

## Wavelet Solutions of Parabolic Equations

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**Abstract** Gronwall inequality based technique of Mattos Lopes [3] for frequency domain and wavelet-Galerkin solution in scaling space of the standard parabolic problem has been developed and extended to analyze non standard Cauchy problem for parabolic heat conduction.

**Keywords** Wavelet-Galerkin Technique; Inverse Ill-posed Problem; Cauchy Problem; Sideways Heat Equation; Gronwall Inequality.

### 1 Introduction

In a number of industrial problems, one needs to find the surface temperature when the surface itself is inaccessible for measurement. In such cases, on the basis of internal measurement at a fixed location inside the body, the surface temperature is computed. The situation can just be reverse, the problem is then called *inverse* one; the example can be of determining temperature on both sides of a thick wall if one side is inaccessible to measurement. The situation can be moduled as to determine the temperature  $u(x, t) \in L^2(0, \infty)$  for  $0 \leq x < 1$  from temperature measurement  $g(t) = u(1, t) \in L^2(0, \infty)$  at  $x = 1$  such that it satisfies

$$\begin{cases} u_{xx} = u_t, & 0 \leq x < 1, t \geq 0 \\ u(x, 0) = 0, & 0 \leq x \leq 1 \\ u(1, t) = g(t), & t \geq 0, u|_{x \rightarrow \infty} \text{ bounded} \end{cases} \quad (1)$$

Equation (1) is called the *Sideways Heat Equation*, and is an example of an ill-posed Cauchy problem, i.e., solution  $u(x, t)$  if exists, does not depend continuously on the initial data: a small disturbance in data may cause a dramatically large error in the solution  $u(x, t)$  for  $0 \leq x < 1$ . Extend  $u(x, t)$  and  $g(t)$  to the whole of  $t$ -axis by defining

$$u(x, t), g(t), F(t) = u(0, t) \in L^2(R)$$

to be zero for  $t < 0$ . Ill-posedness of (1) is due to increasing of  $\hat{g}_\varepsilon(\omega)$  in the frequency domain for noisy temperature  $g_\varepsilon(t)$ , where  $g_\varepsilon \in L^2(R)$  is the measured data with data error

$$\|g - g_\varepsilon\| \leq \varepsilon \quad (2)$$

for some constant  $\varepsilon > 0$ .

Impose *a priori* bound on the solution at  $x = 0$ , i.e.,

$$\|u(0, t)\| \leq M. \quad (3)$$

The problem (1) has the stability in the sense that any two solutions  $u$  and  $\hat{u}$  of (1) satisfies [1]

$$\|u(x, t) - \hat{u}(x, t)\| \leq 2M^{1-x}\varepsilon^x, 0 \leq x < 1.$$

The problem (1) with conditions (2) and (3) was approximated for the first time by Reginska [4] by multiscale analysis and wavelet techniques of measured data. For stability results and optimal error estimates one can refer to Carasso [1], Elden et al. [2], and Seidman and Elden. [5].

The frequency space solution  $\hat{u}(x, \omega) \in L^2(\mathbb{R})$  of (1) is given by

$$\hat{u}(x, \omega) = e^{(1-x)\sqrt{i\omega}} \hat{g}(\omega), \quad (4)$$

where  $\sqrt{i\omega} = \sqrt{|\omega|}/2$  is the principal square-root of  $i\omega$ . Also  $\hat{f}(\omega) = \hat{u}(0, \omega) = e^{\sqrt{i\omega}} \hat{g}(\omega)$ .

Since  $\sqrt{i\omega}$  tends to infinity as  $|\omega| \rightarrow \infty$ , the problem thus is ill-posed. Further, the existence of the solution in  $L^2(\mathbb{R})$  depends on fast decay of  $\hat{g}_\varepsilon$  at high frequencies.

The solution  $u(x, t)$  to (1) is

$$u(x, t) = \int_{-\infty}^{\infty} e^{i\omega t} e^{(1-x)\sqrt{i\omega}} \hat{g}(\omega) d\omega. \quad (5)$$

By Parseval formula,

$$\|u(x, t)\|^2 = \|\hat{u}(x, \omega)\|^2 = \int_{-\infty}^{\infty} e^{(1-x)\sqrt{2|\omega|}} |\hat{g}(\omega)|^2 d\omega.$$

This shows the rapid decay of  $\hat{g}(\omega)$  at high frequencies, i.e. as  $\omega \rightarrow \infty$ . "If the initial data  $g(t)$  is noisy, the Fourier transform (Ft)  $\hat{g}(\omega)$  will have high frequency components" (Vani et al. [6]).

