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S-semipermutable and Weakly *S*-permutable Subgroups

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Abstract S-semipermutable subgroup and weakly s-permutable subgroup are two different generalizations of s-permutable subgroup. In this paper, we investigate the influence of s-semipermutable and weakly s-permutable subgroups on the structure of finite groups. We give some conditions of p-nilpotency and supersolvability under assumption that some primary subgroups (for example, maximal subgroups or minimal subgroups of Sylow subgroups) are either s-semipermutable or weakly s-permutable. Meanwhile, some results are extended by using formation theory.

Keywords s-semipermutable; weakly s-permutable; p-nilpotent; supersolvable

1 Introduction

All groups considered in this paper will be finite. Our notation is standard and taken mainly from B. Huppert [1] and W. Guo [2].

G always means a group, $|G|$ is the order of G, $\pi(G)$ denotes the set of all primes dividing $|G|$ and G_p is a Sylow p-subgroup of G for some $p \in \pi(G)$. Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ a formation provided that (1) if $G \in \mathcal{F}$ and $H \trianglelefteq G$, then $G/H \in \mathcal{F}$, and (2) if G/M and G/N are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for any normal subgroups M, N of G. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In this paper, $\mathcal{U}$ will denote the class of all supersolvable groups. Clearly, $\mathcal{U}$ is a saturated formation.

Two subgroups H and K of G are said to be permutable if $HK = KH$. A subgroup H of a group G is said to be s-permutable (or s-quasinormal, π-quasinormal) in G if H permutes with all Sylow subgroups of G, i.e, $HS = SH$ for any Sylow subgroup S of G. This concept was introduced by Kegel [3] and was investigated by many authors. More recently, Zhang and Wang [4] generalized s-permutable subgroups to s-semipermutable subgroups.

Definition 1.1. *A subgroup H of a group G is said to be s-semipermutable in G if $HG_p = G_pH$ for any Sylow p-subgroup G_p of G with $(p, |H|) = 1$.*

Wang and Wang [5] proved the following theorem:

Theorem 1.2. *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are s-semipermutable in G, then G is p-nilpotent.*

As another generalization of s-permutable subgroups, Skiba [6] introduced the following concept:

Definition 1.3. *A subgroup H of a group G is called weakly s-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the* *subgroup of H generated by all those subgroups of H which are s-quasinormal in G.*

In fact, this concept is also a generalization of c-normal subgroups given in Wang [7] (A subgroup H of a group G is said to be c-normal in G if there is a normal subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$, where H_G denotes the core of H in G). There are examples to show that weakly s-permutable subgroups are not s-semipermutable subgroups and in general the converse is also false. The aim of this article is to unify and improve some earlier results using s-semipermutable and weakly s-permutable subgroups.

2 Preliminaries

Lemma 2.1. *Suppose that H is an s-semipermutable subgroup of a group G and N is a normal subgroup of G. Then*
(1) *H is s-semipermutable in K whenever $H \leq K \leq G$.*
(2) *If H is p-group for some prime $p \in \pi(G)$, then HN/N is s-semipermutable in G/N.*
(3) *If $H \leq O_p(G)$, then H is s-permutable in G.*

Proof: (a) is [3, Property 1], (b) is [3, Property 2], and (c) is [3, Lemma 3].

Lemma 2.2. ([6], Lemma 2.10) *Let H be a weakly s-permutable subgroup of a group G.*
(1) *If $H \leq L \leq G$, then H is weakly s-permutable in L.*
(2) *If $N \trianglelefteq G$ and $N \leq H \leq G$, then H/N is weakly s-permutable in G/N.*
(3) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is weakly s-permutable in G/N.*

Lemma 2.3. ([8], A, 1.2) *Let U, V, and W be subgroups of a group G. Then the following statements are equivalent:*
(1) *$U \cap VW = (U \cap V)(U \cap W)$.*
(2) *$UV \cap UW = U(V \cap W)$.*

Lemma 2.4. ([9],Lemma 2.2) *If P is an s-permutable p-subgroup of a group G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 2.5. ([5] Theorem 3.3) *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime dividing $|G|$. If every maximal subgroup of P is s-semipermutable in G, then G is p-nilpotent.*

Lemma 2.6. ([10],Lemma 2.6) *Let H be a solvable normal subgroup of a group $G (H \neq 1)$. If every minimal normal subgroup of G which is contained in H is not contained in $\Phi(G)$, then the Fitting subgroup $F(H)$ of H is the direct product of minimal normal subgroups of G which are contained in H.*

Lemma 2.7. ([6] Lemma 2.16) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups. Suppose that G is a group with a normal subgroup N such that*

$G/N \in \mathcal{F}$. If N is cyclic, then $G \in \mathcal{F}$.

