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A New Approach for Computing Zadeh's Extension Principle

[1]**M. Z. Ahmad and** [2]**M. K. Hasan**
[1]Institute for Engineering Mathematics, Universiti Malaysia Perlis
02000 Kuala Perlis, Perlis, Malaysia
[2]Department of Industrial Computing, Faculty of Information Science and Technology
Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor, Malaysia
e-mail: [1]mzaini@unimap.edu.my, [2]khatim@ftsm.ukm.my

Abstract Zadeh's extension principle is one of the most fundamental principles in fuzzy set theory. It provides a powerful technique in order to extend a real continuous function to a function accepting fuzzy sets as arguments. If the function is monotone, then the endpoints of the output can be determined quite easily. However, the difficulty arises when the function is non-monotone. In that case, the computation of the output is not an easy task. The purpose of this paper is to provide a new method to reduce this difficulty. The method is based on the implementation of optimisation technique over the α-cuts of fuzzy set. By doing so, the endpoints of the output can be approximated. The method proposed in this paper is easy to implement and can be applied to many practical applications. Several examples are given to illustrate the effectiveness of the proposed method.

Keywords Continuous Function, Fuzzy Set, Optimisation, Zadeh's Extension Principle.

1 Introduction

Fuzzy set theory was formalised by Zadeh in his paper work entitled "Fuzzy Sets" [15]. Since its introduction in 1965, the theory of fuzzy sets has been deeply developed and it has influenced in many fields of application. One of the most important tools in fuzzy set theory is Zadeh's extension principle. It enables to extend any real continuous function to a function accepting fuzzy set as its arguments. If the function is monotone then the extending process is straightforward, i.e. by computing the endpoints of the α-cuts. However, if the function is non-monotone, then the extending process becomes more complicated. This is because the function values at the endpoints of the α-cuts are not the correct endpoints of the output.

There are several methods proposed in the literature to compute real continuous function accepting fuzzy sets as its arguments. For example, fuzzy arithmetic based on LR-fuzzy numbers [4] and interval arithmetic [6]. The latter method, however, shows a serious problem since it has overestimation in computation. This is because the same fuzzy set in fuzzy arithmetic is computed separately even it occurs multiple times in a single expression. In order to avoid this problem, Dong and Wong [3] have proposed a new method called fuzzy weighted average. However, the proposed method is only applicable for monotone functions. For non-monotone functions, the method has been extended by Wood et al. [14], which consists an additional method to locate extreme points. Usually, the extreme points of non-monotone functions are obtained by either analytically or numerically, depending on the functions under consideration. However, in general, this is not an easy task as well. It requires a global optimisation technique. Until now, there is no efficient method to solve global optimisation with low computational complexity. Klir [7] have suggested a different approach to reduce overestimation in the standard fuzzy arithmetic. He proposed a theoretical framework called constrained fuzzy arithmetic, which takes into account the dependencies problem among fuzzy sets. Some new techniques can be found in the literature such as the transformation method [5], the L-U representation [12] and the spline approximation method [2]. However, these proposed methods increased computational complexity when applied to non-monotone functions as well. To overcome these problems, a new high accuracy technique with low computational complexity need to be investigated.

This paper is organised as follows: in Section 2, we recall some basic definitions and theoretical background we need throughout this paper. In Section 3, we present a new strategy for computing Zadeh's extension principle, which is the main contribution in this paper. In Section 4, several examples are given to show the capability of our proposed method. Finally, it is followed by a conclusion in Section 5.

2 Preliminaries

In the following, we recall some basic definitions and theoretical background we need throughout this paper.

2.1 Basic notion of fuzzy sets

Definition 1. *Let S be a nonempty set called universe. A fuzzy set U in S is a function from the universe S to the unit interval $[0,1]$ that maps an element s to $U(s)$*

$$U : S \to [0,1]\,, \tag{1}$$

where $U(s)$ is the degree of membership of s in U. The nearer the value of $U(s)$ to 1, the higher the degree of membership of s in U. In contrast, the nearer the value of $U(s)$ to 0, the lower the degree of membership of s in U.

