MATEMATIKA, 2010, Volume 26, Number 1, 107–114

Construction of Quaternion-Valued Wavelets

Mawardi Bahri
Department of Mathematics, Hasanuddin University,
Tamalanrea Makassar, Indonesia
e-mail: mawardibahri@gmail.com

Abstract In this paper, we introduce quaternion-valued wavelets in the context of the duplex matrix-valued function. We then formulate quaternion scaling and wavelet functions using quaternion multiresolution analysis (QMRA). With these formulations, we obtain coefficients of highpass and lowpass filters of QMRA.

Keywords Quaternion-valued wavelets; quaternion multiresolution analysis; highpass and lowpass filters.

1 Introduction

The concept of matrix-valued wavelets has been recently introduced by [1, 3] by utilizing the theory of paraunitary marix filterbanks. Matix-valued wavelets have applications such as video images, multispectral images and color images.

Recently, the concept of generalizing classical wavelets to quantum algebra has gained lot of popularity. He and Zhao [11, 12] constructed the continuous quaternion wavelet transform of quaternion-valued function. They also demonstrated a number of properties of these extended wavelets using the classical Fourier transform (FT). In [2], Traversoni proposed the discrete quaternion wavelet transform using the quaternionic Fourier transform and latter these were applied by Corrochano [8] and Zhou et al. [9].

The purpose of this paper is to construct discrete quaternion wavelet transform based on the complex duplex matrix-valued function. The Fourier transform in a duplex complex matrix introduced in this paper is not the same with that in matrix-valued function of [?], which presents a difference for the design of filters.

2 Basics

The quaternion algebra [10] was first invented by Sir Hamilton in 1843 and is denoted by $\mathbb{H}$ in his honor. It is an extension of complex numbers to a four-dimensional (4-D) algebra. Every element of $\mathbb{H}$ is a linear combination of a real scalar and three imaginary units $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ with real coefficients

$$\mathbb{H} = \{q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R}\}, \tag{1}$$

which obey Hamilton's multiplication rules

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \quad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}, \quad \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1. \tag{2}$$

Because $\mathbb{H}$ is according to (2) non-commutative, one cannot directly extend various results on complex numbers to quaternions. For simplicity, we express a quaternion q as sum of a scalar q_0 , and a pure 3D quaternion $\boldsymbol{q}$,

$$q = q_0 + \boldsymbol{q} = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3, \tag{3}$$

where the scalar part q_0 is also denoted by $\mathrm{Sc}(q)$. The conjugate of a quaternion q is obtained by changing the sign of the pure part, i. e.

$$\bar{q} = q_0 - \boldsymbol{q} = q_0 - q_1\boldsymbol{i} - q_2\boldsymbol{j} - q_3\boldsymbol{k}. \tag{4}$$

The quaternion conjugation (4) is a linear anti-involution

$$\overline{\overline{p}} = p, \quad \overline{p+q} = \overline{p} + \overline{q}, \quad \overline{pq} = \overline{q}\,\overline{p}, \qquad \forall p, q \in \mathbb{H}. \tag{5}$$

Using (2) the multiplication of the two quaternions $q = q_o + \boldsymbol{q}$ and $p = p_o + \boldsymbol{p}$ can be expressed as

$$qp = q_0p_0 + \boldsymbol{q}\cdot\boldsymbol{p} + q_0\boldsymbol{p} + p_0\boldsymbol{q} + \boldsymbol{q}\times\boldsymbol{p}, \tag{6}$$

where we recognize the scalar product $\boldsymbol{q}\cdot\boldsymbol{p} = -(q_1p_1 + q_2p_2 + q_3p_3)$ and the antisymmetric cross type product $\boldsymbol{q}\times\boldsymbol{p} = \boldsymbol{i}(q_2p_3 - q_3p_2) + \boldsymbol{j}(q_3p_1 - q_1p_3) + \boldsymbol{k}(q_1p_2 - q_2p_1)$. The scalar part of the product is

$$\mathrm{Sc}(qp) = q_0p_0 + \boldsymbol{q}\cdot\boldsymbol{p}, \tag{7}$$

and the pure part is

$$q_0\boldsymbol{p} + p_0\boldsymbol{q} + \boldsymbol{q}\times\boldsymbol{p}. \tag{8}$$

