath \alpha} g'(z) \left( \frac{p(z) + h}{1 + h} \right), \]  

where \( g(z) \in K(\beta), p(z) \in P \), and \( h \) is defined by (1). Taking derivatives of (2), we find that

\[ f''(z) = e^{-\imath \alpha} \left\{ g''(z) \left( \frac{p(z) + h}{1 + h} \right) + \frac{g'(z)}{(1 + h)} (p'(z)) \right\} \]

and hence

\[ 1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{zg''(z)}{g'(z)} + \frac{zp'(z)}{p(z) + h}. \]  

In determining the radius of convexity for the class \( G(\alpha, \delta) \) we note that for \( |z| = r \), (3) yields

\[ \min_{f \in C(\beta,\delta)} \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} = \min_{g \in K(\beta)} \text{Re} \left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} + \min_{p \in P} \text{Re} \left\{ \frac{zp'(z)}{p(z) + h} \right\}. \]  

and from a simple calculation, it can be shown that

\[ \min_{\substack{|z| = r \\ \text{for} \\ g \in K(\beta)}} \text{Re} \left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} = 1. \]

Thus, the radius of convexity of \( G(\alpha, \delta) \) is seen to be the smallest positive \( r \) for which

\[ 1 + \min_{\substack{|z| = r \\ \text{for} \\ p \in P}} \text{Re} \left\{ \frac{zp'(z)}{p(z) + h} \right\} = 0. \]
2  Main Results

The following result is obtained using the above argument which is a result given by Silverman [7] and Daud and Shaharuddin [2].

**Theorem 1.** Suppose \( p(z) \in P, \) \( h \) is defined by (1) and

\[
a = \frac{1 + r^{2m}}{1 - r^{2m}}.
\]

Then

\[
\text{Re} \left\{ \frac{zp'(z)}{p(z) + h} \right\} \geq \begin{cases} 
\frac{-2r}{(1+r)(1+h-(1-h)r)} & (0 \leq r \leq r_\delta) \\
2\sqrt{h^2 + ah - a - 2h} & (r_\delta < r < 1),
\end{cases}
\]

where \( r_\delta \) is the unique root of the equation \((1 - 2\delta) r^3 - 3 (1 - 2\delta) r^2 + 3r - 1 = 0\) in the interval \((0,1]\). This result is sharp.

For the proof, refer to Silverman [7] when \( \delta \) taken only positive values, whereas Daud and Shaharuddin [2] completed it when \( \delta \) taken any negative values.

**Theorem 2.** Suppose \( r_\delta \) is the unique root of \( t(r) = (1 - 2\delta) r^3 - 3 (1 - 2\delta) r^2 + 3r - 1 \) in the interval \((0,1]\). Set

\[
r(\beta, \delta) = \frac{1}{(1 - 2\delta) + \sqrt{4\delta^2 - 6\delta + 2}}.
\]

Then the radius of convexity of \( G(\alpha, \delta) \) is \( r(\beta, \delta) \) when \( 0 < r(\alpha, \beta) \leq r_\delta \), and is otherwise the smallest root greater then \( r_\delta \) of the polynomial equation \( v(r) = (1 - 2\delta) r^4 + 2 \delta r^2 - \delta \). This result is sharp for all \( \delta \).

**Proof.** By applying Theorem 1 to (5), the radius of convexity of \( G(\alpha, \delta) \) is the smallest positives root of

\[
\begin{cases} 
1 - \frac{2r}{(1+r)(1+h-(1-h)r)} = 0 & (0 \leq r < r_\delta) \\
1 + 2\sqrt{h^2 + ah - a - 2h} = 0 & (r_\delta < r < 1)
\end{cases}
\]

where \( h \) is defined by (1). The first expression in (6) may be written as

\[
\frac{-(1 - 2\delta) r^2 - 2 (1 - 2\delta) r + 1}{(1 + r) [(1 + h) - (1 - h) r]} = 0,
\]

whose roots are

\[
\frac{(1 - 2\delta) \pm \sqrt{(1 - 2\delta)^2 + (1 - 2\delta)}}{-(1 - 2\delta)} = \frac{1}{(1 - 2\delta) \pm \sqrt{4\delta^2 - 6\delta + 2}}.
\]

If both roots are positive, the minimum root is \( r(\alpha, \delta) \). Similarly, a computation shows that \( r^* \) is a root of the second expression in (6) if an only if it is a root of \( v(r) \). This completes the proof.
3 Further Results

We let \( R_{cv} [G(\alpha, \delta)] = \sup \{ r : Re \left[ 1 + \frac{z'^0(z)}{r(z)} \right] > 0, |z| = r, f \in G(\alpha, \delta) \} \) be the radius of convexity of \( G(\alpha, \delta) \). The estimation of the lower and upper bound of \( R_{cv} [G(\alpha, \delta)] \) is obtained for a choice of an extreme point.