Definition 2. *Let U be a fuzzy set defined on $\mathbb{R}$. The support of U is the crisp set of all points on $\mathbb{R}$ such that the degree of membership of U is non-zero, that is*

$$supp(U) = \{s \in \mathbb{R} \mid U(s) > 0\}\ .$$

Definition 3. *Let U be a fuzzy set defined on $\mathbb{R}$. The core of U is the crisp set of all points on $\mathbb{R}$ such that the degree of membership of U is 1, that is*

$$core(U) = \{s \in \mathbb{R} \mid U(s) = 1\}\ .$$

Definition 4. *Let U be a fuzzy set on $\mathbb{R}$. U is called a fuzzy interval if:*

(i) *U is normal: there exists $s_0 \in \mathbb{R}$ such that $U(s_0) = 1$;*

(ii) *U is convex: for all $s, t \in \mathbb{R}$ and $0 \leq \lambda \leq 1$, it holds that*

$$U(\lambda s + (1-\lambda)t) \geq \min\left(U(s), U(t)\right);$$

(iii) *U is upper semi-continuous: for any $s_0 \in \mathbb{R}$, it holds that*

$$U(s_0) \geq \lim_{s \to s_0^{\pm}} U(s);$$

(iv) *$[U]^0 = \overline{\{s \in \mathbb{R} \mid U(s) > 0\}}$ is a compact subset of $\mathbb{R}$.*

The α-cut of a fuzzy interval U, with $0 < \alpha \leq 1$ is the crisp set

$$[U]^\alpha = \{s \in \mathbb{R} \mid U(s) \geq \alpha\} \ .$$

For a fuzzy interval U, its α-cuts are closed intervals in $\mathbb{R}$. Let denote them by

$$[U]^\alpha = [u_1^\alpha, u_2^\alpha] \, .$$

Definition 5. *A fuzzy interval U is called a triangular fuzzy interval if its membership function has the following form:*

$$U(s) = \begin{cases} 0 & , \textit{if } s < a \,, \\ \frac{s-a}{b-a} & , \textit{if } a \leq s \leq b \,, \\ \frac{c-s}{c-b} & , \textit{if } b \leq s \leq c \,, \\ 0 & , \textit{if } s > c \,, \end{cases}$$

and its α-cuts are simply $[U]^\alpha = [a + \alpha(b-a), c - \alpha(c-b)]$, $\alpha \in (0, 1]$.

In this paper, the set of all fuzzy intervals is denoted by $\mathcal{F}(\mathbb{R})$.

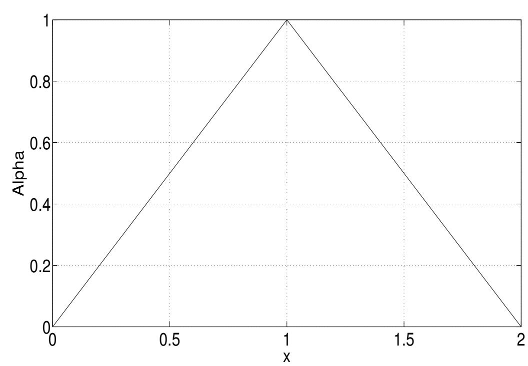

Figure 1: Triangular fuzzy interval $U(0, 1, 2)$.

2.2 Fuzzy interval arithmetic

Consider two fuzzy intervals U and V. The fuzzy intervals U and V can be decomposed into the sets of intervals I_1 and I_2, respectively with

$$I_1 = \left\{[u_1^{\alpha_1}, u_2^{\alpha_1}], [u_1^{\alpha_2}, u_2^{\alpha_2}], ..., [u_1^{\alpha_j}, u_2^{\alpha_j}], ..., [u_1^{\alpha_n}, u_2^{\alpha_n}]\right\} \ , \quad j = 1, 2, ..., n \, , \tag{2}$$

$$I_2 = \left\{[v_1^{\alpha_1}, v_2^{\alpha_1}], [v_1^{\alpha_2}, v_2^{\alpha_2}], ..., [v_1^{\alpha_j}, v_2^{\alpha_j}], ..., [v_1^{\alpha_n}, v_2^{\alpha_n}]\right\}, \quad j = 1, 2, ..., n. \tag{3}$$

The basic arithmetic operations of U and V can be defined by applying interval arithmetic [9] seperately to each $\alpha_j, j = 1, 2, ..., n$, in the following ways:

(i) Addition : $[U + V]^{\alpha_j} = [u_1^{\alpha_j} + v_1^{\alpha_j}, u_2^{\alpha_j} + v_2^{\alpha_j}]$,

(ii) Subtraction : $[U - V]^{\alpha_j} = [u_1^{\alpha_j} - v_2^{\alpha_j}, u_2^{\alpha_j} - v_1^{\alpha_j}]$,

(iii) Multiplication : $[U \times V]^{\alpha_j} = [c_1^{\alpha_j}, c_2^{\alpha_j}]$, where

$$c_1^{\alpha_j} = \min\left(u_1^{\alpha_j} v_1^{\alpha_j}, u_1^{\alpha_j} v_2^{\alpha_j}, u_2^{\alpha_j} v_1^{\alpha_j}, u_2^{\alpha_j} v_2^{\alpha_j}\right),$$

$$c_2^{\alpha_j} = \max\left(u_1^{\alpha_j} v_1^{\alpha_j}, u_1^{\alpha_j} v_2^{\alpha_j}, u_2^{\alpha_j} v_1^{\alpha_j}, u_2^{\alpha_j} v_2^{\alpha_j}\right).$$

(iv) Division : $\left[\frac{U}{V}\right]^{\alpha_j} = [d_1^{\alpha_j}, d_2^{\alpha_j}]$, with $0 \notin [V]^0$, where

$$d_1^{\alpha_j} = \min\left(\frac{u_1^{\alpha_j}}{v_2^{\alpha_j}}, \frac{u_1^{\alpha_j}}{v_1^{\alpha_j}}, \frac{u_2^{\alpha_j}}{v_2^{\alpha_j}}, \frac{u_2^{\alpha_j}}{v_1^{\alpha_j}}\right),$$

$$d_2^{\alpha_j} = \max\left(\frac{u_1^{\alpha_j}}{v_2^{\alpha_j}}, \frac{u_1^{\alpha_j}}{v_1^{\alpha_j}}, \frac{u_2^{\alpha_j}}{v_2^{\alpha_j}}, \frac{u_2^{\alpha_j}}{v_1^{\alpha_j}}\right).$$

2.3 Zadeh's extension principle

In [15], Zadeh proposed the so-called extension principle which becomes an important tool in fuzzy set theory and its applications. The principal idea of Zadeh's extension principle is that each function $f : X \to Y$ induces another function $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(Y)$ defined for each fuzzy interval U in X by

$$\hat{f}(U)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} U(x) & , \text{if } y \in \text{range}(f), \\ 0 & , \text{if } y \notin \text{range}(f). \end{cases} \tag{4}$$

In this case, the function $\hat{f}$ is said to be obtained from f by Zadeh's extension principle. In general, the computation of $\hat{f}$ is not an easy task. An exception occurs when f is monotone. For example, if $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = e^x,$$

and the fuzzy interval U defined by

$$U(x) = \begin{cases} 0 & , \text{if } x < 0, \\ x & , \text{if } 0 \leq x \leq 1, \\ -x + 2 & , \text{if } 1 \leq x \leq 2, \\ 0 & , \text{if } x > 2, \end{cases} \tag{5}$$

then we can easily obtain $\hat{f} : \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$ as follow:

$$\hat{f}(U)(y) = \begin{cases} 0 & \text{, if } y < 1\,, \\ \log(y) & \text{, if } 1 \le y \le e^1\,, \\ -\log(y) + 2 & \text{, if } e^1 \le y \le e^2\,, \\ 0 & \text{, if } y > e^2\,. \end{cases} \tag{6}$$

Furthermore, in [11] if $f : \mathbb{R} \to \mathbb{R}$ is a real continuous function, then $\hat{f} : \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$ is well-defined function and

$$[\hat{f}(U)]^\alpha = f([U]^\alpha), \quad \forall \alpha \in [0,1], \quad U \in \mathcal{F}(\mathbb{R})\,, \tag{7}$$

where $f([U]^\alpha) = \{f(x) \mid x \in [U]^\alpha\}$. Consequently, if U is a fuzzy interval with the closure of its support is $[U]^0 = [u_1, u_2]$ and f is a real continuous function, then we have that

$$\left[\hat{f}(U)\right]^0 = \left[\min_{x \in [u_1,u_2]} f(x), \max_{x \in [u_1,u_2]} f(x)\right]. \tag{8}$$

Therefore, in order to find the endpoints of (8), we need optimisation technique.