In particular, if both q and p are pure quaternions, (6) reduces to

$$\boldsymbol{q}\boldsymbol{p} = \boldsymbol{q}\cdot\boldsymbol{p} + \boldsymbol{q}\times\boldsymbol{p}. \tag{9}$$

According to (6), the multiplication of a quaternion q and its conjugate can be expressed as

$$\begin{aligned} q\bar{q} &= q_0q_0 - \boldsymbol{q}\cdot\boldsymbol{q} + q_0(-\boldsymbol{q}) + q_0\boldsymbol{q} + \boldsymbol{q}\times(-\boldsymbol{q}) \\ &= q_o^2 + q_1^2 + q_2^2 + q_3^2. \end{aligned} \tag{10}$$

This leads to the modulus $|q|$ of a quaternion q defined as

$$|q| = \sqrt{q\overline{q}} = \sqrt{q_o^2 + q_1^2 + q_2^2 + q_3^2}. \tag{11}$$

Using the conjugate (4) and the modulus of q, we can define the inverse of $q \in \mathbb{H}\setminus\{0\}$ as

$$q^{-1} = \frac{\bar{q}}{|q|^2} \tag{12}$$

which shows that $\mathbb{H}$ is a normed division algebra. Furthermore, we get $|q^{-1}| = |q|^{-1}$. It is straightforward to see that, with (5) and (11), the following modulus properties hold

$$|pq| = |p||q|, \quad |p| = |\overline{p}|, \quad \forall p, q \in \mathbb{H}. \tag{13}$$

Note that the unit orthogonal imaginary units $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ in (2) can be represented using Pauli spin matrices

$$\boldsymbol{i} = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \boldsymbol{j} = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \boldsymbol{k} = i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{14}$$

where σ_1, σ_2, and σ_3 are the usual Pauli matrices and i is the complex unit imaginary. It means that quaternions can be represented as

$$q = \begin{pmatrix} q_0 + iq_3 & iq_1 - q_2 \\ iq_1 + q_2 & q_0 - iq_3 \end{pmatrix}. \tag{15}$$

It is convenient to introduce the inner product of two quaternion functions, $f, g : \mathbb{R}^2 \longrightarrow \mathbb{H}$, as follows:

$$(f, g)_{L^2(\mathbb{R}^2;\mathbb{H})} = \int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\, d^2\boldsymbol{x}. \tag{16}$$

In particular, if $f = g$, then we obtain the associated norm

$$\|f\|_{L^2(\mathbb{R}^2;\mathbb{H})} = (f, f)^{1/2}_{L^2(\mathbb{R}^2;\mathbb{H})} = \left(\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2\, d^2\boldsymbol{x}\right)^{1/2}. \tag{17}$$

The quaternion module $L^2(\mathbb{R}^2; \mathbb{H})$ is then defined as

$$L^2(\mathbb{R}^2; \mathbb{H}) = \{f : \mathbb{R}^2 \longrightarrow \mathbb{H}, \|f\|_{L^2(\mathbb{R}^2;\mathbb{H})} < \infty\}. \tag{18}$$

3 Quaternion-Valued Multiresolution Analysis

Before we give the definition for biorthogonal quaternion-valued wavelets, we introduce quaternion-valued multiresolution analysis (QMRA). Let

$$L^2(\mathbb{R}, \mathbb{C}^{2\times 2}) = \left\{\boldsymbol{f}(x) = \begin{pmatrix} f_1(x) & -\bar{f}_2(x) \\ f_2(x) & \bar{f}_1(x) \end{pmatrix} : x \in \mathbb{R}, f_k \in L^2(\mathbb{R}; \mathbb{C}), k, l = 1, 2\right\}, \tag{19}$$

denotes the space of quaternion matrix-valued functions defined on $\mathbb{R}$ with values in $\mathbb{C}^{2\times 2}$.