The Meyer scaling function  $\varphi$  and wavelet function  $\psi$  are well localized, i.e., they have compact support in frequency (Ft) domain (and not in time-domain) and decay very fast. "The orthogonal projections onto Meyer scaling (and wavelet) spaces can be considered as low-pass filter, filtering off the high frequencies" [3]. In other words, "the high frequency components, which normally present data error, are filtered away by expanding the data function in wavelet basis" (Vani et al. [6]).

Another version of (1) as given in Mattos and Lopes[3], is

$$\begin{aligned} k(x)u_{xx}(x, t) &= u_t(x, t), \quad t \geq 0, \quad 0 \leq x < 1 \quad [0 < \alpha \leq k(x) < \infty] \\ u(0, t) &= g(t), \quad u_x(0, t) = 0 \end{aligned}$$

where  $k(x)$  is assumed to be smooth.

Literature finds yet another type of Cauchy problem in non standard parabolic equation form [2]:

$$\begin{aligned} (k(x)u_x(x, t))_x &= u_t(x, t), \quad t \geq 0, \quad 0 \leq x < 1 \\ u(0, t) &= g(t), \quad u_x(0, t) = 0, \quad u(x, 0) = 0. \end{aligned}$$

where coefficient  $k(x)$  satisfies  $0 < \alpha \leq k(x) < \infty$ .  $\square$

## 2 Wavelet-Galerkin Solution of Non-Standard Parabolic Equations

**Theorem 1.** (Mattos and Lopes, [3], pp. 216-217) The operator  $D_j(x)$  defined by

$$[(D_j)_{lk}(x)]_{l,k \in Z} = \left[ \frac{1}{k(x)} \langle \varphi'_{jl}, \varphi_{jk} \rangle \right]_{l,k \in Z}$$

satisfies the following three conditions:

- (i)  $(D_j)_{lk} = -(D_j)_{kl}$ , i.e., matrix  $(D_j)$  is skew symmetric
- (ii)  $(D_j)_{lk} = (D_j)_{(l-k)0}$ , i.e.,  $(D_j)_{lk}$  are equal along diagonals
- (iii)  $\|D_j(x)\| \leq \frac{\pi 2^j}{k(x)}$ .

The theorem below is a version of Gronwall inequality.

**Theorem 2.** (Mattos and Lopes, [3], pp. 215-216). Let  $u$  and  $v$  be positive continuous functions,  $x \geq a$  and  $c > 0$ . If

$$u(x) = c + \int_a^x \int_a^s v(\tau) u(\tau) d\tau ds.$$

then

$$u(x) \leq c \exp \left( \int_a^x \int_a^s v(\tau) d\tau ds \right).$$

K

We state and prove the following theorem and find the solution of (9) in frequency domain as well as in scaling spaces:

**Theorem 3.** Let  $W(x)$  be continuous functions,  $x \geq 0$ ,  $\gamma = W(0) > 0$ , and  $k'(x)$  is the derivative of  $k(x)$ . If

$$W(x) = \gamma + \int_0^x \int_0^s \left[ \frac{l}{k(\tau)} W(\tau) - \frac{k'(\tau)}{k(\tau)} W'(\tau) \right] d\tau ds. \quad (6)$$

then

$$W(x) \leq \gamma \exp \int_0^x \int_0^s \left[ \frac{l}{k(\tau)} + \frac{k'^2(\tau)}{4k^2(\tau)} \right] d\tau ds. \quad (7)$$

*Proof.* From (6),

$$W'' = \frac{lW}{k} - \frac{k'}{k} W' \Rightarrow \frac{W''}{W} + \frac{k'}{k} \frac{W'}{W} = \frac{l}{k}. \quad (8)$$

Now  $\left(\frac{W'}{W}\right)' = \frac{WW'' - W'^2}{W^2} = \frac{W''}{W} - \left(\frac{W'}{W}\right)^2$  implies

$$\frac{W''}{W} = \left(\frac{W'}{W}\right)' + \left(\frac{W'}{W}\right)^2.$$

Therefore,

$$\frac{W''}{W} + \frac{k'}{k} \frac{W'}{W} = \left(\frac{W'}{W}\right)' + \left(\frac{W'}{W}\right)^2 + \frac{k'}{k} \frac{W'}{W} = \left[\frac{W'}{W} + \frac{1}{2k}\right]^2 + \left(\frac{W'}{W}\right)' - \frac{k'^2}{4k^2} \geq \left(\frac{W'}{W}\right)' - \frac{k'^2}{4k^2}.$$