Definition 6. *Let $f : \mathbb{R} \to \mathbb{R}$ be a real continuous function defined on $[u_1, u_2]$. An optimisation problem is concerned with finding the minimum and maximum values of f on the interval $[u_1, u_2]$.*

If the problem seeks a minimum value for f on the interval $[u_1, u_2]$, then the problem is simply called a minimisation problem and is denoted by

$$b_{\min} = \min_{x \in [u_1,u_2]} f(x)\,. \tag{9}$$

If the problem seeks a maximum value for f on the interval $[u_1, u_2]$, then the problem is simply called a maximisation problem and is denoted by

$$b_{\max} = \max_{x \in [u_1,u_2]} f(x)\,. \tag{10}$$

Definition 7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a real continuous function defined on $[u_1, u_2]$. We say that*

(i) *f attains a global minimum at $x_{\min}$ if $f(x_{\min}) \le f(x)$ for all $x \in [u_1, u_2]$,*

(ii) *f attains a global maximum at $x_{\max}$ if $f(x_{\max}) \ge f(x)$ for all $x \in [u_1, u_2]$.*

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a real continuous function defined on $[u_1, u_2]$. Define the function $-f : \mathbb{R} \to \mathbb{R}$ so that its value for any point $x \in [u_1, u_2]$ is $-f$. Then*

(i) *$x_{\min}$ is a global minimum of f on the interval $[u_1, u_2]$ if and only if it is a global maximum of $-f$ on the interval $[u_1, u_2]$,*

(ii) *$x_{\max}$ is a global maximum of f on the interval $[u_1, u_2]$ if and only if it is a global minimum of $-f$ on the interval $[u_1, u_2]$.*

Proof. See [13]. □

3 α-Discretisation and Interval Partitioning

Let U be a fuzzy interval and its α-cuts are denoted by $[U]^\alpha = [u_1^\alpha, u_2^\alpha]$. First, we discretise $\alpha \in [0,1]$ up to n points. These points are equally spaced, that is the points spacing is $h = 1/n$. After discretisation, we have a set of α as follows:

$$\alpha = \{\alpha_1, \alpha_2, ..., \alpha_{i-1}, \alpha_i, \alpha_{i+1}, ..., \alpha_{n-1}, \alpha_n\} \ , \tag{11}$$

where $\alpha_1 = 0$, $\alpha_n = 1$ and $i = 2, ..., n-1$. From (11), we have the following set of α-cuts:

$$I = \{[U]^{\alpha_1}, [U]^{\alpha_2}, ..., [U]^{\alpha_{i-1}}, [U]^{\alpha_i}, [U]^{\alpha_{i+1}}, ..., [U]^{\alpha_{n-1}}, [U]^{\alpha_n}\} \ . \tag{12}$$

For the different α-cuts of U the following property holds [8]:

$$[U]^{\alpha_1} \supseteq [U]^{\alpha_2} \supseteq ... \supseteq [U]^{\alpha_{i-1}} \supseteq [U]^{\alpha_i} \supseteq [U]^{\alpha_{i+1}} \supseteq ... \supseteq [U]^{\alpha_{n-1}} \supseteq [U]^{\alpha_n} \ , \tag{13}$$

for $i = 2, ..., n-1$. From (13), it is clear that one α-cut includes another α-cut. So, the diameter of the α-cuts becomes smaller and smaller, and finally converges to a single point as α approaches to 1. In contrary, the diameter of the α-cuts becomes wider and wider as α approaches to 0 (see Fig. 2).

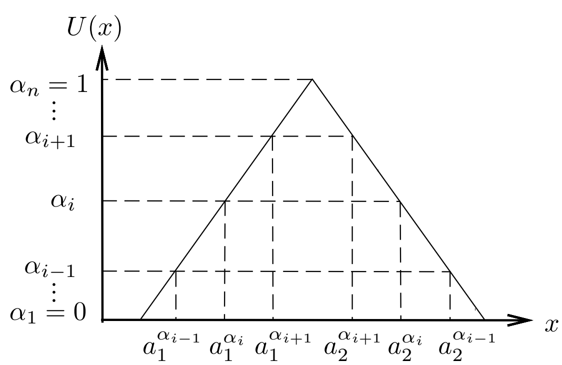

Figure 2: α-discretisation and interval partitioning.