The matrix Fourier transform $\boldsymbol{f}$ on $\mathbb{R}$ is defined by

$$\hat{\boldsymbol{f}}(\omega) = \int_{\mathbb{R}} \boldsymbol{f}(x) D(e^{-i\omega x}, e^{i\omega x})\, dx, \tag{20}$$

where $D(e^{-ik\omega}, e^{ik\omega})$ is a $2x2$ diagonal matrix. The inverse of the above matrix Fourier transform is given by

$$\boldsymbol{f}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\boldsymbol{f}}(\omega) D(e^{i\omega x}, e^{-\omega x})\, d\omega, \tag{21}$$

By inserting (20) to (19) we easily obtain

$$\hat{\boldsymbol{f}}(\omega) = \begin{pmatrix} f_1(x)e^{-i\omega x}\, dx & -\bar{f}_2(x)e^{i\omega x}\, dx \\ f_2(x)e^{-i\omega x}\, dx & \bar{f}_1(x)e^{i\omega x}\, dx \end{pmatrix} = \begin{pmatrix} \hat{f}_1(\omega) & -\hat{\bar{f}}_2(\omega) \\ \hat{f}_2(\omega) & \hat{\bar{f}}_1(\omega) \end{pmatrix}. \tag{22}$$

Definition 1 *A quaternion-valued multiresolution analysis (QMRA) is the decomposition of the space $L^2(\mathbb{R}, \mathbb{C}^{2\times 2})$ into a chain of closed subspaces $\boldsymbol{V}_j$ called scaling spaces*

$$\ldots \subset \boldsymbol{V}_{-2} \subset \boldsymbol{V}_{-1} \subset \boldsymbol{V}_0 \subset \boldsymbol{V}_1 \subset \boldsymbol{V}_2 \ldots$$

such that the following four axioms are satisfied:

Axiom 1 *(completeness)*

$$\overline{\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j} = \overline{\lim_{j\to\infty} \boldsymbol{V}_j} = L^2(\mathbb{R}, \mathbb{C}^{2\times 2}), \qquad \bigcap_{j\in\mathbb{Z}} \boldsymbol{V}_j = \lim_{j\to-\infty} \boldsymbol{V}_j = \{\mathbf{0}_2\}.$$

Axiom 1 means that the space $L^2(\mathbb{R})$ is the closure of the union of all $\boldsymbol{V}_j$ and the intersection of all $\boldsymbol{V}_j$ is empty.

Axiom 2 *(scale invariance)*

$$\boldsymbol{f}(x) \in \boldsymbol{V}_j \Leftrightarrow \boldsymbol{f}(2x) \in \boldsymbol{V}_{j+1}, j \in \mathbb{Z}.$$

Axiom 3 *(translation invariance)*

$$f(x) \in \boldsymbol{V}_0 \Leftrightarrow f(x-k) \in \boldsymbol{V}_0, \forall k \in \mathbb{Z}.$$

Axiom 4 *(translation invariant basis) There exists a function $\Phi(x) \in \boldsymbol{V}_0$ such that*

$$\{\Phi(x-k),\ k \in \mathbb{Z}\}$$

is an orthonormal basis in $\boldsymbol{V}_0$. The function $\Phi(x)$ is called the quaternion scaling function in QMRA. Orthonormality means that

$$\int_{\mathbb{R}} \Phi(x-k_1)\overline{\Phi(x-k_2)}\, dx = \delta_{k_1,k_2} I_2, \quad \forall k_1, k_2 \in \mathbb{Z}, \tag{23}$$

where δ_{k_1,k_2} is the Kronecker delta and I_2 is 2×2 identity matrix .