Lemma 2.8. ([11], Lemma 2.4) *Let H be a normal subgroup of a group G such that G/H is p-nilpotent and let P be a Sylow p-subgroup of H, where p is the smallest prime divisor of $|G|$. If $|P| \leq p^2$ and G is A_4-free, then G is p-nilpotent.*

Lemma 2.9.([12], Lemma 3.16)*Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type. Also let P be a normal p-subgroup of a group G such that $G/P \in \mathcal{F}$. If G is A_4-free and $|P| \leqslant p^2$, then $G \in \mathcal{F}$.*

3 Main results

Theorem 3.1. *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime divisor of $|G|$. If every maximal subgroup of P is either s-semipermutable or weakly s-permutable in G, then G is p-nilpotent.*

Proof: Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(i) G has a unique minimal normal subgroup N and G/N is p-nilpotent. Moreover $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G. Consider G/N. We will show that G/N satisfies the hypothesis of the theorem. Let M/N be a maximal subgroup of PN/N. It is easy to see $M = P_1N$ for some maximal subgroup P_1 of P. It follows that $P_1 \cap N = P \cap N$ is a Sylow p-subgroup of N. If P_1 is s-semipermutable in G, then M/N is s-semipermutable in G/N by Lemma 2.1. If P_1 is weakly s-permutable in G, then there is a subnormal subgroup T of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{sG}$. So $G/N = M/N \cdot TN/N = P_1N/N \cdot TN/N$. Since

$$(|N : P_1 \cap N|, |N : T \cap N|) = 1,$$

we have

$$(P_1 \cap N)(T \cap N) = N = N \cap G = N \cap P_1T.$$

By Lemma 2.3, $(P_1N) \cap (TN) = (P_1 \cap T)N$. It follows that

$$(P_1N/N) \cap (TN/N) = (P_1N \cap TN)/N = (P_1 \cap T)N/N \leq (P_1)_{sG}N/N \leq (P_1N/N)_{sG}.$$

Hence M/N is weakly s-permutable in G/N. Therefore, G/N satisfies the hypothesis of the theorem. The choice of G yields that G/N is p-nilpotent. Consequently the uniqueness of N and the fact that $\Phi(G) = 1$ are obvious.

(ii) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ by step (1). since

$$G/O_{p'}(G) \cong (G/N)/(O_{p'}(G)/N)$$

is p-nilpotent, hence G is p-nilpotent, a contradiction.

(iii) $O_p(G) = 1$.

If $O_p(G) \neq 1$, Step (1) yields $N \leq O_p(G)$ and $\Phi(O_p(G)) \leq \Phi(G) = 1$. Therefore, G has a maximal subgroup M such that $G = MN$ and $G/N \cong M$ is p-nilpotent. Since $O_p(G) \cap M$ is normalized by N and M, $O_p(G) \cap M$ is normal in G. The uniqueness of N yields $N = O_p(G)$. Clearly, $P = N(P \cap M)$. Furthermore $P \cap M < P$, thus there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. Hence $P = NP_1$. By the hypothesis, P_1 is either s-semipermutable or weakly s-permutable in G. If we assume P_1 is s-semipermutable in G, then $P_1 M_q$ is a group for $q \neq p$. Hence

$$P_1 < M_p, M_q | q \in \pi(M), q \neq p >= P_1 M$$

is a group. Then $P_1 M = M$ or G by maximality of M. If $P_1 M = G$, then $P = P \cap P_1 M = P_1(P \cap M) = P_1$, a contradiction. If $P_1 M = M$, then $P_1 \leq M$. Therefore, $P_1 \cap N = 1$ and N is of prime order. Then the p-nilpotency of G/N implies the p-nilpotency of G, a contradiction. Therefore we may assume P_1 is weakly s-permutable in G. Then there is a subnormal subgroup T of G such that $G = P_1 T$ and $P_1 \cap T \leq (P_1)_{sG} \leq O_p(G) = N \leq O^p(G)$ since N is the unique minimal normal subgroup of G. Since $|G : T|$ is a number of p-power, $O^p(G) \leq T$. Hence