Since one α-cut includes another α-cut, $[U]^{\alpha}$ can be constructed as follows:

$$
\begin{aligned}
&[U]^{\alpha_n} = [u^{\alpha_n}]\,, && (14)\\
&[U]^{\alpha_{n-1}} = [u_1^{\alpha_{n-1}}, u^{\alpha_n}] \cup [u^{\alpha_n}, u_2^{\alpha_{n-1}}]\,, && (15)\\
&\vdots\\
&[U]^{\alpha_{i+1}} = [u_1^{\alpha_{i+1}}, u_1^{\alpha_{i+2}}] \cup [u_1^{\alpha_{i+2}}, u_2^{\alpha_{i+2}}] \cup [u_2^{\alpha_{i+2}}, u_2^{\alpha_{i+1}}]\,, && (16)\\
&[U]^{\alpha_i} = [u_1^{\alpha_i}, u_1^{\alpha_{i+1}}] \cup [u_1^{\alpha_{i+1}}, u_2^{\alpha_{i+1}}] \cup [u_2^{\alpha_{i+1}}, u_2^{\alpha_i}]\,, && (17)\\
&[U]^{\alpha_{i-1}} = [u_1^{\alpha_{i-1}}, u_1^{\alpha_i}] \cup [u_1^{\alpha_i}, u_2^{\alpha_i}] \cup [u_2^{\alpha_i}, u_2^{\alpha_{i-1}}]\,, && (18)\\
&\vdots\\
&[U]^{\alpha_2} = [u_1^{\alpha_2}, u_1^{\alpha_3}] \cup [u_1^{\alpha_3}, u_2^{\alpha_3}] \cup [u_2^{\alpha_3}, u_2^{\alpha_2}]\,, && (19)\\
&[U]^{\alpha_1} = [u_1^{\alpha_1}, u_1^{\alpha_2}] \cup [u_1^{\alpha_2}, u_2^{\alpha_2}] \cup [u_2^{\alpha_2}, u_2^{\alpha_1}]\,, && (20)
\end{aligned}
$$

for $i = 2, 3, ..., n-1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a real continuous function. Given a fuzzy interval U in $\mathbb{R}$ and we want to find a fuzzy interval $B = f(U)$ that is induced by f. From α-discretisation and interval partitioning of the α-cuts (see Eqs. (14)–(20)), we can find $B = f(U)$ as follows:

$$
\begin{aligned}
&b^{\alpha_n} = f(u^{\alpha_n})\,, && (21)\\
&b_1^{\alpha_{n-1}} = \min\left(\min_{x\in[u_1^{\alpha_{n-1}}, u^{\alpha_n}]} f(x), \min_{x\in[u^{\alpha_n}, u_2^{\alpha_{n-1}}]} f(x)\right)\,, && (22)\\
&\vdots\\
&b_1^{\alpha_{i+1}} = \min\left(\min_{x\in[u_1^{\alpha_{i+1}}, u_1^{\alpha_{i+2}}]} f(x), \min_{x\in[u_1^{\alpha_{i+2}}, u_2^{\alpha_{i+2}}]} f(x), \min_{x\in[u_2^{\alpha_{i+2}}, u_2^{\alpha_{i+1}}]} f(x)\right)\,, && (23)\\
&b_1^{\alpha_i} = \min\left(\min_{x\in[u_1^{\alpha_i}, u_1^{\alpha_{i+1}}]} f(x), \min_{x\in[u_1^{\alpha_{i+1}}, u_2^{\alpha_{i+1}}]} f(x), \min_{x\in[u_2^{\alpha_{i+1}}, u_2^{\alpha_i}]} f(x)\right)\,, && (24)\\
&b_1^{\alpha_{i-1}} = \min\left(\min_{x\in[u_1^{\alpha_{i-1}}, u_1^{\alpha_i}]} f(x), \min_{x\in[u_1^{\alpha_i}, u_2^{\alpha_i}]} f(x), \min_{x\in[u_2^{\alpha_i}, u_2^{\alpha_{i-1}}]} f(x)\right)\,, && (25)\\
&\vdots\\
&b_1^{\alpha_2} = \min\left(\min_{x\in[u_1^{\alpha_2}, u_1^{\alpha_3}]} f(x), \min_{x\in[u_1^{\alpha_3}, u_2^{\alpha_3}]} f(x), \min_{x\in[u_2^{\alpha_3}, u_2^{\alpha_2}]} f(x)\right)\,, && (26)\\
&b_1^{\alpha_1} = \min\left(\min_{x\in[u_1^{\alpha_1}, u_1^{\alpha_2}]} f(x), \min_{x\in[u_1^{\alpha_2}, u_2^{\alpha_2}]} f(x), \min_{x\in[u_2^{\alpha_2}, u_2^{\alpha_1}]} f(x)\right)\,, && (27)
\end{aligned}
$$

