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A Certain Class of Multivalent Prestarlike Functions Defined by Dziok-Srivastava Linear Operator

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Abstract In this paper, we have introduced a new class of p-valent prestarlike functions with negative coefficients defined by Dziok-Srivastava linear operator. Growth and distortion theorems have been proved in terms of Saigo fractional integral operator. Class preserving integral operator and radius of convexity of this class have also been investigated. Some special cases of the results have also been proved.

Keywords Prestarlike functions; Dziok-Srivastava linear operator; Fractional integral operator.

1 Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n\, z^n, \tag{1.1}$$

which are analytic and univalent in the unit disc $E = \{z : |z| < 1\}$. Let T denote the subclass of S consisting of functions analytic and univalent which can be expressed in the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n\, z^n \ , \qquad a_n \geq 0. \tag{1.2}$$

A function $f(z) \in S$ is said to be univalent starlike of order $\alpha(0 \leq \alpha < 1)$, denoted by $S(\alpha)$, if and only if

$$\text{Re}\{\frac{zf'(z)}{f(z)}\} > \alpha, \qquad z \in E, \tag{1.3}$$

and it is called convex of order α $(0 \leq \alpha < 1)$, denoted by $K(\alpha)$, if and only if

$$\text{Re}\{1 + \frac{zf''(z)}{f'(z)}\} > \alpha, \qquad z \in E. \tag{1.4}$$

Let $f(z)$ be defined by (1.1), and

$$\phi(z) = z + \sum_{n=2}^{\infty} d_n\, z^n. \tag{1.5}$$

Then the convolution or Hadamard product of $f(z)$and $\phi(z)$ is given by

$$(f*\phi)(z) = z+\sum_{n=2}^{\infty} a_n d_n \; z^n. \tag{1.6}$$

A function $f(z) \in S$ is said to be pre-starlike of order α $(0 \le \alpha < 1)$, denoted by $R(\alpha)$, if and only if

$$f(z)*z(1-z)^{2\alpha-2} \in S(\alpha). \tag{1.7}$$

Let

$$S^*(\alpha) = S(\alpha)\bigcap T \ , \ K^*(\alpha) = K(\alpha)\bigcap T \ , \ \text{ and } \ R^*(\alpha) = R(\alpha)\bigcap T.$$

These classes have been studied by [1–5]. The function

$$S_\gamma(z) = z(1-z)^{-2(1-\gamma)} \ , \ 0 \le \gamma < 1, \tag{1.8}$$

is the familiar extremal function for the class $S^*(\gamma)$. Setting

$$C(\gamma, n) = \frac{\prod_{i=2}^{n} \ (i-2\gamma)}{(n-1)!} \ , \quad n \ge 2, \tag{1.9}$$

then $S_\gamma(z)$ can be written in the form

$$S_\gamma(z) = z+\sum_{n=2}^{\infty} C(\gamma, n) \; z^n. \tag{1.10}$$

We note that $C(\gamma, n)$ is a decreasing function in γ and that

$$\lim_{n\to\infty} C(\gamma, n) = \begin{cases} \infty \ , & \gamma < 1/2 \\ 1 \ , & \gamma = 1/2 \\ 0 \ , & \gamma > 1/2 \end{cases} \tag{1.11}$$

The function $f(z)$ is said to be subordinate to $g(z)$ in E written $f(z) < g(z)$, if there exist a function $w(z)$ analytic in E such that $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = g(w(z))$.