$$P_1 \cap T \leq (P_1)_{sG} \leq O^p(G) \cap P_1 \leq T \cap P_1,$$

and so $P_1 \cap T = (P_1)_{sG} = O^p(G) \cap P_1$. Consequently, $G = PO^p(G)$ implies that $(P_1)_{sG} \trianglelefteq G$ by Lemma 2.4. By the minimality of N, we have $(P_1)_{sG} = N$ or $(P_1)_{sG} = 1$. If $(P_1)_{sG} = N$, then $N \leq P_1$ and $P = NP_1 = P_1$, a contradiction. Thus $P_1 \cap T = (P_1)_{sG} = 1$, and so $|T|_p = p$. Then T is p-nilpotent. Let $T_{p'}$ be the normal p-complement of T. Then $T_{p'}$ is subnormal in G and $T_{p'}$ is a p'-Hall subgroup of G. It follows that $T_{p'}$ is the normal p-complement of G, a contradiction.

(iv) The final contradiction.

If P has a maximal subgroup P_1 which is weakly s-permutable in G, then there is a subnormal subgroup T of G such that $G = P_1 T$ and $P_1 \cap T \leq (P_1)_{sG} \leq O_p(G) = 1$. Then $P_1 \cap T = 1$. Hence $|T|_p = p$. Therefore, T is p-nilpotent. Thus G is p-nilpotent, a contradiction. Now we may assume that all maximal subgroups of P are s-semipermutable in G. Then G is p-nilpotent by Lemma 2.5, a contradiction.

Corollary 3.2. *Let p be a prime dividing the order of a group G, where p is the smallest prime divisor of $|G|$ and H a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is either s-semipermutable or weakly s-permutable in G, then G is p-nilpotent.*

Proof: By Lemma 2.1 and 2.2, every maximal subgroup of P is either s-semipermutable or weakly s-permutable in H. By Theorem 3.1, H is p-nilpotent. Now, let $H_{p'}$ be the normal p-complement of H. Then $H_{p'} \trianglelefteq G$. Assume $H_{p'} \neq 1$ and consider $G/H_{p'}$. Applying Lemma 2.1 and 2.2 it is easy to see that $G/H_{p'}$ satisfies the hypotheses for the normal subgroup

$H/H_{p'}$. Therefore by induction $G/H_{p'}$ is p-nilpotent and so G is p-nilpotent. Hence we may assume $H_{p'} = 1$ and therefore $H = P$ is a p-group. Since G/H is p-nilpotent, we can consider K/H be the normal p-complement of G/H. By Schur-Zassenhaus's theorem, there exists a Hall p'-subgroup $K_{p'}$ of K such that $K = HK_{p'}$. A new application of Theorem 3.1 yields K is p-nilpotent and so $K = H \times K_{p'}$. Hence $K_{p'}$ is a normal p-complement of G. This completes the proof.

Corollary 3.3. ([12], Theorem 3.4) *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are c-normal in G, then G is p-nilpotent.*

Corollary 3.4. *Suppose that every maximal subgroup of any Sylow subgroup of a group G is either s-semipermutable or weakly s-permutable in G, then G is a Sylow tower group of supersolvable type.*

Proof: Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. Then every maximal subgroup of P is either s-semipermutable or weakly s-permutable in G. By Theorem 3.1, G is p-nilpotent. Let U be the normal p-complement of G. By Lemma 2.1 and 2.2, U satisfies the hypothesis of the Corollary. Therefore it follows by induction that U, and hence G is a Sylow tower group of supersolvable type.

Theorem 3.5. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. A group $G \in \mathcal{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and every maximal subgroup of any Sylow subgroup of H is either s-semipermutable or weakly s-permutable in G.*

Proof: The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order.

(i) By Lemma 2.1 and 2.2, every maximal subgroup of any Sylow subgroup of H is either s-quasinormally embedded or weakly s-permutable in H. By Corollary 3.4, H is a Sylow tower group of supersolvable type. Let p be the largest prime divisor of $|H|$ and let P be a Sylow p-subgroup of H. Then P is normal in G. Let N be a minimal normal subgroup of G contained in P. We consider G/N. It is easy to see that $(G/N, H/N)$ satisfies the hypothesis of the Theorem. By the minimality of G, we have $G/N \in \mathcal{F}$. Since $\mathcal{F}$ is a saturated formation, N is the unique minimal normal subgroup of G contained in P and $N \not\leq \Phi(G)$. By Lemma 2.6, it follows that $P = F(P) = N$.