and,

$$b_2^{\alpha_{n-1}} = \max\left(\max_{x\in[u_1^{\alpha_{n-1}},u^{\alpha_n}]} f(x), \max_{x\in[u^{\alpha_n},u_2^{\alpha_{n-1}}]} f(x)\right) , \tag{28}$$

$$\vdots$$

$$b_2^{\alpha_{i+1}} = \max\left(\max_{x\in[u_1^{\alpha_{i+1}},u_1^{\alpha_{i+2}}]} f(x), \max_{x\in[u_1^{\alpha_{i+2}},u_2^{\alpha_{i+2}}]} f(x), \max_{x\in[u_2^{\alpha_{i+2}},u_2^{\alpha_{i+1}}]} f(x)\right) , \tag{29}$$

$$b_2^{\alpha_i} = \max\left(\max_{x\in[u_1^{\alpha_i},u_1^{\alpha_{i+1}}]} f(x), \max_{x\in[u_1^{\alpha_{i+1}},u_2^{\alpha_{i+1}}]} f(x), \max_{x\in[u_2^{\alpha_{i+1}},u_2^{\alpha_i}]} f(x)\right) , \tag{30}$$

$$b_2^{\alpha_{i-1}} = \max\left(\max_{x\in[u_1^{\alpha_{i-1}},u_1^{\alpha_i}]} f(x), \max_{x\in[u_1^{\alpha_i},u_2^{\alpha_i}]} f(x), \max_{x\in[u_2^{\alpha_i},u_2^{\alpha_{i-1}}]} f(x)\right) , \tag{31}$$

$$\vdots$$

$$b_2^{\alpha_2} = \max\left(\max_{x\in[u_1^{\alpha_2},u_1^{\alpha_3}]} f(x), \max_{x\in[u_1^{\alpha_3},u_2^{\alpha_3}]} f(x), \max_{x\in[u_2^{\alpha_3},u_2^{\alpha_2}]} f(x)\right) , \tag{32}$$

$$b_2^{\alpha_1} = \max\left(\max_{x\in[u_1^{\alpha_1},u_1^{\alpha_2}]} f(x), \max_{x\in[u_1^{\alpha_2},u_2^{\alpha_2}]} f(x), \max_{x\in[u_2^{\alpha_2},u_2^{\alpha_1}]} f(x)\right) , \tag{33}$$

where $b_1^{\alpha_i}$ and $b_2^{\alpha_i}$ are the lower and upper bound of the fuzzy interval B, respectively.

The optimisation problems above will be performed by using Brent's method [1]. The idea of Brent's method is to find the minimum of a parabola through three points. If the function to be minimised is nicely parabolic near to the minimum, then the parabola fitted through any three points in a single leap to the minimum. In the worst possible case, where the parabolic interpolation is acceptable but useless, then the method will approximately alternate between parabolic interpolation and golden section search [10]. However, in case where the function is monotonically increasing or decreasing, then the minimum is obtained at the endpoints of interval without using Brent's method. To find the maximum, we refer to the Theorem 1. Please note that the global minimum and global maximum are only obtained at α_1 and they are denoted by $b_1^{\alpha_1}$ and $b_2^{\alpha_1}$, respectively.

4 Examples

In this section, we implement the proposed method to compute Zadeh's extension principle for some real continuous functions.

Example 1.

