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Some Bounds of Radius of Convexity of a Class of Analytic Functions

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Abstract Let $G(\alpha,\delta)$ denote the class of function f, $f(0)=f'(0)-1=0$ for which $Re\left\{e^{i\alpha}f'(z)\right\}>\delta$ in $D=\{z:|z|<1\}$ where $|\alpha|\le\pi$ and $\cos\alpha>\delta$. We obtain some sharp results related to its radius of convexity.

Keywords Analytic functions; convex; and radius of convexity.

1 Introduction

We denote $G(\alpha,\delta)$ the class of normalised analytic functions f in the unit disc D where

$$f(z)=z+a_2z^2+\cdots+a_nz^n+\cdots$$

satisfying $\mathrm{Re}\left\{e^{i\alpha}f'(z)\right\}>\delta$ where $|\alpha|\le\pi$ and $\cos\alpha>\delta$. This class of functions has been studied by many researches among others MacGregor [5] who extensively looked into its basic properties for $G(0,0)$, Goel and Mehrok [3] for $G(0,\delta)\,(\delta\ge 0)$ and Silverman and Silvia [6] for $G(\alpha,0)$. Daud [1] obtained some basic properties for $G(\alpha,\delta)$ including its representation theorem, extremals and argument. In this paper we have extended the later work by looking at, in particular, the radius of convexity of the class.

Let S be the class of normalized univalent functions analytic in the unit disc. Let $K(\beta)$ denote the subclass of S consisting of functions $g(z)$ for which

$$\mathrm{Re}\left\{1+\frac{g''(z)}{g'(z)}\right\}\ge\beta\quad(0\le\beta\le1).$$

This class is called convex of orderβ. In particular $f(z)$ is in the class $G(\alpha,\delta)$ if there exists a function $g(z)\in K(\beta)$ such that

$$\mathrm{Re}\left\{e^{i\alpha}\frac{f'(z)}{g'(z)}\right\}>\delta\quad(0\le\delta\le1,|z|<1).$$

Define a function p is in the class P if and only if

$$p(z)=\int\limits_{|x|=1}\frac{1+zx}{1-zx}d\mu(x),$$

for some probability measure μ. Denote by P_δ the functions $p(z)$ that are analytic in $|z|<1$ and satisfy the conditions $p(0)=1$ and $\mathrm{Re}\;p(z)>\delta$, and set $P_0=p$. It is well known that a function $q(z)$ is in P_δ if and only if there exists a function $p(z)\in P$ such that

$$q(z)=(1-\delta)\,p(z)+\delta=\frac{p(z)+h}{1+h},$$

where

$$h = \frac{\delta}{1-\delta}. \tag{1}$$

Thus if $f(z) \in C(\beta, \delta)$, then we may write

$$e^{i\alpha}\frac{f'(z)}{g'(z)} = \frac{p(z)+h}{1+h}$$

so that

$$f'(z) = e^{-i\alpha} g'(z)\left(\frac{p(z)+h}{1+h}\right), \tag{2}$$

where $g(z) \in K(\beta)$,$p(z) \in P$, and h is defined by (1). Taking derivatives of (2), we find that

$$f''(z) = e^{-i\alpha}\left\{g''(z)\left(\frac{p(z)+h}{1+h}\right) + \frac{g'(z)}{(1+h)}(p'(z))\right\}$$

$$zf''(z) = e^{-i\alpha} z\left\{g''(z)\left(\frac{p(z)+h}{1+h}\right) + \frac{g'(z)}{(1+h)}(p'(z))\right\}$$

$$\frac{zf''(z)}{f'(z)} = \frac{zg''(z)}{g'(z)} + \frac{zp'(z)}{p(z)+h}$$

and hence

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{zg''(z)}{g'(z)} + \frac{zp'(z)}{p(z)+h}. \tag{3}$$