**Theorem 3.** Let \( A = \cos \alpha - \delta \).

(i) \( R_{cv} [G(\alpha, \delta)] \geq (\sqrt{2} - 1) \left| \frac{2(A+\delta)A-1}{A-\sqrt{1-A(2\delta+A)}} \right| \).

(ii) If \( \delta < \frac{1}{4} \),

\[
R_{cv} [G(\alpha, \delta)] \leq \left[ 1 + 2 \sqrt{\frac{A}{1 - 4\delta A}} \left\{ \sqrt{A} - \sqrt{(A + \delta) + \sqrt{1 - 4\delta A}} \right\} \right]^{1/2} < 1. \tag{7}
\]

**Proof.** Let \( f \in G(\alpha, \delta) \). A theorem of Daud [1] gives us \( Re f'(z) \geq 0 \) for

\[
|z| < \frac{A - \sqrt{1 - A(2\delta + A)}}{2(A + \delta) A - 1}.
\]

Put \( \lambda = \frac{A - \sqrt{1 - A(2\delta + A)}}{2(A + \delta) A - 1} \) and let \( g(z) = \frac{f(\lambda z)}{\lambda} \in G(0, 0) \). It is known from MacGregor[4] that \( g \) is convex for \( |z| < \sqrt{2} - 1 \). Thus \( f \) is convex if

\[
|z| < \frac{\sqrt{2} - 1}{\lambda} = \left( \sqrt{2} - 1 \right) \left| \frac{2(A + \delta) A - 1}{A - \sqrt{1 - A(2\delta + A)}} \right|.
\]

We consider an extreme point \( g(z) = -e^{-i\alpha} (e^{-i\alpha} - 2\delta) z - 2e^{-i\alpha} A \log (1 - z) \) of \( G(\alpha, \delta) \) as in Daud [1]. Then

\[
g'(z) = -e^{-i\alpha} (e^{-i\alpha} - 2\delta) + 2e^{-i\alpha} A \left( \frac{1}{1 - z} \right)
\]

and

\[
zg'(z) = -e^{-i\alpha} (e^{-i\alpha} - 2\delta) z + 2e^{-i\alpha} A \left( \frac{z}{1 - z} \right).
\]

So

\[
g'(z) + zg''(z) = (zg'(z))' = -e^{-i\alpha} (e^{-i\alpha} - 2\delta) + 2e^{-i\alpha} A \left( \frac{1}{1 - z} \right)^2
\]

and this expression is zero if and only if

\[
z = 1 \pm \sqrt{1 + \frac{e^{2i\alpha}}{1 - 2\delta e^{i\alpha}}} \tag{8}
\]

These values are well defined since the condition \( \delta < \frac{1}{4} \) implies \( 1 - 2\delta e^{i\alpha} \neq 0 \). Now let \( \rho = \rho(\alpha, \delta) \) be the smallest of the modulii of these two roots. Then we must have \( R_{cv} [G(\alpha, \delta)] \leq \rho \). To see this, we argue as follows. If \( g'(z) = 0 \) for some \( z \) satisfying \( |z| < \rho \), then

\[
1 + \frac{zg''(z)}{g'(z)} \tag{9}
\]
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is not defined, and \( g \) is not convex on the disc \(|z| < \rho\). If \( g'(z) \neq 0 \) on the disc, then (9) is zero on \(|z| = \rho\) and \( g \) is not convex on any larger disc.