For $\alpha_i \in \mathbb{C}$ $(i = 1, 2, 3, ..., l)$ and $\beta_j \in \mathbb{C} - \{0, -1, -2, ...\}$ $(j = 1, 2, 3, ..., m)$, the generalized hypergeometric function is defined by

$${}_lF_m(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} . \frac{z^n}{n!} \tag{1.12}$$

$$(l \le m+1; m \in N_0 = \{0, 1, 2...\}) \ ,$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1; & n{=}0 \\ a(a+1)(a+2)...(a+n-1), & n \in N = 1, 2, ... \end{cases} \tag{1.13}$$

Corresponding to the function $h(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; z) = z\ {}_lF_m(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; z)$, the Dziok-Srivastava operator [6] $H_m^L(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m)$ is defined by

$$H_m^L(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m)f(z) = h(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; z) * f(z) \tag{1.14}$$

$$= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \cdots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_m)_{n-1}}\ a_n\ \frac{z^n}{(n-1)!}.$$

It is well known [6] that

$$\alpha_1 H_m^L(\alpha_1 + 1, ..., \alpha_l; \beta_1, ..., \beta_m)f(z) = z[H_m^L(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m)f(z)]' + (\alpha_1 - 1)H_m^L(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m)f(z)\ . \tag{1.15}$$

To make the notation simple, we write

$$H_m^L[\alpha_1]f(z) = H_m^L(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m)f(z).$$

We note that special cases of the Dziok-Srivastava operator $H_m^L[\alpha_1]$ include the Hohlov linear operator [7], the Carlson-Shafer operator [8], the Ruschweyh derivative operator [4], and many others.

In [9] Shenan has introduced the class $S_m^l(\alpha_l, \beta_m, \beta, \alpha)$ of uniformly convex functions defined by Dziok-Srivastava operator $H_m^L[\alpha_1]$. In the present paper we defined a new class of analytic functions defined by $H_m^L[\alpha_1]$ and investigate several interesting results of this class.

Definition 1.1 For A, B arbitrary fixed real number, $-1 \le B < A \le 1$, a function $f(z) \in T$ defined by (1.2) is said to be in the class $S_m^l[A, B, \alpha, \gamma]$ if it satisfies

$$\alpha_1 \frac{H_m^L[\alpha_1 + 1]\phi_\gamma(z)}{H_m^L[\alpha_1]\phi_\gamma(z)} + 1 - \alpha_1 \prec \frac{1 + [(A - B)(1 - \alpha) + B]z}{1 + Bz}, \quad (z \in E), \tag{1.16}$$

where $0 \le \alpha < 1, 0 \le \gamma < 1$.
The condition (1.16) is equivalent to

$$\left| \frac{z\frac{[H_m^L[\alpha_1+1]\phi_\gamma(z)]'}{H_m^L[\alpha_1]\phi_\gamma(z)} - 1}{B + (A - B)(1 - \alpha) - Bz\frac{[H_m^L[\alpha_1+1]\phi_\gamma(z)]'}{H_m^L[\alpha_1]\phi_\gamma(z)}} \right| < 1, \quad (z \in E) \tag{1.17}$$

where

$$\phi_\gamma(z) = (f * S_\gamma)(z) = z - \sum_{n=2}^{\infty} a_n C(\gamma, n)\ z^n. \tag{1.18}$$

It may be noted that the class $S_m^l[A, B, \alpha, \gamma]$ is very general, since it includes the classes of starlike, convex and pre-starlike functions by assigning specific values to A, B, m, l, α and γ. For example
(i) for $A = -B = l = 1, m = 0$, and $\alpha_1 = 1$, the class $S_m^l[A, B, \alpha, \gamma]$ reduces to the class of γ-prestarlike functions of order α, which was introduced by Sheil-Small [10].
(ii) For $A = -B = l = 1, m = 0, \gamma = \frac{1}{2}$ and $\alpha_1 = 1$, we obtain the class $S^*(\alpha)$ of starlike functions of order α.
(iii) For $A = -B = l = 1, m = 0, \gamma = \frac{1}{2}$ and $\alpha_1 = 2$, we obtain the class $K^*(\alpha)$ of convex

functions of order α.
Several other classes which are studied by various researchers such as [3], [6] can be obtained from the above class $S^l_m[A, B, \alpha, \gamma]$.