In determining the radius of convexity for the class $G(\alpha, \delta)$ we note that for $|z| = r$, (3) yields

$$\min_{f \in C(\beta,\delta)} \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} = \min_{g \in K(\beta)} \operatorname{Re}\left\{1 + \frac{zg''(z)}{g'(z)}\right\} + \min_{p \in P} \operatorname{Re}\left\{\frac{zp'(z)}{p(z)+h}\right\}. \tag{4}$$

and from a simple calculation, it can be shown that

$$\min_{\substack{|z|=r \\ g \in K(\beta)}} \operatorname{Re}\left\{1 + \frac{zg''(z)}{g'(z)}\right\} = 1.$$

Thus, the radius of convexity of $G(\alpha, \delta)$ is seen to be the smallest positive r for which

$$1 + \min_{\substack{|z|=r \\ p \in P}} \operatorname{Re}\left\{\frac{zp'(z)}{p(z)+h}\right\} = 0. \tag{5}$$

2 Main Results

The following result is obtained using the above argument which is a result given by Silverman [7] and Daud and Shaharuddin [2].

Theorem 1. *Suppose $p(z) \in P, h$ is defined by (1) and*

$$a = \frac{1 + r^{2m}}{1 - r^{2m}}.$$

Then

$$\operatorname{Re}\left\{\frac{zp'(z)}{p(z)+h}\right\} \geq \begin{cases} -\frac{2r}{(1+r)[1+h-(1-h)r]} & (0 \leq r \leq r_\delta) \\ 2\sqrt{h^2+ah}-a-2h & (r_\delta < r < 1), \end{cases}$$

where r_δ is the unique root of the equation $(1-2\delta)\, r^3 - 3\,(1-2\delta)\, r^2 + 3r - 1 = 0$ in the interval $(0,1]$. This result is sharp.

For the proof, refer to Silverman [7] when δ taken only positive values, whereas Daud and Shaharuddin [2] completed it when δ taken any negative values.

Theorem 2. *Suppose r_δ is the unique root of $t\,(r) = (1-2\delta)\, r^3 - 3\,(1-2\delta)\, r^3 + 3r - 1$ in the interval $(0,1]$. Set*

$$r\,(\beta,\delta) = \frac{1}{(1-2\delta)+\sqrt{4\delta^2-6\delta+2}}.$$

Then the radius of convexity of $G\,(\alpha,\delta)$is $r\,(\beta,\delta)$ when $0 < r\,(\alpha,\beta) \leq r_\delta$, and is otherwise the smallest root greater then r_δ of the polynomial equation $v\,(r) = (1-2\delta)\; r^4 + 2\,\delta r^2 - \delta$. This result is sharp for all δ.

Proof. By applying Theorem 1 to (5), the radius of convexity of $G\,(\alpha,\delta)$ is the smallest positives root of

$$\begin{cases} 1 - \frac{2r}{(1+r)[(1+h)-(1-h)r]} = 0 & (0 \leq r < r_\delta) \\ 1 + 2\sqrt{h^2+ah} - a - 2h = 0 & (r_\delta < r < 1) \end{cases} \tag{6}$$

where h is defined by (1). The first expression in (6) may be written as

$$\frac{-\,(1-2\delta)\, r^2 - 2\,(1-2\delta)\, r + 1}{(1+r)\,[(1+h)-(1-h)\, r]} = 0,$$

whose roots are

$$\frac{(1-2\delta) \mp \sqrt{(1-2\delta)^2 + (1-2\delta)}}{-\,(1-2\delta)} \quad = \quad \frac{1}{(1-2\delta) \pm \sqrt{4\delta^2-6\delta+2}}.$$

If both roots are positive, the minimum root is $r\,(\alpha,\delta)$. Similarly, a computation shows that r^* is a root of the second expression in (6) if an only if it is a root of $v\,(r)$. This completes the proof.

3 Further Results

We let $R_{cv}\left[G(\alpha,\delta)\right] = \sup\left\{r : Re\left[1+\frac{zf''(z)}{f'(z)}\right] > 0, |z| = r, f \in G(\alpha,\delta)\right\}$ be the radius of convexity of $G(\alpha,\delta)$. The estimation of the lower and upper bound of $R_{cv}\left[G(\alpha,\delta)\right]$ is obtained for a choice of an extreme point.