(ii) Since $N \trianglelefteq G$, we may take a maximal N_1 of N such that $N_1 \trianglelefteq G_p$, where G_p is a Sylow p-subgroup of G. Then N_1 is either s-semipermutable or weakly s-permutable in G. If N_1 is weakly s-permutable in G, then there is a subnormal subgroup T of G such that $G = N_1T$ and $N_1 \cap T \leq (N_1)_{sG}$. Thus $G = NT$ and $N = N \cap N_1T = N_1(N \cap T)$. This implies that $N \cap T \neq 1$. But since $N \cap T$ is normal in G and N is minimal normal in G, $N \cap T = N$. It follows that $T = G$ and so $N_1 = (N_1)_{sG}$ is s-permutable in G. By Lemma 2.4, $O^p(G) \leq N_G(N_1)$. Thus $N_1 \trianglelefteq G_pO^p(G) = G$. It follows that $N_1 = 1$

and so $|N| = p$. By Lemma 2.7, $G \in \mathcal{F}$, a contradiction. If N_1 is s-semipermutable in G, then N_1 is s-permutable in G by Lemma 2.1 and it follows the same contradiction.

Corollary 3.6. ([10], Theorem 3.3) *Let H be a normal subgroup of a group G such that G/H is supersolvable. If every maximal subgroup of any Sylow subgroup of H is c-normal in G, then G is supersolvable.*

Theorem 3.7. *Let p is the smallest prime dividing the order of a group $|G|$ and P a Sylow p-subgroup of G. If G is A_4-free and every 2-maximal subgroups of P is either s-semipermutable or weakly s-permutable in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(i) By Lemma 2.8, $|P| \geqslant p^3$ and so every 2-maximal subgroups P_2 of P is non-identity.

(ii) G is not a non-abelian simple group.

Suppose G is simple. Let P_2 a 2-maximal subgroup of P. If P_2 is weakly s-permutable in G, then there is a subnormal subgroup T of G such that $G = P_2T$ and $P_2 \cap T \leq (P_2)_{sG} \leq O_p(G) = 1$. Since G is simple, we have $P_2 = P_2 \cap T = P_2 \cap G = 1$. By Lemma 2.8, G is p-nilpotent, a contradiction. Hence we may assume P_2 is s-semipermutable in G. Suppose Q is a Sylow q-subgroup with $q \neq p$. Then $P_2Q^g = Q^gP_2$ for any $g \in G$. Since G is simple, we have $G = P_2Q$ from [1, VI, 4.10], a contradiction.

(iii) G has a unique minimal normal subgroup N such that G/N is p-nilpotent, moreover $\Phi(G) = 1$.

(iv) $O_{p'}(G) = 1$.

(v) $O_p(G) = 1$.

If $O_p(G) \neq 1$, Step (3) yields $N \leq O_p(G)$ and $\Phi(O_p(G)) \leq \Phi(G) = 1$. Therefore, G has a maximal subgroup M such that $G = MN$ and $G/N \cong M$ is p-nilpotent. Since $O_p(G) \cap M$ is normalized by N and M, hence by G, the uniqueness of N yields $N = O_p(G)$. Clearly, $P = N(P \cap M)$. Furthermore $P \cap M < P$. If $P \cap M$ is a maximal subgroup of P, then N is a subgroup of order p. By applying [16, Lemma 2.8], we obtain that $N \leq Z(G)$. Since G/N is p-nilpotent, it follows that G is p-nilpotent, a contradiction. Therefore $P \cap M$ is contained in a 2-maximal subgroup P_2 of P. By the hypothesis, P_2 is either s-semipermutable or weakly s-permutable in G. If we assume P_2 is s-semipermutable in G, then P_2M_q is a group for $q \neq p$. Hence

$$P_2 < M_p, M_q | q \in \pi(M), q \neq p >= P_2M$$

is a group. Then $P_2M = M$ or G by maximality of M. If $P_2M = G$, then $P = P \cap P_2M = P_2(P \cap M) = P_2$, a contradiction. If $P_2M = M$, then $P_2 \leq M$. Therefore,