Among several interesting definitions of fractional integrals given in the literature (cf., e.g., [11–13]), we find it be convenient to recall here the following definition.
Definition 1.2 For real numbers $\beta > 0$, δ, and η, the fractional integral operator $I^{\beta,\delta,\eta}_{0,z}$ is defined by

$$I^{\beta,\delta,\eta}_{0,z} f(z) = \frac{z^{-\beta-\delta}}{\Gamma(\beta)} \int_0^z (z-t)^{\beta-1} {}_2F_1(\beta+\delta, -\eta; 1-\frac{t}{z}) f(t)\, dt. \tag{1.19}$$

for $\beta > 0$ and $k > \max(0, \delta - \eta) - 1$, where $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z - t)^{\beta-1}$ is removed by requiring $\log(z - t)$, to be real when $(z - t) > 0$, provided further that $f(z) = O(|z|^K)$, $z \to 0$. It is easy to observe that,

$$I^{\beta,-\beta,\eta}_{0,z} f(z) = D^{-\beta}_z f(z), \quad (\beta > 0), \tag{1.20}$$

where D^{-p}_z is the fractional integral operator considered by Owa [14].
Lemma 1.1 [6, p.415, lemma 3] *If* $\beta > 0$ *and* $k > \delta - \eta - 1$ *then*

$$I^{\beta,\delta,\eta}_{0,z} z^K = \frac{\Gamma(K + 1)\Gamma(K - \delta + \eta + 1)}{\Gamma(K - \delta + 1)\Gamma(K + \beta + \eta + 1)}\, z^{K-\delta} \tag{1.21}$$

2 Coefficient Estimates

Theorem 2.1 *A function* $f(z)$ *defined by* (1.2) *belongs to the class* $S^l_m[A, B, \alpha, \gamma]$ *if and only if*

$$\sum_{n=2}^{\infty} [(1-B)(n-1)+(A-B)(1-\alpha)]\, C(\gamma, n) \frac{\phi(n)}{(n-1)!}\, a_n \le (A-B)\, (1-\alpha)\ , \tag{2.1}$$

where

$$\phi(n) = \frac{(\alpha_1)_{n-1} \cdots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_m)_{n-1}}\ , \tag{2.2}$$

and $C(\gamma, n)$ *is given by* (1.11). *The result is sharp.*
Proof. Assuming that (2.1) holds and $|z| = 1$, then from (1.17) and (1.14) we have

$$|z(H^l_m[\alpha_1]\phi_\gamma(z))' - H^L_m[\alpha_1]\phi_\gamma(z)|$$

$$- |\{(B + (A - B)(1 - \alpha))H^L_m[\alpha_1]\phi_\gamma(z)\} - Bz(H^L_m[\alpha_1]\phi_\gamma(z))'|$$

$$= |\sum_{n=2}^{\infty}(n-1)C(\gamma, n)\frac{\phi(n)}{(n-1)!}\, a_n\, z^n| - |(A - B)(1 - \alpha)z$$

$$+\sum_{n=2}^{\infty}\left\{B(n-1)-(A-B)(1-\alpha)\right\}C(\gamma,n)\frac{\phi(n)}{(n-1)!}\ a_n\ z^n|$$

$$\leq \sum_{n=2}^{\infty}(n-1)C(\gamma,n)\frac{\phi(n)}{(n-1)!}\ a_n - \sum_{n=2}^{\infty}\left\{B(n-1)-(A-B)(1-\alpha)\right\}$$

$$C(\gamma,n)\frac{\phi(n)}{(n-1)!}\ a_n-(A-B)(1-\alpha)$$

$$=\sum_{n=2}^{\infty}\left\{(1-B)(n-1)+(A-B)(1-\alpha)\right\}C(\gamma,n)\frac{\phi(n)}{(n-1)!}\ a_n-(A-B)(1-\alpha)$$

$$\leq 0.$$

Hence by maximum modulus principle $f(z)\in S_m^l[A,B,\alpha,\gamma]$.
Conversely, assume that $f(z)$ is in the class $S_m^l[A,B,\alpha,\gamma]$. Then

$$\left|\frac{\frac{z(H_m^L[\alpha_1]\phi_\gamma(z))'}{H_m^L[\alpha_1]\phi_\gamma(z)}-1}{B+(A-B)(1-\alpha)-B\frac{z(H_m^L[\alpha_1]\phi_\gamma(z))'}{H_m^L[\alpha_1]\phi_\gamma(z)}}\right|<1\ ,\ z\in U. \tag{2.3}$$