Theorem 3. *Let* $A = \cos\alpha - \delta$.

(i) $R_{cv}\left[G(\alpha,\delta)\right] \geq \left(\sqrt{2}-1\right)\left|\frac{2(A+\delta)A-1}{A-\sqrt{1-A(2\delta+A)}}\right|$,

(ii) *If* $\delta < \frac{1}{3}$,

$$R_{cv}\left[G(\alpha,\delta)\right] \leq \left[1+2\sqrt{\frac{A}{1-4\delta A}\left\{\sqrt{A}-\sqrt{(A+\delta)+\sqrt{1-4\delta A}}\right\}}\right]^{1/2} < 1. \tag{7}$$

Proof. Let $f \in G(\alpha,\delta)$. A theorem of Daud [1] gives us $Ref'(z) \geq 0$ for

$$|z| < \frac{A-\sqrt{1-A(2\delta+A)}}{2(A+\delta)A-1}.$$

Put $\lambda = \frac{A-\sqrt{1-A(2\delta+A)}}{2(A+\delta)A-1}$ and let $g(z) = \frac{f(\lambda(z))}{\lambda} \in G(0,0)$. It is known from MacGregor[4] that g is convex for $|z| < \sqrt{2}-1$. Thus f is convex if

$$|z| < \frac{\sqrt{2}-1}{\lambda} = \left(\sqrt{2}-1\right)\left|\frac{2(A+\delta)A-1}{A-\sqrt{1-A(2\delta+A)}}\right|.$$

We consider an extreme point $g(z) = -e^{-i\alpha}\left(e^{-i\alpha}-2\delta\right)z - 2e^{-i\alpha}A\log(1-z)$ of $G(\alpha,\delta)$ as in Daud [1]. Then

$$g'(z) = -e^{-i\alpha}\left(e^{-i\alpha}-2\delta\right) + 2e^{-i\alpha}A\left(\frac{1}{1-z}\right)$$

and

$$zg'(z) = -e^{-i\alpha}\left(e^{-i\alpha}-2\delta\right)z + 2e^{-i\alpha}A\left(\frac{z}{1-z}\right).$$

So

$$g'(z) + zg''(z) = \left(zg'(z)\right)' = -e^{-i\alpha}\left(e^{-i\alpha}-2\delta\right) + 2e^{-i\alpha}A\left(\frac{1}{1-z}\right)^2$$

and this expression is zero if and only if

$$z = 1 \pm \sqrt{1+\frac{e^{i2\alpha}}{1-2\delta e^{i\alpha}}}. \tag{8}$$

These values are well defined since the condition $\delta < \frac{1}{3}$ implies $1-2\delta e^{i\alpha} \neq 0$. Now let $\rho = \rho(\alpha,\delta)$ be the smallest of the modulii of these two roots. Then we must have $R_{cv}\left[G(\alpha,\delta)\right] \leq \rho$. To see this, we argue as follows. If $g'(z) = 0$ for some z satisfying $|z| < \rho$, then

$$1 + \frac{zg''(z)}{g'(z)} \tag{9}$$

is not defined, and g is not convex on the disc $|z| < \rho$. If $g'(z) \neq 0$ on the disc, then (9) is zero on $|z| = \rho$ and g is not convex on any larger disc.

To determine ρ, we first find Rez, Imz for the z in (9). If $\alpha = 0$ or $\alpha = \pm\pi$, we have

$$\rho = \sqrt{\frac{2(1-\delta)}{1-2\delta}} - 1 \quad , \quad \rho = \sqrt{\frac{2(1+\delta)}{1+2\delta}} - 1$$

respectively, and these are the values given by (7). Assume now that $\cos\alpha \neq 1$. Let

$$\zeta^2 = 1 + \frac{e^{i\alpha}}{1-2\delta e^{i\alpha}}$$

and put

$$\zeta = x + iy \quad , (x, y \text{ real}).$$

We have

$$\frac{e^{i2\alpha}}{1-2\delta e^{i\alpha}} = \frac{\cos 2\alpha + i\sin 2\alpha}{1-2\delta(\cos\alpha + i\sin\alpha)} \quad = \frac{\cos 2\alpha - 2\delta\cos\alpha + i(\sin 2\alpha - 2\delta\sin\alpha)}{1-4\delta\cos\alpha + 4\delta^2}$$