$P_2 \cap N = 1$. Since $P = NP_2$, we have $|N| = p^2$. Then the p-nilpotency of G/N implies the p-nilpotency of G by Lemma 2.8, a contradiction. Now we suppose P_2 is weakly s-permutable in G. Then there is a subgroup T of G such that $G = P_2T$ and $P_2 \cap T \leq (P_2)_{sG}$. From Lemma 2.4 we have $O^p(G) \leq N_G((P_2)_{sG})$. Since $(P_2)_{sG}$ is subnormal in G,

$$P_2 \cap T \leq (P_2)_{sG} \leq O_p(G) = N.$$

Thus, $(P_2)_{sG} \leq P_1 \cap N$, where p_1 is a maximal subgroup of P which contains P_2. Then

$$(P_2)_{sG} \leq ((P_2)_{sG})^G = ((P_2)_{sG})^{O^p(G)P} = ((P_2)_{sG})^P \leq (P_1 \cap N)^P = P_1 \cap N \leq N.$$

It follows that $((P_2)_{sG})^G = 1$ or $((P_2)_{sG})^G = P_1 \cap N = N$. If $((P_2)_{sG})^G = P_1 \cap N = N$, then $N \leq P_1$ and $P = NP_1 = P_1$, a contradiction. If $((P_2)_{sG})^G = 1$, then $P_2 \cap T = 1$ and so $|T|_p = p^2$. Hence T is p-nilpotent by Lemma 2.8. Since T is subnormal in G, we have G is p-nilpotent, a contradiction.

(vi) The final contradiction.

If we suppose $NP < G$, then NP satisfies the hypothesis of the theorem. The choice of G yields that N is p-nilpotent, a contradiction with steps (2) and (3). Therefore we may assume $G = NP$. Since G/N is a p-subgroup, we may assume G has a normal subgroup M such that $|G : M| = p$ and $N \leq M$. Hence the maximal subgroups of Sylow p-subgroup $P \cap M$ of M are the 2-maximal subgroups of Sylow p-subgroup P of G. By Lemma 2.1 and 2.2, every maximal subgroup of Sylow p-subgroup $P \cap M$ is either s-semipermutable or weakly s-permutable in M. Now applying Theorem 3.1, we get M is p-nilpotent, and so G is p-nilpotent, a contradiction.

Corollary 3.8. ([12], Theorem 3.2) *Let p be the smallest prime dividing the order of a group $|G|$ and P a Sylow p-subgroup of G. If G is A_4-free and every 2-maximal subgroups of P is c-normal in G, then G is p-nilpotent.*

Corollary 3.9. *Let p be the smallest prime dividing the order of a group G and G is A_4-free. Assume that H is a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every 2-maximal subgroup of P is either s-semipermutale or weakly s-permutable in G, then G is p-nilpotent.*

Using similar arguments as in the proof of Theorem 3.5 and Lemma 2.9, we also obtain the following.

Theorem 3.10. *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type and G is A_4-free. Then $G \in \mathcal{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and every 2-maximal subgroup of any Sylow subgroup of H is either s-semipermutale or weakly s-permutable in G.*

Theorem 3.11. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. A group $G \in \mathcal{F}$ if and only if there is a normal subgroup E of G such that $G/E \in \mathcal{F}$ and every cyclic subgroup $< x >$ of any Sylow subgroup of E with prime order or order 4 (if the Sylow 2-subgroups are non-abelian) is either s-semipermutable or weakly s-permutable* *in* G.

Proof: We need only to prove the sufficiency part since the necessity part is evident. Suppose that the assertion is false and let G be a counterexample of minimal order. Then

(i) E is solvable.

Let K be any proper subgroup of E. Then $|K| < |G|$ and $K/K \in \mathcal{U}$. Let $< x >$ be any cyclic subgroup of any Sylow subgroup of K with prime order or order 4(if the Sylow 2-subgroups are non-abelian). It is clear that $< x >$ is also a cyclic subgroup of a Sylow subgroup of E with prime order or order 4. By the hypothesis, $< x >$ is either s-semipermutable or weakly s-permutable in G. By Lemma 2.1 and 2.2, $< x >$ is either s-semipermutable or weakly s-permutable in K. This shows that the hypothesis still holds for $(\mathcal{U}, K)$. By the choice of G, K is supersolvable. By [12, Theorem 3.11.9], E is solvable.