This implies

$$\frac{|\sum_{n=2}^{\infty}(n-1)C(\gamma,n)\frac{\phi(n)}{(n-1)!}\ a_n\ z^n|}{|(A-B)(1-\alpha)z+\sum_{n=2}^{\infty}\left\{B(n-1)-(A-B)(1-\alpha)\right\}C(\gamma,n)\frac{\phi(n)}{(n-1)!}\ a_n\ z^n|}<1$$

Since $|\mathrm{Re}(z)|\leq|z|$ for any z, we find from (2.3) that

$$\mathrm{Re}\left\{\frac{\sum_{n=2}^{\infty}(n-1)C(\gamma,n)\frac{\phi(n)}{(n-1)!}\ a_n\ z^n}{(A-B)(1-\alpha)z+\sum_{n=2}^{\infty}\{B(n-1)-(A-B)(1-\alpha)\}C(\gamma,n)\frac{\phi(n)}{(n-1)!}\ a_n\ z^n}\right\}<1 \tag{2.4}$$

Now choosing, the value of z on the real axis so that $\frac{z(H_m^L[\alpha_1]\phi_\gamma(z))'}{H_m^L[\alpha_1]\phi_\gamma(z)}$ is real, then upon clearing the denominator in (2.4) and letting $z\to 1$ through real values, we have

$$\sum_{n=2}^{\infty}(n-1)C(\gamma,n)\frac{\phi(n)}{(n-1)!}\ a_n\leq (A-B)(1-\alpha)$$

$$+\sum_{n=2}^{\infty}\left\{B(n-1)-(A-B)(1-\alpha)\right\}C(\gamma,n)\frac{\phi(n)}{(n-1)!}\ a_n$$

which gives the desired assertion (2.1).

Finally, we note that equality in (2.1) holds for the function

$$f(z)=z-\frac{(A-B)(1-\alpha)(n-1)!}{\left\{(1-B)(n-1)+(A-B)(1-\alpha)\right\}C(\gamma,n)\phi(n)\ a_n}\ z^n. \tag{2.5}$$

3 Distortion and Growth Theorems

Theorem 3.1 *Let the function* $f(z)$ *defined by* (1.2) *be in the class* $S_m^l[A,B,\alpha,\gamma]$ *. Then*

$$\left| I_{0,z}^{\beta,\delta,\eta} f(z) \right| \geq \frac{\Gamma(2)\Gamma(2-\delta+\eta)}{\Gamma(2-\delta)\Gamma(2+\beta+\eta)}|z|^{1-\delta}$$

$$\times \left\{ 1 - \frac{(A-B)(1-\alpha)(2-\delta+\eta)\prod_{j=1}^{m}\beta_j}{[(1-B)+(A-B)(1-\alpha)](1-\gamma)(2-\delta)(2+\beta+\eta)\prod_{j=1}^{m}\alpha_j}|z| \right\} (3.1),$$

and

$$\left| I_{0,z}^{\beta,\delta,\eta} f(z) \right| \leq \frac{\Gamma(2)\Gamma(2-\delta+\eta)}{\Gamma(2-\delta)\Gamma(2+\beta+\eta)}|z|^{1-\delta}$$

$$\times \left\{ 1 + \frac{(A-B)(1-\alpha)(2-\delta+\eta)\prod_{j=1}^{m}\beta_j}{[(1-B)+(A-B)(1-\alpha)](1-\gamma)(2-\delta)(2+\beta+\eta)\prod_{j=1}^{m}\alpha_j}|z| \right\} .(3.2)$$