To determine \( \rho \), we first find \( \text{Re} z, \text{Im} z \) for the \( z \) in (9). If \( \alpha = 0 \) or \( \alpha = \pm \pi \), we have

\[
\rho = \sqrt{\frac{2 (1 - \delta)}{1 - 2\delta}} - 1, \quad \rho = \sqrt{\frac{2 (1 + \delta)}{1 + 2\delta}} - 1
\]

respectively, and these are the values given by (7). Assume now that \( \cos \alpha \neq 1 \). Let

\[
\zeta^2 = 1 + \frac{e^{i\alpha}}{1 - 2\delta e^{i\alpha}}
\]

and put

\[
\zeta = x + iy \quad (x, y \text{ real}).
\]

We have

\[
\frac{e^{i2\alpha}}{1 - 2\delta e^{i\alpha}} = \frac{\cos 2\alpha + i \sin 2\alpha}{1 - 2\delta (\cos \alpha + i \sin \alpha)} = \frac{\cos 2\alpha - 2\delta \cos \alpha + i (\sin 2\alpha - 2\delta \sin \alpha)}{1 - 4\delta \cos \alpha + 4\delta^2}
\]

which gives

\[
\zeta^2 = x^2 - y^2 + i2xy = 1 + \frac{\cos 2\alpha - 2\delta \cos \alpha}{1 - 4\delta \cos \alpha + 4\delta^2} + \frac{i \sin 2\alpha - 2\delta \sin \alpha}{1 - 4\delta \cos \alpha + 4\delta^2}
\]

so that

\[
x^2 - y^2 = 1 + \frac{\cos 2\alpha - 2\delta \cos \alpha}{1 - 4\delta \cos \alpha + 4\delta^2} = \frac{2A (A - \delta)}{1 - 4A\delta}
\]

(10)

and

\[
2xy = \frac{\sin 2\alpha - 2\delta \sin \alpha}{1 - 4\delta \cos \alpha + 4\delta^2} = \frac{2A \sqrt{1 - (A + \delta)^2}}{1 - 4A\delta}
\]

(11)

Thus

\[
x^2 = \frac{A^2 \left(1 - (A + \delta)^2\right)}{x^2 (1 - 4A\delta)^2} = \frac{2A (A - \delta)}{1 - 4A\delta}
\]

which we rewrite as

\[
(1 - 4A\delta)^2 x^4 - 2A (A - \delta) (1 - 4A\delta) x^2 - A^2 \left(1 - (A + \delta)^2\right) = 0.
\]

An elementary argument shows that \( 1 - 4A\delta > 0 \) when \( \delta < \frac{1}{4} \), and so

\[
x^2 = \frac{2A (A - \delta) (1 - 4A\delta) \pm \sqrt{(2A (A - \delta) (1 - 4A\delta))^2 + 4A^2 (1 - 4A\delta)^2 (1 - (A + \delta)^2)}}{2 (1 - 4A\delta)^2}
\]

\[
= A (A - \delta) \pm A \sqrt{1 - 4A\delta}.
\]
The product of these roots is negative since $|A + \delta| = |\cos \alpha| < 1$, and $A - \delta + \sqrt{1 - 4A\delta}$ is positive since $\delta < \frac{1}{3}$, so we must have

$$x^2 = \frac{A(A - \delta) + A\sqrt{1 - 4A\delta}}{1 - 4A\delta}$$

which gives

$$x = \pm \sqrt{\frac{A(A - \delta) + A\sqrt{1 - 4A\delta}}{1 - 4A\delta}}.$$ 

(12)

Substituting $x$ in (11), we obtain

$$y = \pm \sqrt{\frac{A \left(1 - (A + \delta)^2\right)}{(1 - 4A\delta)\left[(A - \delta) + \sqrt{1 - 4A\delta}\right]},}$$

(13)

and the possible values of $\zeta = x + iy$ are determined. It follows from (8) that

$$\rho = \rho(\alpha, \delta) = \sqrt{(1-x)^2 + y^2}$$

where $x$ and $y$ are the positive alternatives in (12) and (13). Thus

$$|\rho|^2 = \left[1 - \frac{A(A - \delta) + A\sqrt{1 - 4A\delta}{1 - 4A\delta}}{1 - 4A\delta}\right]^2 + \frac{A \left(1 - (A + \delta)^2\right)}{(1 - 4A\delta)\left[(A - \delta) + \sqrt{1 - 4A\delta}\right]}$$

$$= 1 + 2 \left[\frac{A}{1 - 4A\delta}\right]^\frac{1}{2} \left[\sqrt{A - \left(A - \delta + \sqrt{1 - 4A\delta}\right)}\right]^\frac{1}{2}$$

as required. The expression in the second bracket is negative when $\delta < \frac{1}{3}$, so we have $\rho < 1$. This completes the proof.