Since $\boldsymbol{V}_0 \subset \boldsymbol{V}_1$, any function in $\boldsymbol{V}_0$ can be expanded in terms of the basis functions $\Phi_{1,k} = \sqrt{2}\,\Phi(2x-k)$ of $\boldsymbol{V}_1$. In particular, the scaling function $\Phi \in V_0$ can be expanded in terms of $\{\Phi_{1,k}\}$ as

$$\Phi(x) = \sqrt{2}\sum_k H_k \Phi(2x-k), \tag{24}$$

where H_k are all 2×2 constant matrices. Equation (24) is called a quaternion dilation equation and $\Phi(x)$ is quaternion scaling functions.

Proposition 1 *The scaling coefficients $\{H_k\}$ of the quaternion dilation equation (24) satisfy the normality condition, i.e.*

$$\sum_k H_k \overline{H_{k-2l}} = I_2 \delta_{0l},\ \forall\ l \in \mathbb{Z}. \tag{25}$$

Proof. Notice first that by repeated applications of (24) we have

$$\begin{aligned}\Phi(x)\overline{\Phi(x-l)} &= \sqrt{2}\sum_k H_k \Phi(2x-k)\overline{\Phi(x-l)} \\ &= 2\sum_k H_k \Phi(2x-k)\overline{\sum_m H_m \Phi(2x-2l-m)}. \end{aligned} \tag{26}$$

By integrating both sides of (26) with respect to x we get

$$\begin{aligned} I_2\delta_{0l} &= 2\sum_k H_k \left(\sum_m \frac{1}{2}\int_{\mathbb{R}} \Phi(2x-k)\overline{\Phi(2x-2l-m)}\, d(2x)\overline{H_m}\right) \\ &= \sum_k \sum_m H_k I_2 \delta_{k,2l+m}\overline{H_m} \\ &= \sum_k H_k \overline{H_{k-2l}}. \end{aligned} \tag{27}$$

The last line is obtained by taking $m = k - 2l$. This completes the proof of (25). □

As an easy consequence of Proposition 1, we obtain

$$\sum_k H_k^2 = I_2, \qquad \text{and} \qquad \sum_k H_k \overline{H_{k-2l}} = \mathbf{0}_2 \quad \text{for } l \in \mathbb{Z}\setminus\{0\}. \tag{28}$$

Similarly, according to (24) we have the *quaternion wavelet equation*

$$\Psi(x) = \sum_{k\in\mathbb{Z}} < \Phi_{1,k}, \Psi > \Phi_{1,k} = \sqrt{2}\sum_{k\in\mathbb{Z}} G_k \Phi(2x-k). \tag{29}$$

Proposition 2 *The coefficients $\{G_k\}_{k\in\mathbb{Z}}$ of the wavelet equation (29) satisfy*

$$\sum_k G_k = \mathbf{0}_2. \tag{30}$$

The integer translated quaternion wavelet functions $\Psi(x-l)$ form an orthonormal set

$$\int_{\mathbb{R}} \Psi(x)\overline{\Psi(x-l)}\, dx = I_2\delta_{0l},\ \forall l \in \mathbb{Z}. \tag{31}$$

Therefore

$$\sum_k G_k \overline{G_{k-2l}} = I_2\delta_{0l}. \tag{32}$$

Let us define

$$m_0(\omega) = \sum_k \frac{H_k}{\sqrt{2}} D(e^{-ik\omega}, e^{ik\omega}). \tag{33}$$

If m_0 is a 2π-periodic function in $L^2(\mathbb{R}, \mathbb{C}^{2\times 2})$, the function m_0 is called the *lowpass filter* associated with the scaling function Φ. The dilation equation (24) becomes

$$\hat{\Phi}(\omega) = m_0(\frac{\omega}{2})\hat{\Phi}(\frac{\omega}{2}). \tag{34}$$