$(z \in E_0, \beta > 0, \delta < 2, \beta+\eta > -2, \delta-\eta < 2$ and $\delta(\beta+\eta) \leq 3\beta)$,
where

$$E_0 = \begin{cases} E & (\delta \leq 1) \\ E\backslash\{0\} & (\delta > 1) \end{cases} \tag{3.3}$$

The equalities in (3.1) *and* (3.2) *are attained by the function*

$$f(z) = z - \frac{(A-B)(1-\alpha)}{2[(1-B)+(A-B)(1-\alpha)](1-\gamma)\phi(2)}z^2. \tag{3.4}$$

Proof. Since the function $f(z)$ defined by (1.2) is in the class $S_m^l[A,B,\alpha,\gamma]$, we have from Theorem 2.1,

$$\sum_{n=2}^{\infty} [(1-B)(n-1)+(A-B)(1-\alpha)]\ C(\gamma,n)\frac{\phi(n)}{(n-1)!}\ a_n \leq (A-B)\ (1-\alpha).$$

Now,

$$[(1-B)+(A-B)(1-\alpha)]\phi(2)C(\gamma,2)\sum_{n=2}^{\infty}a_n$$

$$= \sum_{n=2}^{\infty}[(1-B)+(A-B)(1-\alpha)]\phi(2)C(\gamma,2)a_n$$

$$\leq \sum_{n=2}^{\infty}[(n-1)(1-B)+(A-B)(1-\alpha)]\frac{\phi(n)}{(n-1)!}C(\gamma,n)a_n$$
$$\leq (A-B)\ (1-\alpha)$$

and therefore

$$\sum_{n=2}^{\infty} a_n \leq \frac{(A-B)(1-\alpha)}{2[(1-B)+(A-B)(1-\alpha)]\phi(2)(1-\gamma)}\ . \tag{3.5}$$

By virtue of Lemma 1.1, we have

$$I_{0,z}^{\beta,\delta,\eta} f(z) = \frac{\Gamma(2)\Gamma(2-\delta+\eta)}{\Gamma(2-\delta)\Gamma(2+\beta+\eta)} z^{1-\delta}$$
$$-\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(n-\delta+\eta+1)}{\Gamma(n-\delta+1)\Gamma(n+\beta+\eta+1)} a_n z^{n-\delta}. \tag{3.6}$$

Now define the function $\Phi(z)$ by

$$\Phi(z) = \frac{\Gamma(2-\delta)\Gamma(2+\beta+\eta)}{\Gamma(2)\Gamma(2-\delta+\eta)} z^{\delta} I_{0,z}^{\beta,\delta,\eta} f(z)$$
$$= z - \sum_{n=2}^{\infty} \Psi(n) a_n z^n \tag{3.7}$$

where, for convenience

$$\Psi(n) = \frac{(2)_{n-1}(2-\delta+\eta)_{n-1}}{(2-\delta)_{n-1}(2+\beta+\eta)_{n-1}}, \quad n \geq 2. \tag{3.8}$$

It is easily seen from the assumptions in (3.3) that $\Psi(n)$ is non-increasing for integers $n \geq 2,$ and we have

$$0 < \Psi(n) \leq \Psi(2) = \frac{2(2-\delta+\eta)}{(2-\delta)(2+\beta+\eta)} \tag{3.9}$$