which gives

$$\zeta^2 = x^2 - y^2 + i2xy \quad = 1 + \frac{\cos 2\alpha - 2\delta\cos\alpha}{1-4\delta\cos\alpha+4\delta^2} + i\frac{\sin 2\alpha - 2\delta\sin\alpha}{1-4\delta\cos\alpha+4\delta^2}$$

so that

$$x^2 - y^2 = 1 + \frac{\cos 2\alpha - 2\delta\cos\alpha}{1-4\delta\cos\alpha+4\delta^2} = \frac{2A(A-\delta)}{1-4A\delta} \tag{10}$$

and

$$2xy = \frac{\sin 2\alpha - 2\delta\sin\alpha}{1-4\delta\cos\alpha+4\delta^2} \quad = \frac{2A\sqrt{1-(A+\delta)^2}}{1-4A\delta}. \tag{11}$$

Thus

$$x^2 - \frac{A^2\left(1-(A+\delta)^2\right)}{x^2(1-4A\delta)^2} = \frac{2A(A-\delta)}{1-4A\delta}$$

which we rewrite as

$$(1-4A\delta)^2 x^4 - 2A(A-\delta)(1-4A\delta)x^2 - A^2\left(1-(A+\delta)^2\right) = 0.$$

An elementary argument shows that $1 - 4A\delta > 0$ when $\delta < \frac{1}{3}$, and so

$$x^2 = \frac{2A(A-\delta)(1-4A\delta) \pm \sqrt{(2A(A-\delta)(1-4A\delta))^2 + 4A^2(1-4A\delta)^2\left(1-(A+\delta)^2\right)}}{2(1-4A\delta)^2}$$

$$= \frac{A(A-\delta) \pm A\sqrt{1-4A\delta}}{1-4A\delta}.$$

The product of these roots is negative since $|A+\delta| = |\cos\alpha| < 1$, and $A-\delta+\sqrt{1-4A\delta}$ is positive since $\delta < \frac{1}{3}$, so we must have

$$x^2 = \frac{A\left(A-\delta\right)+A\sqrt{1-4A\delta}}{1-4A\delta}$$

which gives

$$x = \pm\sqrt{\frac{A\left(A-\delta\right)+A\sqrt{1-4A\delta}}{1-4A\delta}}. \tag{12}$$

Substituting x in (11), we obtain

$$y = \pm\sqrt{\frac{A\left(1-\left(A+\delta\right)^2\right)}{\left(1-4A\delta\right)\left(\left(A-\delta\right)+\sqrt{1-4A\delta}\right)}}, \tag{13}$$

and the possible values of $\zeta = x+iy$ are determined. It follows from (8) that

$$\rho = \rho\left(\alpha,\delta\right) = \sqrt{\left(1-x\right)^2+y^2}$$

where x and y are the positive alternatives in (12) and (13). Thus

$$\begin{aligned}|\rho|^2 &= \left[\frac{1-\left[A\left(A-\delta\right)+A\sqrt{1-4A\delta}\right]^{\frac{1}{2}}}{1-4A\delta}\right]^2 + \frac{A\left(1-\left(A+\delta\right)^2\right)}{\left(1-4A\delta\right)\left[\left(A-\delta\right)+\sqrt{1-4A\delta}\right]} \\ &= 1+2\left[\frac{A}{1-4A\delta}\right]^{\frac{1}{2}}\left[\sqrt{A}-\left(A-\delta+\sqrt{1-4A\delta}\right)^{\frac{1}{2}}\right]\end{aligned}$$

as required. The expression in the second bracket is negative when $\delta < \frac{1}{3}$, so we have $\rho < 1$. This completes the proof.

We note that the restriction on δ in Theorem 3(ii) of the theorem is inevitable, since the function g used in the proof to obtain upper bound for $R_{cv}\left[G\left(\alpha,\delta\right)\right]$ is actually convex in the case $\alpha = 0$, $\delta = \frac{1}{2}$. The following result is obtained by using a different choice of$g$.