Equation (34) follows from

$$\begin{aligned}
\hat{\Phi}(\omega) &= \int_{\mathbb{R}} \Phi(x) D(e^{-i\omega x}, e^{i\omega x})\, dx \\
&\overset{(24)}{=} \sum_k \sqrt{2}\, H_k \int_{\mathbb{R}} \Phi(2x-k)\, D(e^{-i\omega x}, e^{i\omega x})\, dx \\
&= \sum_k \frac{H_k}{\sqrt{2}}\, D(e^{-\frac{ik\omega}{2}}, e^{\frac{ik\omega}{2}}) \int_{\mathbb{R}} \Phi(2x-k)\, D(e^{-i(2x-k)\frac{\omega}{2}}, e^{i(2x-k)\frac{\omega}{2}})\, d(2x-k) \\
&= \sum_k \frac{H_k}{\sqrt{2}}\, D(e^{-\frac{ik\omega}{2}}, e^{\frac{ik\omega}{2}}) \hat{\Phi}(\frac{\omega}{2}) \\
&= m_0(\frac{\omega}{2}) \hat{\Phi}(\frac{\omega}{2}). \qquad (35)
\end{aligned}$$

Without loss of generality we assume $\mathbf{\Phi}(0) = I_2$, iterating (34) leads to

$$\hat{\Phi}(\omega) = m_0(\frac{\omega}{2}) m_0(\frac{\omega}{4}) \cdots = \prod_{n=1}^{\infty} m_0(\frac{\omega}{2^n}). \qquad (36)$$

Similarly, we define the highpass filter

$$m_1(\omega) = \sum_k \frac{G_k}{\sqrt{2}}\, D(e^{-i\omega x}, e^{i\omega x}), \qquad (37)$$

where m_1 is also a 2π-periodic function. The Fourier transform of the wavelet equation (29) becomes therefore

$$\hat{\Psi}(\omega) = m_1(\frac{\omega}{2}) \hat{\Phi}(\frac{\omega}{2}). \qquad (38)$$

The following proposition provides (in the Fourier domain) a necessary condition for the orthonormality of the scaling function and its integer translations.

Proposition 3 *If the quaternion scaling function $\Phi(x) \in L^2(\mathbb{R})$ and if $\Phi(x)$ is orthonormal to its integer translations $\{\Phi(x-k), k \in \mathbb{Z}\}$ then*

$$\sum_{l=-\infty}^{\infty} \hat{\Phi}(\omega + 2\pi l) \overline{\hat{\Phi}(\omega + 2\pi l)} = I_2. \qquad (39)$$

Proof Using the inverse matrix Fourier transform (21) gives ($k \in \mathbb{Z}$)

$$\begin{aligned}
\delta_{k,0} I_2 &= \int_{\mathbb{R}} \Phi(x) \overline{\Phi(x-k)}\, dx \\
&= \int_{\mathbb{R}} [\frac{1}{2\pi} \int_{\mathbb{R}} \hat{\Phi}(\omega)\, D(e^{i\omega x}, e^{-i\omega x})\, d\omega]\, \overline{\Phi(x-k)}\, dx \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\Phi}(\omega) \int_{\mathbb{R}} \overline{\Phi(x-k)\, D(e^{-i\omega(x-k)}, e^{i\omega(x-k)})}\, d(x-k)\, D(e^{i\omega k}, e^{-i\omega k})\, d\omega \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\Phi}(\omega) \overline{\hat{\Phi}(\omega)}\, D(e^{i\omega k}, e^{-i\omega k})\, d\omega \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sum_{l=-\infty}^{\infty} \hat{\Phi}(\omega + 2\pi l) \overline{\hat{\Phi}(\omega + 2\pi l)} D(e^{i\omega k}, e^{-i\omega k})\, d\omega. \qquad (40)
\end{aligned}$$