Making use of (3.5) and (3.9) in (3.7), we see that

$$|\Phi(z)| = |z| - |z|^2 \sum_{n=2}^{\infty} \Psi(n) a_n$$
$$\geq |z| - \Psi(2)|z|^2 \sum_{n=2}^{\infty} a_n$$
$$\geq |z| - \frac{(A-B)(1-\alpha)(2-\delta+\eta)\prod_{j=1}^{m}\beta_j}{[(1-B)+(A-B)(1-\alpha)](1-\gamma)(2-\delta)(2+\beta+\eta)\prod_{j=1}^{m}\alpha_j}|z|^2$$

which implies the assertion (3.1) of Theorem 3.1.
Also we have,

$$|\Phi(z)| \leq |z|+|z|^2 \sum_{n=2}^{\infty} \Psi(n) a_n$$

$$\leq |z| + \Psi(2)|z|^2 \sum_{n=2}^{\infty} a_n$$

$$\leq |z|+ \frac{(A-B)(1-\alpha)(2-\delta+\eta) \prod_{j=1}^{m} \beta_j}{[(1-B)+(A-B)(1-\alpha)](1-\gamma)(2-\delta)(2+\beta+\eta) \prod_{j=1}^{m} \alpha_j} |z|^2$$

which implies the assertion (3.2) of Theorem 3.1.

Corollary 3.1 *Let the function $f(z)$ defined by* (1.2) *be in the class $S_m^l[A, B, \alpha, \gamma]$. Then*

$$|D_z^{-\beta} f(z)| \geq \frac{\Gamma(2)|z|^{1+\beta}}{\Gamma(2+\beta)} \times \left\{ 1 - \frac{(A-B)(1-\alpha) \prod_{j=1}^{m} \beta_j}{[(1-B)+(A-B)(1-\alpha)](1-\gamma)(2+\beta) \prod_{j=1}^{m} \alpha j} |z| \right\}, \tag{3.10}$$

and

$$|D_z^{-\beta} f(z)| \leq \frac{\Gamma(2)|z|^{1+\beta}}{\Gamma(2+\beta)} \times \left\{ 1 + \frac{(A-B)(1-\alpha) \prod_{j=1}^{m} \beta_j}{[(1-B)+(A-B)(1-\alpha)](1-\gamma)(2+\beta) \prod_{j=1}^{m} \alpha_j} |z| \right\}. \tag{3.11}$$

$(z \in E; \beta > 0)$. *Equalities in* (3.10) *and* (3.11) *are attained by the function given by* (3.4).

Proof. In view of the relationship (1.20) by setting $\delta = -\beta$ in Theorem 2.1, Corollary 3.1 follows readily.

Corollary 3.2 *Let the function $f(z)$ defined by* (1.2) *be in the class $S_m^l[A, B, \alpha, \gamma]$, then for $|z| = r$,*

$$r - \frac{(A-B)(1-\alpha) \prod_{j=1}^{m} \beta_j}{2[(1-B)+(A-B)(1-\alpha)](1-\gamma) \prod_{j=1}^{m} \alpha_j} |r|^2 \leq |f(z)|$$

$$\leq r + \frac{(A-B)(1-\alpha)\prod_{j=1}^{m}\beta_j}{2[(1-B)+(A-B)(1-\alpha)](1-\gamma)\prod_{j=1}^{m}\alpha_j}|r|^2. \tag{3.12}$$

The estimate is sharp for the function $f(z)$ given by (3.4).
Proof. Set $\beta = 0$ in Corollary 3.1.

4 Integral Operators

Theorem 4.1 *Letc be a real number such that $c > -1$, if $f(z) \in S_m^l[A, B, \alpha, \gamma]$, then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt \tag{4.1}$$

also belongs to $S_m^l[A, B, \alpha, \gamma]$.
Proof. Let $f(z)$ defined by (1.2) be in the class $S_m^l[A, B, \alpha, \gamma]$. Then from the representation of $F(z)$, we have

$$F(z) = z - \sum_{n=2}^{\infty} b_n\, z^n, \tag{4.2}$$

where

$$b_n = \frac{c+1}{c+n}\, a_n < a_n. \tag{4.3}$$

Therefore, by (4.3) and Theorem 2.1,

$$\sum_{n=2}^{\infty}\,[(1-B)(n-1)+(A-B)(1-\alpha)]\; C(\gamma,n)\frac{\phi(n)}{(n-1)!}\, b_n$$

$$< \sum_{n=2}^{\infty}\,[(1-B)(n-1)+(A-B)(1-\alpha)]\; C(\gamma,n)\frac{\phi(n)}{(n-1)!}\, a_n \leq (A-B)\,(1-\alpha).$$

Since $f(z) \in S_m^l[A, B, \alpha, \gamma]$, by Theorem 2.1, $F(z) \in S_m^l[A, B, \alpha, \gamma]$.