Theorem 4. $R_{cv}\left[G\left(\alpha,\delta\right)\right] \leq \left(\frac{1}{nA}\right)^{1/(n-1)}$ *where n is the smallest integer satisfying $n \geq 2, n > \frac{1}{A}$ .This result is sharp.*

The proof is immediate once we have proved

Lemma 5. *The class $G\left(\alpha,\delta\right)$ contains the function*

$$g\left(z\right) = z + \left(\frac{A}{n}\right)z^n \quad \left(z \in D\right)$$

where $n \geq 2$ is an integer, and the radius of the largest disc in which g is convex is

$$\left(\frac{1}{nA}\right)^{1/(n-1)},$$

where $n > \frac{1}{A}$. This result is sharp. Proof of the lemma. Let $g(z) = z + \left(\frac{A}{n}\right) z^n \quad (z \in D)$. Then $g'(z) = 1 + Az^{n-1}$ and so

$$Re\left[e^{i\alpha} g'(z)\right] = Re\left[e^{i\alpha}\left(1 + Az^{n-1}\right)\right]$$
$$= \cos\alpha + ARe\left(e^{i\alpha} z^{n-1}\right)$$
$$> \cos\alpha - A$$
$$= \delta.$$

Hence $g \in G(\alpha, \delta)$. Now $g''(z) = (n-1)Az^{n-2}$ which gives

$$1 + \frac{zg''(z)}{g'(z)} = 1 + \frac{z(n-1)Az^{n-2}}{1 + Az^{n-1}} \quad = n + \frac{1-n}{1 + Az^{n-1}}.$$

So,

$$Re\left[1 + \frac{zg''(z)}{g'(z)}\right] > 0$$

if and only if

$$Re\left[n + \frac{1-n}{1 + Az^{n-1}}\right] > 0,$$

and simplifying the above inequality , we have

$$(n-1)\left(1 + Re\overline{Az^{n-1}}\right) < n\left|1 + Az^{n-1}\right|^2,$$

$$(n-1) + (n-1)Re\overline{Az^{n-1}} < n\left[1 + 2Re\overline{Az^{n-1}} + \left|Az^{n-1}\right|^2\right],$$

$$n\left|Az^{n-1}\right|^2 + (n+1)Re\overline{Az^{n-1}} + 1 > 0.$$

This inequality is true on $|z| = \rho$ if and only if $n\left(A\rho^{n-1} - 1\right)^2 - (n+1)A\rho^{n-1} + 1 \geq 0$ or $\left(nA\rho^{n-1} - 1\right)\left(A\rho^{n-1} - 1\right) \geq 0$, and when $n > \frac{1}{A}$ it follows that g is convex on $|z| < \left(\frac{1}{nA}\right)^{\frac{1}{(n-1)}}$, but not on any larger disc.

Following Silverman and Silvia [6], we next consider polynomials of degree n in $G(\alpha, \delta)$. We find sufficient conditions in terms of α, δ, n for these to be convex. **Theorem 6.** *Suppose $p_n(z) = z + \sum_{k=2}^{n} c_k z^k$ is in $G(\alpha, \delta)$. Then p_n is convex if $\cos\alpha \leq \frac{1}{(n+2)(n-1)} + \delta$. This result is sharp.*

Proof. Let $f(z) = z + \sum_{k=2}^{n} c_k z^k$. Then by a result of Kobori [4], $f$is convex if $\sum_{k=2}^{\alpha} k^2 |a_k| \leq 1$ for the p_n in the statement. We have, using the result on coefficient bound of $G(\alpha, \delta)$ in Daud [1],

$$\sum_{k=2}^{n} k^2 |c_k| \leq \sum_{k=2}^{n} 2k(\cos\alpha - \delta)$$
$$= 2\left[\frac{n(n-1)}{2} - 1\right](\cos\alpha - \delta)$$
$$\leq 1$$

if $\cos\alpha \leq \frac{1}{(n+2)(n-1)} + \delta$. This completes the proof of the theorem.