Thus, from (8),

$$\left(\frac{W'}{W}\right)' \leq \frac{l}{k} + \frac{k'^2}{4k^2}.$$

This implies

$$\frac{W'}{W} \leq \int_0^x \left[ \frac{l}{k} + \frac{k'^2}{4k^2} \right] d\tau.$$

Integrating,

$$\begin{aligned} \ln W(x) - \ln \gamma &\leq \int_0^x \int_0^s \left[ \frac{l}{k(\tau)} + \frac{k'^2}{4k^2(\tau)} \right] d\tau ds, \\ \text{i.e., } \ln \frac{W(x)}{\gamma} &\leq \int_0^x \int_0^s \left[ \frac{l}{k(\tau)} + \frac{k'^2}{4k^2(\tau)} \right] d\tau ds. \end{aligned}$$

This implies

$$W(x) \leq \gamma \exp \int_0^x \int_0^s \left[ \frac{l}{k(\tau)} + \frac{k'^2}{4k^2(\tau)} \right] d\tau ds. \quad \square$$

Consider the following heat conduction problem

$$\begin{cases} k(x)u_{xx}(x, t) + k'u_x(x, t) - u_t(x, t) = 0, & t \geq 0, 0 \leq x < 1 \\ u(0, t) = g(t) \\ u_x(0, t) = 0 \end{cases} \quad (9)$$

for  $0 < \alpha \leq k(x) \leq \beta < \infty$ ,  $0 < \delta \leq k'(x) \leq \vartheta < \infty$ .

## 2.1 Solution of (9) in Frequency Domain

$$\begin{cases} k(x)\hat{u}_{xx}(x, \omega) + k'(x)\hat{u}_x - i\omega\hat{u}(x, \omega) = 0, & \omega \in R, 0 \leq x < 1 \\ \hat{u}(0, \omega) = \hat{g}(\omega) \\ \hat{u}_x(0, \omega) = 0 \end{cases} \quad (10)$$

$$\hat{u}'' = i\frac{\omega}{k}\hat{u} - \frac{1}{k}\hat{u}'. \quad (11)$$

$$\hat{u}(x, \omega) = \hat{g}(\omega) + \int_0^x \int_0^s \left[ \frac{i\omega}{k(\tau)}\hat{u}(\tau, \omega) - \frac{k'(\tau)}{k(\tau)}\frac{d}{dx}\hat{u}(\tau, \omega) \right] d\tau ds. \quad (12)$$

Also from (12),

$$|\hat{u}(x, \omega)| \leq |\hat{g}(\omega)| + \int_0^x \int_0^s \left| i\frac{\omega}{k(\tau)}\hat{u}(\tau, \omega) - \frac{k'(\tau)}{k(\tau)}\frac{d}{dx}\hat{u}(\tau, \omega) \right| d\tau ds.$$

From Theorem 3

$$|\hat{u}(x, \omega)| \leq |\hat{g}(\omega)| \exp \left[ \int_0^x \int_0^s \left[ \frac{1}{k(\tau)} |\omega| + \frac{k'^2(\tau)}{4k^2(\tau)} \right] d\tau ds \right].$$

## 2.2 Galerkin Solution in Scaling Spaces

Approximating solution of (9) in scaling spaces

$$\begin{aligned}
& \langle k(x)u_{xx} + k'(x)u_x - u_t, \varphi_{jk} \rangle = 0 \\
& \langle u(0, t), \varphi_{jk} \rangle = \langle P_j g, \varphi_{jk} \rangle \\
& \langle u_x(0, t), \varphi_{jk} \rangle = \langle 0, \varphi_{jk} \rangle, \quad k \in Z
\end{aligned} \tag{13}$$

where  $\varphi_{jk}$  is the orthonormal basis of  $V_j$  given by the scaling function  $\varphi$ .