Theorem 4.2 *Let c be a real number such that $c > -1$. If $f(z) \in S_m^l[A, B, \alpha, \gamma]$, then the function $f(z)$ defined by* (4.1) *is univalent in $|z| < R^*$, where*

$$R^* = \inf_{n\geq 2}\left(\frac{C(\gamma,n)[(1-B)(n-1)+(A-B)(1-\alpha)]\prod_{i=1}^{l}(\alpha_i)_{n-1}}{n(c+n)(A-B)(1-\alpha)\prod_{j=1}^{m}(\beta_j)_{n-1}(n-1)!}\right)^{\frac{1}{n-1}} \tag{4.4}$$

The result is sharp.
Proof. In order to get the required result it suffices to show that

$$|f'(z)-1| < 1, |z| < R^*, \tag{4.5}$$

where $f(z)$ is defined as in (4.2). (4.5) is satisfied if

$$\sum_{n=2}^{\infty} n\left(\frac{c+n}{1+c}\right) a_n\ |z|^{n-1} < 1. \tag{4.6}$$

In view of Theorem 2.1 we have

$$\sum_{n=2}^{\infty} \frac{[(1-B)(n-1)+(A-B)(1-\alpha)]\ C(\gamma,n)\phi(n)}{(A-B)\ (1-\alpha)(n-1)!}\ a_n \le 1,$$

so that (4.6) is satisfied if

$$n\left(\frac{c+1}{c+n}\right)\ |z|^{n-1}$$

$$< \left(\frac{[(1-B)(n-1)+(A-B)(1-\alpha)]C(\gamma,n)\prod_{i=1}^{l}(\alpha_i)_{n-1}}{(A-B)(1-\alpha)\prod_{j=1}^{m}(\beta_j)_{n-1}(n-1)!}\right)$$

or $|z| < R^*$, where R^* is defined by (4.4).
Sharpness follows for the function $F(z)$ defined as in (2.3).

5 Radius of Convexity

Theorem 5.1. *Let the function $f(z)$, defined by* (1.2), *be in the class $S_m^l[A,B,\alpha,\gamma]$. Then $f(z)$ is convex in the disc $|z| < r_1$, where*

$$r_1 = \inf_{n\ge 2} \left(\frac{C(\gamma,n)[(1-B)(n-1)+(A-B)(1-\alpha)]\prod_{i=1}^{l}(\alpha_i)_{n-1}}{n^2(A-B)(1-\alpha)\prod_{j=1}^{m}(\beta_j)_{n-1}(n-1)!}\right)^{\frac{1}{n-1}}. \tag{5.1}$$

T*he result is sharp for the function $f(z)$ given by* (2.5).
Proof. To establish the required result it is sufficient to show that

$|\frac{zf''(z)}{f'(z)}| \le 1$ for $|z| < 1$, or equivalently

$$\frac{\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n|z|^{n-1}} \le 1$$

or

$$\sum_{n=2}^{\infty} n^2 a_n|z|^{n-1} \le 1 \tag{5.2}$$

By virtue of Theorem 2.1, (5.2) is true if

$$|z| \leq \left(\frac{C(\gamma, n)[(1-B)(n-1) + (A-B)(1-\alpha)] \prod_{i=1}^{l} (\alpha_i)_{n-1}}{n^2(A-B)(1-\alpha) \prod_{j=1}^{m} (\beta_j)_{n-1}(n-1)!} \right)^{\frac{1}{n-1}}.$$

Thus $f(z)$ is univalent convex of order α in $|z| < r_1$, where r_1 is given by (5.1).

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