We note that the restriction on $\delta$ in Theorem 3(ii) of the theorem is inevitable, since the function $g$ used in the proof to obtain upper bound for $R_{cv}[G(\alpha, \delta)]$ is actually convex in the case $\alpha = 0, \delta = \frac{1}{2}$. The following result is obtained by using a different choice of $g$.

**Theorem 4.** $R_{cv}[G(\alpha, \delta)] \leq \left(\frac{1}{nA}\right)^{1/(n-1)}$ where $n$ is the smallest integer satisfying $n \geq 2, n > \frac{1}{\alpha}$. This result is sharp.

The proof is immediate once we have proved

**Lemma 5.** The class $G(\alpha, \delta)$ contains the function

$$g(z) = z + \left(\frac{A}{n}\right) z^n \quad (z \in D)$$

where $n \geq 2$ is an integer, and the radius of the largest disc in which $g$ is convex is

$$\left(\frac{1}{nA}\right)^{1/(n-1)}.$$
where \( n > \frac{1}{A} \). This result is sharp. Proof of the lemma. Let \( g(z) = z + \left( \frac{A}{n} \right) z^n \) \( (z \in D) \). Then \( g'(z) = 1 + Az^{n-1} \) and so
\[
\text{Re} \left[ e^{i\alpha} g'(z) \right] = \text{Re} \left[ e^{i\alpha} (1 + Az^{n-1}) \right] 
= \cos \alpha + A \text{Re} \left( e^{i\alpha} z^{n-1} \right) 
> \cos \alpha - A 
= \delta.
\]
Hence \( g \in G(\alpha, \delta) \). Now \( g''(z) = (n-1)Az^{n-2} \) which gives
\[
1 + zg''(z)g'(z) = 1 + \frac{z(n-1)Az^{n-2}}{1 + Az^{n-1}} = n + \frac{1 - n}{1 + Az^{n-1}}.
\]
So,
\[
\text{Re} \left[ 1 + zg''(z)g'(z) \right] > 0
\]
if and only if
\[
\text{Re} \left[ n + \frac{1 - n}{1 + Az^{n-1}} \right] > 0,
\]
and simplifying the above inequality, we have
\[
(n-1) \left( 1 + \text{Re} Az^{n-1} \right) < n \left| 1 + Az^{n-1} \right|^2,
\]
\[
(n-1) + (n-1) \text{Re} Az^{n-1} < n \left[ 1 + 2 \text{Re} Az^{n-1} + \left| Az^{n-1} \right|^2 \right],
\]
\[
n \left| Az^{n-1} \right|^2 + (n+1) \text{Re} Az^{n-1} + 1 > 0.
\]
This inequality is true on \( |z| = \rho \) if and only if \( n \left( (\rho^{n-1} - 1)^2 - (n+1) \rho^{n-1} + 1 \right) \geq 0 \) or \( (n\rho^{n-1} - 1)(\rho^{n-1} - 1) \geq 0 \), and when \( n > \frac{1}{A} \) it follows that \( g \) is convex on \( |z| < \left( \frac{1}{nA} \right)^{\frac{1}{n-1}} \), but not on any larger disc.

Following Silverman and Silvia [6], we next consider polynomials of degree \( n \) in \( G(\alpha, \delta) \). We find sufficient conditions in terms of \( \alpha, \delta, n \) for these to be convex. \textbf{Theorem 6.} Suppose \( p_n(z) = z + \sum_{k=2}^{n} c_k z^k \) is in \( G(\alpha, \delta) \). Then \( p_n \) is convex if \( \cos \alpha \leq \frac{1}{(n+2)(n-1)} + \delta \). This result is sharp.

\textbf{Proof.} Let \( f(z) = z + \sum_{k=2}^{n} c_k z^k \). Then by a result of Kobori [4], \( f \) is convex if \( \sum_{k=2}^{n} k^2 |a_k| \leq 1 \) for the \( p_n \) in the statement. We have, using the result on coefficient bound of \( G(\alpha, \delta) \) in Daud [1],
\[
\sum_{k=2}^{n} k^2 |c_k| \leq \sum_{k=2}^{n} 2k (\cos \alpha - \delta) 
\]
\[
= 2 \left[ \frac{n(n-1)}{2} - 1 \right] (\cos \alpha - \delta) 
\leq 1.
\]
if \( \cos \alpha \leq \frac{1}{(\alpha + 2)(\alpha - 1)} + \delta \). This completes the proof of the theorem.