Let the approximate solution be  $u_j(x, t) \in V_j$  be

$$u_j(x, t) = \sum_{l \in Z} W_l(x) \varphi_{jl}(t).$$

From (13)

$$\langle k(x) \sum_{l \in z} W_l''(x) \varphi_{jl}(t) + k'(x) \sum_{l \in Z} W_l'(x) \varphi_{jl} - \sum_{l \in z} W_l(x) \varphi'_{jl}(t), \varphi_{jk} \rangle = 0.$$

That is,

$$\begin{aligned}
& k(x)W_k'' + k'(x)W_k'(x) - \sum_l W_l(x) \langle \varphi'_{jl}, \varphi_{jk} \rangle = 0 \quad \text{or} \\
& \frac{d^2 W_k}{dx^2} + \frac{k'(x)}{k(x)} W_k'(x) - \frac{1}{k(x)} \sum_l W_l(x) \langle \varphi'_{jl}, \varphi_{jk} \rangle = 0 \\
& \frac{d^2 W_k}{dx^2} + \frac{k'(x)}{k(x)} W_k'(x) - \sum_l W_l(x) (D_j)_{lk}(x) = 0, \quad k \in Z
\end{aligned}$$

That is we get infinite dimensional differential equation

$$\frac{d^2 W}{dx^2} + \frac{k'}{k} \frac{dW}{dx} - D_j(x)W = 0 \quad \text{with} \quad W(0) = \gamma, \quad W'(0) = 0, \tag{14}$$

where  $\gamma = P_j g = \sum_{z \in Z} \gamma_z \varphi_{jz} = \sum_{z \in Z} \langle g, \varphi_{jz} \rangle \varphi_{jz}$ .

Solution of (2.9) is analogous to the solution of (13)

$$W(x) = \gamma + \int_0^x \int_0^s \left[ D_j(\tau)W(\tau) - \frac{k'(\tau)}{k(\tau)} W'(\tau) \right] d\tau ds, \tag{15}$$

$$\|W(x)\| \leq \|\gamma\| + \int_0^x \int_0^s \left\| D_j(\tau)W(\tau) - \frac{k'(\tau)}{k(\tau)} W'(\tau) \right\| d\tau ds.$$

Thus, by Theorem 3 and using Theorem 1,

$$\|W(x)\| \leq \|\gamma\| \exp \int_0^x \int_0^s \left[ \frac{2^{-j}\pi}{k(\tau)} + \frac{k'^2(\tau)}{4k^2(\tau)} \right] d\tau ds. \tag{16}$$

### 3 Numerical Example

We consider the following test problem:

$$(x+a)^2 u_{xx}(x,t) + 2(x+a)u_x(x,t) - u_t(x,t) = 0, \quad t \geq 0, \quad 0 \leq x < 1 \quad (17)$$

where  $u(0,t) = g(t)$ ,  $u_x(0,t) = 0$ .

By taking approximating solution  $u_j(x,t) \in V_j$  of equation (17) in scaling spaces, we get an infinite dimensional differential equation

$$\frac{d^2 W}{dx^2} + \frac{2(x+a)dW}{(x+a)^2 dx} - lW = 0$$

with  $W(0) = \gamma$ ,  $W'(0) = 0$ ,  $l = 2^{-j}\pi$ , where  $j$  may large, and  $\gamma = \sum_{z \in Z} \langle g, \varphi_{jz} \rangle \varphi_{jz}$ .

Exact solution of the problem is

$$W(x) = \frac{\gamma}{\alpha - \beta} \left[ \frac{\alpha e^{\beta \log(x+a)}}{e^{\beta \log a}} - \frac{\beta e^{\alpha \log(x+a)}}{e^{\alpha \log a}} \right],$$

where

$$\alpha = \frac{-1 + \sqrt{1 + 4l}}{2} \quad \text{and} \quad \beta = \frac{-1 - \sqrt{1 + 4l}}{2}.$$

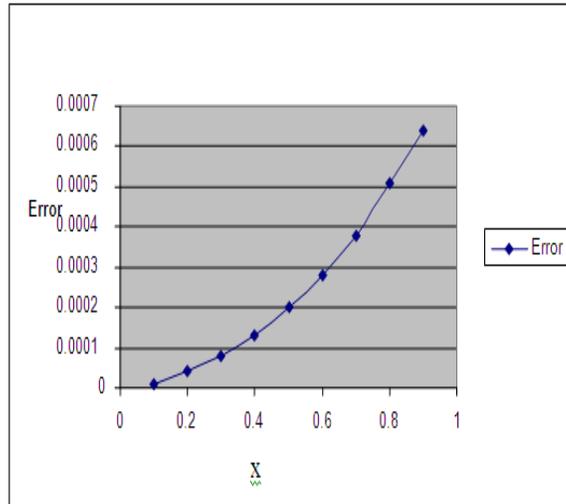
Wavelet Galerkin solution in scaling space is

$$\|W(x)\| \leq \|\gamma\| \exp(l+1) \left( \frac{x}{a} - \log(x+a) + \log a \right).$$

**Remarks:** With increasing  $a$  and  $j = 10$ , the solution is found to be better. The values of exact and approximate solution for  $a = 15, 20, 25$  and  $j = 10$  show that the errors decreases as  $a$  increases. For  $j$  below and beyond 10 the bad result is obtained.