The last line in equation (40) follows because $e^{i\omega k} = e^{i(\omega+2\pi l)k}, \forall l, k \in \mathbb{Z}$, and represents the $-k$th Fourier coefficient of a periodic function $f(\omega) = \sum_{l=-\infty}^{\infty} \hat{\phi}(\omega + 2\pi l)\overline{\hat{\phi}(\omega + 2\pi l)}$. Since the $k = 0$ coefficient of $f(\omega)$ equals I_2 and all the other coefficients are zero matrix, this implies that

$$\sum_{l=-\infty}^{\infty} \hat{\Phi}(\omega + 2\pi l)\overline{\hat{\Phi}(\omega + 2\pi l)} = I_2.$$

Therefore (39) in Proposition 3 is proved. □

4 Conclusion

Using the basic concepts of the complex duplex matrix-valued function and its Fourier transform we introduced quaternion-valued wavelets. We then constructed quaternion-valued multiresolution analysis. Using the spectral representation of the Fourier transform, we derived several important properties such as the highpass and lowpass filters. In the future we will show that the construction can also be extended using the a complex representation of a quaternion matrix.

Acknowledgment

This paper was written while the author was a postdoctoral fellow at the Universiti Sains Malaysia, Malaysia.

References

[1] A. T. Walden and A. Serroukh, *Wavelet Analysis of Matrix-Valued Times-series*, Proc. R. Soc. Lond. A, 458(2002), 157-179.

[2] L. Traversoni, *Imaging Analysis Using Quaternion Wavelets*, in Geometric Algebra with Applications in Science and Engineering, E. B. Corrochano and G. Sobczyk (ed.), Birkhäuser, 2001, 326-343.

[3] X. G. Xia and B. W. Sutter, *Vector-Valued Wavelets and Vector Filter Banks*, IEEE Transaction on Signal Processing, 44(3)(1996), 508–518.

[4] B. Mawardi, E. Hitzer, A. Hayashi and R. Ashino, *An Uncertainty Principle for Quaternion Fourier Transform*, Comput. Math. Appl. 56(9)(2008), 2411–2417.

[5] B. Mawardi, E. Hitzer, R. Ashino and R. Vaillancourt, *Windowed Fourier Transform of Two-Dimensional Quaternionic Signals*, Applied Mathematics and Computation, 2010. In Press.

[6] B. Mawardi and E. Hitzer, *Clifford Algebra $Cl_{3,0}$-Valued Wavelet Transformation, Clifford Wavelet Uncertainty Inequality and Clifford Gabor Wavelets*, Int. J. Wavelets Multiresolut. Inf. Process., 5(6)(2007), 997–1019.

[7] E. Hitzer and B. Mawardi, *Clifford Fourier Transform on Multivector Fields and Uncertainty Principle for Dimensions $n = 2$ (mod 4) and $n = 3$ (mod 4)*, Adv. in Appl. Clifford Algebras , 18(3–4)(2008), 715–736.

[8] E. B. Corrochano, *The Theory and Use of the Quaternion Wavelet Transform*, Journal of Mathematical Imaging and Vision, 24(1)(2006), 19–35.

[9] J. Zhou, Y. Xu and X. Yang, *Quaternion Wavelet Phase Based Stereo Matching for Uncalibrated Images*, Pattern Recognition Letters, 28(12)(2007), 1509–1522.

[10] K. Gürlebeck and W. Sprössig, *Quaternionic and Clifford Calculus for Physicists and Engineers*, John Wiley and Sons, England, 1997.

[11] J. X. He, *Continuous Wavelet Transform on the Space* $L^2(\mathbb{R}, \mathbb{H}; dx)$, Applied Mathematics Letters, 17(1)(2001), 111–121.

[12] J. Zhao, L. Peng, *Quaternion-Valued Admissible Wavelets Associated with the 2-Dimensional Euclidean Group with Dilations*, Journal of Natural Geometry, 20(1)(2001), 21–32.