We consider the following triangular fuzzy interval U defined by

$$U(x) = \begin{cases} 0 & \text{, if } x < 0\,, \\ \frac{2}{3}x & \text{, if } 0 \leq x \leq \frac{3}{2}\,, \\ -\frac{2}{3}x + 2 & \text{, if } \frac{3}{2} \leq x \leq 3\,, \\ 0 & \text{, if } x > 3\,. \end{cases} \tag{34}$$

The α-cuts of U are given by

$$[U]^\alpha = \left[\frac{3}{2}\alpha, 3 - \frac{3}{2}\alpha\right]. \tag{35}$$

We take the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = (x-1)^2\,, \tag{36}$$

and we want to find $\hat{f}(U) = (U-1)^2$. Since f is non-monotone, then the function values at the endpoints of the α-cuts are not the correct endpoints of the output. From calculus, this function has a minimum point at $x = 1$. So, the correct range of $\hat{f}(U)$ is defined on the interval $[0, 4]$. From Zadeh's extension principle, the analytical solution is given by

$$\hat{f}(U)(y) = \begin{cases} 0 & \text{, if } y < 0\,, \\ \max\left(-\frac{2}{3}\sqrt{y} + \frac{2}{3}, \frac{2}{3}\sqrt{y} + \frac{2}{3}\right) & \text{, if } 0 \leq y \leq \frac{1}{4}\,, \\ \max\left(-\frac{2}{3}\sqrt{y} + \frac{2}{3}, -\frac{2}{3}\sqrt{y} + \frac{4}{3}\right) & \text{, if } \frac{1}{4} \leq y \leq 1\,, \\ -\frac{2}{3}\sqrt{y} + \frac{4}{3} & \text{, if } 1 \leq y \leq 4\,, \\ 0 & \text{, if } y > 4\,. \end{cases} \tag{37}$$

The graphs of U, $f(x)$ and $\hat{f}(U)$ are depicted in Figures 3(a), 3(b) and 3(c), respectively. In Figure‘3(c) we compare the analytical solution with the approximation solution obtained by using the method proposed in this paper. We observe that they are equal. In term of computational complexity, the proposed method requires only 61 function evaluations with $n = 11$.

Example 2.

We consider the following triangular fuzzy interval U defined by

$$U(x) = \begin{cases} 0 & \text{, if } x < \frac{3}{4}\pi\,, \\ \frac{4}{3\pi}x - 1 & \text{, if } \frac{3}{4}\pi \leq x \leq \frac{3}{2}\pi\,, \\ -\frac{4}{3\pi}x + 3 & \text{, if } \frac{3}{2}\pi \leq x \leq \frac{9}{4}\pi\,, \\ 0 & \text{, if } x > \frac{9}{4}\pi\,. \end{cases} \tag{38}$$

The α-cuts of U are given by

$$[U]^\alpha = \left[\frac{3\pi}{4}\alpha + \frac{3\pi}{4}, -\frac{3\pi}{4}\alpha + \frac{9\pi}{4}\right]. \tag{39}$$

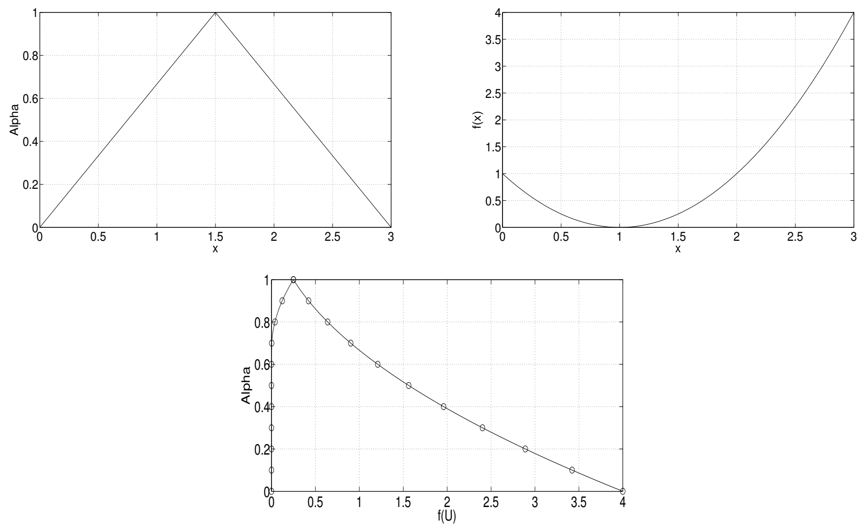

Figure 3: (a) Fuzzy interval U; (b) Function handle; and (c) Comparison between analytical solution (solid line) and its approximation (circle mark).