(ii) $G^{\mathcal{F}}$ is a p-group, where $G^{\mathcal{F}}$ is the $\mathcal{F}$-residual of G. Moreover $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G and $\exp(G^{\mathcal{F}}) = p$ or $\exp(G^{\mathcal{F}}) = 4$ (if $p = 2$ and $G^{\mathcal{F}}$ is non-abelian).

Since $G/E \in \mathcal{F}$, $G^{\mathcal{F}} \leq E$. Let M be a maximal subgroup of G such that $G^{\mathcal{F}} \nsubseteq M$ (that is, M is an $\mathcal{F}$-abnormal maximal subgroup of G). Then $G = ME$. We claim that the hypothesis holds for $(\mathcal{F}, M)$. In fact,

$$M/M \cap E \cong ME/E = G/E \in \mathcal{F}$$

and by the similar argument as above, we can prove that the hypothesis holds for $(\mathcal{F}, M)$. By the choice of G, $M \in \mathcal{F}$. Thus (2) holds by [12,Theorem 3.4.2].

(iii) $< x >$ is s-permutable in G for any element $x \in G^{\mathcal{F}}$.

Let $x \in G^{\mathcal{F}}$. Then the order of x is p or 4 by Step (2). By the hypothesis, $< x >$ is either s-semipermutable or weakly s-permutable in G. If $< x >$ is s-semipermutable in G, then $< x >$ is s-permutable in G by Lemma 2.1 since $< x > \leq G^{\mathcal{F}} \leq O_p(G)$. If $< x >$ is weakly s-permutable in G, then there is a subnormal subgroup T of G such that $G =< x > T$ and $< x > \cap T \leq < x >_{sG}$. Hence

$$G^{\mathcal{F}} = G^{\mathcal{F}} \cap G = G^{\mathcal{F}} \cap < x > T =< x > (G^{\mathcal{F}} \cap T).$$

Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is abelian, we have

$$(G^{\mathcal{F}} \cap T)\Phi(G^{\mathcal{F}})/\Phi(G^{\mathcal{F}}) \trianglelefteq G/\Phi(G^{\mathcal{F}}).$$

Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G, $G^{\mathcal{F}} \cap T \leq \Phi(G^{\mathcal{F}})$ or $G^{\mathcal{F}} = (G^{\mathcal{F}} \cap T)\Phi(G^{\mathcal{F}}) = G^{\mathcal{F}} \cap T$. If $G^{\mathcal{F}} \cap T \leq \Phi(G^{\mathcal{F}})$, then $< x >= G^{\mathcal{F}} \trianglelefteq G$. In this case, $< x >$ is s-permutable in G. If $G^{\mathcal{F}} = G^{\mathcal{F}} \cap T$, then $T = G$ and so $< x >=< x >_{sG}$ is s-permutable in G.

(iv) $|G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})| = p$.

Assume that $|G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})| \neq p$ and let $L/\Phi(G^{\mathcal{F}})$ be any cyclic subgroup of $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$. Let $x \in L \backslash \Phi(G^{\mathcal{F}})$. Then $L =< x > \Phi(G^{\mathcal{F}})$. Since $< x >$ is s-permutable in G by step (3), $L/\Phi(G^{\mathcal{F}})$ is s-permutable in $G/\Phi(G^{\mathcal{F}})$. It follows from [5, Lemma 2.11] that $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ has a maximal subgroup which is normal in $G/\Phi(G^{\mathcal{F}})$. But this is impossible since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G. Thus $|G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})| = p$.

(v) The final contradiction.

Since
$$(G/\Phi(G^{\mathcal{F}}))/(G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})) \cong G/G^{\mathcal{F}} \in \mathcal{F},$$
we have that $G/\Phi(G^{\mathcal{F}}) \in \mathcal{F}$ by Lemma 2.7. But $\Phi(G^{\mathcal{F}}) \leq \Phi(G)$ and $\mathcal{F}$ is a saturated formation, therefore $G \in \mathcal{F}$, the final contradiction.

Corollary 3.12. ([13], Theorem 4.2) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. If every cyclic subgroup of $G^{\mathcal{F}}$ of prime order or order 4 is c-normal in G, then $G \in \mathcal{F}$.*

Corollary 3.13. *If every cyclic subgroup of G of prime order or order 4 is weakly s-permutable in G, then G is supersolvable.*

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