The function \( f_1(z) = z \) is a member of each of the classes \( G(\alpha, \delta) \). In fact \( f_1 \) is the only function with this property. To see that this is the case, suppose that some other function \( g \) has the same property, so that \( g' \) takes a value \( \lambda \neq 1 \). Consider the line in the Argand diagram containing \( \lambda \) and 1. Rotate this line about any point between \( \lambda \) and 1 so that in its new position it contains neither of these points. The line now separates the plane into two half planes, and one of these contains 1 but does not contain \( \delta \). The plane can be described as the collection of points \( w \) for which \( \text{Re}^{i\alpha} w > \delta \) and since \( w \) may be taken as 1, we have \( \cos \alpha > \delta \). For this \( \alpha \) and \( \delta \), any function in \( G(\alpha, \delta) \) fails to take value \( \lambda \), and so is not in \( G(\alpha, \delta) \). This is the required contradiction.

**Theorem 7.** (i) Let \( \varepsilon > 0 \), and let \( F_{\varepsilon} \) be the family of classes \( G(\alpha, \delta) \) for which \( \cos \alpha \geq \delta + \varepsilon \). Then \( \cap F_{\varepsilon} \) contains a function, which is not convex.

(ii) Let \( F_0 \) be any family of classes \( G(\alpha, \delta) \) with the property that, for any \( \varepsilon > 0 \), there exists a member \( G(\alpha, \delta) \) of \( F_0 \) with \( \cos \alpha < \delta + \varepsilon \). Then \( \cap F_0 \) contains only the function \( f_1(z) = z \), \( z \in D \).

As an immediate corollary to the theorem we have the following result that answers the above question in the case of convexity.

**Corollary 8.** Let \( F \) be the intersection of any family of classes \( G(\alpha, \delta) \). Then either \( F \) contains just the identity function \( f_1(z) = z \), or \( F \) contains a function which is not convex.

Part (i) of the theorem was proved by Silverman and Silvia [6] in the case \( |\alpha| < \frac{\pi}{2} - \varepsilon, |\alpha| \leq \frac{\pi}{2} \). Our proof is rather more geometrical than theirs, using the following property of any half plane \( \text{Re}^{i\alpha} w > \delta \) in the \( w \)-Argand diagram with \( |\alpha| \leq \pi \). \( \cos \alpha > \delta \): the largest radius of any circle centered at 1 whose interior is a subset of the half plane, is \( \cos \alpha > \delta \).

**Proof of theorem.**

(i) Let \( f(z) = z + \lambda z^n \). A slight modification to the last part of proof of Lemma 5 shows that \( f \) is convex if and only if \( |\lambda| < \frac{1}{n^2} \). Choose \( n \) so that \( \varepsilon > \frac{1}{n^2} \) and then \( \lambda \) so that \( \frac{1}{n^2} < n \lambda < \varepsilon \). Then \( \lambda > \frac{1}{n^2} \) and \( f \) is not convex. Also \( f'(z) = 1 + n \lambda z^{n-1} \), so that \( f' \) maps \( D \) onto a circle centered at 1 and radius \( \varepsilon \), and \( f \in G(\alpha, \delta) \) whenever \( \cos \alpha - \delta > \varepsilon \).

(ii) Suppose that \( \cap F_0 \) contains a function \( g \neq f_1 \), so that, by the open mapping theorem, the interior of some disc centered at 1 is a subset of \( g'(D) \). Let the radius of this disc be \( d \), and let \( G(\alpha, \delta) \) be a class in \( F_0 \) for which \( \cos \alpha - \delta < d \). The derivative of any function in \( G(\alpha, \delta) \) maps \( D \) into the half plane \( \text{Re}^{i\alpha} w > \delta \) and the radius of the largest circle, centred at 1, inside this half plane, is \( \cos \alpha - \delta \). This implies \( g \notin \cap F_0 \), which contradicts our assumption. So \( \cap F_0 \) contains only \( f_1 \).

### 4 Conclusion

This paper extends some of the results given by Silverman and Silvia [6] and Daud [1] on bounds for \( R_{cv}[G(\alpha, \delta)] \), the radius of convexity of \( G(\alpha, \delta) \). Using the approach by Silverman [7] and Goel and Mehrok [3], we extend the results when \( \delta \) taken as any negative values in Daud and Shaharuddin [2]. With this results, we emphasis on using the extreme values function to look on the consideration of convexity and starlikeness of the class.
References