The function $f_1(z) = z$ is a member of each of the classes$G(\alpha,\delta)$. In fact f_1 is the only function with this property. To see that this is the case, suppose that some other function g has the same property, so that g' takes a value$\lambda \neq 1$. Consider the line in the Argand diagram containing λ and 1. Rotate this line about any point between λ and 1 so that in its new position it contains neither of these points. The line now separates the plane into two half planes, and one of these contains 1 but does not containλ. The plane can be described as the collection of points w for which $Ree^{i\alpha}w > \delta$ and since w may be taken as 1, we have$\cos\alpha > \delta$. For this α andδ, any function in $G(\alpha,\delta)$ fails to take value λ, and so is not in $G(\alpha,\delta)$. This is the required contradiction.

Theorem 7. *(i) Let*$\varepsilon > 0$*, and let* $F_\in$ *be the family of classes* $G(\alpha,\delta)$ *for which*$\cos\alpha \geq \delta + \varepsilon$*. Then* $\cap F_\in$ *contains a function, which is not convex.*

(ii) Let F_0 *be any family of classes* $G(\alpha,\delta)$ *with the property that, for any*$\varepsilon > 0$*, there exists a member* $G(\alpha,\delta)$ *of* F_0 *with*$\cos\alpha < \delta + \varepsilon$*. Then* $\cap F_0$ *contains only the function*$f_1(z) = z,\ (z \in D)$*.*

As an immediate corollary to the theorem we have the following result that answers the above question in the case of convexity.

Corollary 8. *Let* F *be the intersection of any family of classes*$G(\alpha,\delta)$*. Then either* F *contains just the identity function*$f_1(z) = z$*, or* F *contains a function which is not convex.*

Part (i) of the theorem was proved by Silverman and Silvia [6] in the case$|\alpha| < \frac{\pi}{2} - \varepsilon,\ |\alpha| \leq \frac{\pi}{2}$. Our proof is rather more geometrical than theirs, using the following property of any half plane $Ree^{i\alpha}w > \delta$ in the w-Argand diagram with$|\alpha| \leq \pi,\ \cos\alpha > \delta$: the largest radius of any circle centered at 1 whose interior is a subset of the half plane, is $\cos\alpha > \delta$.

Proof of theorem.

(i) Let $f(z) = z + \lambda z^n$. A slight modification to the last part of proof of Lemma 5 shows that f is convex if and only if $|\lambda| < \frac{1}{n^2}$. Choose n so that $\varepsilon > \frac{1}{n}$ and then λ so that $\frac{1}{n} < n\lambda < \varepsilon$. Then $\lambda > \frac{1}{n^2}$ and f is not convex. Also$f'(z) = 1 + n\lambda z^{n-1}$, so that f' maps D onto a circle centered at 1 and radiusε, and $f \in G(\alpha,\delta)$ whenever$\cos\alpha - \delta > \varepsilon$.

(ii) Suppose that $\cap F_0$ contains a function $g \neq f_1$, so that, by the open mapping theorem, the interior of some disc centered at 1 is a subset of $g'(D)$. Let the radius of this disc be d, and let $G(\alpha,\delta)$ be a class in F_0 for which $\cos\alpha - \delta < d$. The derivative of any function in $G(\alpha,\delta)$ maps D into the half plane $Ree^{i\alpha}w > \delta$ and the radius of the largest circle, centred at 1, inside this half plane, is $\cos\alpha - \delta$. This implies $g \notin \cap F_0$, which contradicts our assumption. So $\cap F_0$ contains only f_1.

4 Conclusion

This paper extends some of the results given by Silverman and Silvia [6] and Daud [1] on bounds for $R_{cv}[G(\alpha,\delta)]$, the radius of convexity of $G(\alpha,\delta)$. Using the approach by Silverman [7] and Goel and Mehrok [3], we extend the results when δtaken as any negative values in Daud and Shaharuddin [2]. With this results, we emphasis on using the extreme values function to look on the consideration of convexity and starlikeness of the class.

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