We take the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \cos(x) \tag{40}$$

and we want to find $\hat{f}(U) = \cos(U)$. The cosine function is periodic with a period of 2π. In this example, we consider the cosine function defined on the interval $[\frac{3}{4}\pi, \frac{9}{4}\pi]$. We know that this function has two extremum points, namely at $x = \pi$ and $x = 2\pi$. So, the correct range of $\hat{f}(U)$ is defined on the interval $[-1, 1]$. Using the technique proposed in Section 3, we obtain the approximation of $\hat{f}(U)$, which is closely equal to the analytical solution. As in Example 1, the total number of function evaluations required in this example is 61 as well with $n = 11$. The analytical solution is given as follow:

$$\hat{f}(U)(y) = \begin{cases} 0 & \text{, if } y < -1\,, \\ \max\left(\frac{4}{3\pi}\cos^{-1}(y) - 1, \frac{4}{3\pi}(2\pi - \cos^{-1}(y)) - 1\right) & \text{, if } -1 \leq y \leq -\frac{1}{\sqrt{2}}\,, \\ \frac{4}{3\pi}(2\pi - \cos^{-1}(y)) - 1 & \text{, if } -\frac{1}{\sqrt{2}} \leq y \leq 0\,, \\ -\frac{4}{3\pi}(2\pi - \cos^{-1}(y)) + 3 & \text{, if } 0 \leq y \leq \frac{1}{\sqrt{2}}\,, \\ \max\left(-\frac{4}{3\pi}(2\pi - \cos^{-1}(y)) + 3, -\frac{4}{3\pi}(2\pi + \cos^{-1}(y)) + 3\right) & \text{, if } \frac{1}{\sqrt{2}} \leq y \leq 1\,, \\ 0 & \text{, if } y > 1\,. \end{cases} \tag{41}$$

The graphs of U, $f(x)$ and $\hat{f}(U)$ are depicted in Figures 4(a), 4(b) and 4(c), respectively.

5 Conclusion

The method presented in this paper greatly improves the solution technique for the computation of functions that take fuzzy sets as their arguments. The convexity of the solution

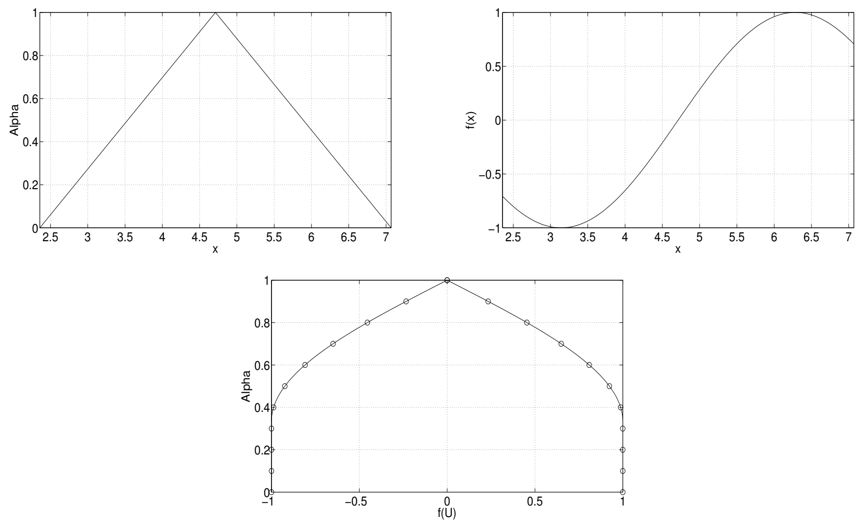

Figure 4: (a) Fuzzy interval U; (b) Function handle; and (c) Comparison between analytical solution (solid line) and its approximation (circle mark).

is also ensured. Furthermore, it is computationally easy to implement and requires only a few function evaluations at every level of α. The proposed method can also be used in order to solve hard global optimisation problems. In the future, the proposed method will be incorporated into classical numerical methods for solving fuzzy differential equations.

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