Table 1: Errors in the solution at the values of  $a = 25, j = 10$

values	Exact Solution	Wavelet Galerkin Solution	Error
0.1	1	1.00001	0.00001
0.2	1.00001	1.00004	0.00004
0.3	1.00001	1.00008	0.00008
0.4	1.00001	1.00013	0.00012
0.5	1	1.0002	0.0002
0.6	1.00001	1.00029	0.00028
0.7	1.00001	1.00039	0.00038
0.8	1	1.00051	0.00051
0.9	1.00001	1.00064	0.00063

Figure 1: Graph of the solution at the values of  $a = 25, j = 10$ Table 2: Errors in the solutions at the values  $a = 20, j = 10$ 

Values	Exact Solution	Wavelet Galerkin Solution	Error
0.1	1	1.00001	0.00001
0.2	0.99999	1.00005	0.00004
0.3	1.00003	1.00011	0.00008
0.4	1	1.0002	0.0002
0.5	0.99998	1.00031	0.00033
0.6	1.00001	1.00044	0.00043
0.7	1	1.0006	0.0006
0.8	1	1.00078	0.00078
0.9	1	1.00098	0.00098

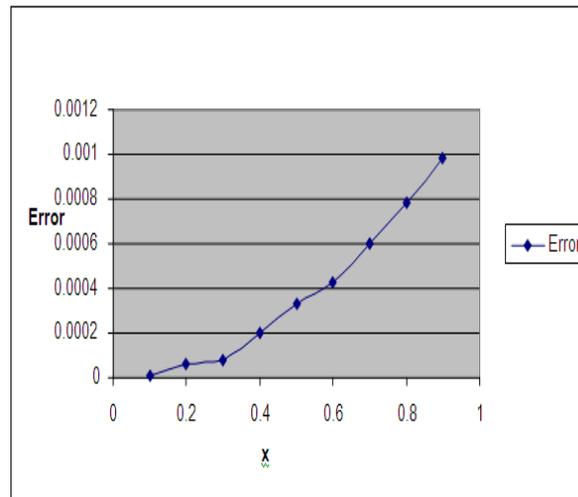


Figure 2: Graph of the solutions at the values  $a = 20, j = 10$

Table 3: Errors in the solutions at the values  $a = 15, j = 10$

Values	Exact Solution	Wavelet Galerkin Solution	Error
0.1	0.99999	1.00003	0.00004
0.2	0.99999	1.00008	0.00009
0.3	0.99999	1.0002	0.00021
0.4	1	1.00035	0.00035
0.5	1	1.00054	0.00054
0.6	1	1.00078	0.00078
0.7	1	1.00106	0.00106
0.8	1	1.00138	0.00138
0.9	1	1.00174	0.00174

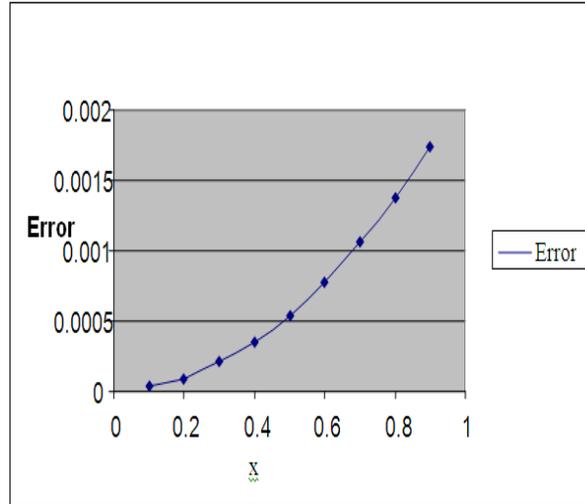


Figure 3: Graph of the solutions at the values  $a = 15, j = 10$

## 4 Conclusion

Wavelet-Galerkin method is one of the important and effective tools applied recent days in obtaining solutions of standard parabolic heat conduction problems. So far, we extended the Gronwall inequality technique of Mattos and Lopes, [3] and applied to non-standard Cauchy problem for parabolic heat conduction. For  $k > 0$ , as in the statement, the above extended technique has far reaching consequence for interval  $0 \leq x < 1$  of existence. A comparable solution to that of exact solution can easily be obtained for certain large values of  $j$ .

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