MATEMATIKA, 2010, Volume 26, Number 2, 167–178

On a Paradox in Linear Plus Linear Fractional Transportation Problem

[1]**Vishwas Deep Joshi and** [2]**Nilama Gupta**
[1,2]Department of Mathematics, Malaviya National Institute of Technology,
J.L.N. Marg, Jaipur-302017, CityplaceRajasthan, country-regionIndia
e-mail: [1]vdjoshi.or@gmail.com, [2]n1_gupt@yahoo.com

Abstract A paradoxical situation arises in a linear plus linear fractional transportation problem (LPLFTP), when value of the objective function falls below the optimal value and this lower value is attainable by transporting larger amount of quantity. In this paper, a new heuristic is proposed for finding initial basic feasible solution for LPLFTP and a sufficient condition for the existence of a paradoxical solution is established in LPLFTP. Two numerical examples are proposed for explanation of the algorithms.

Keywords Transportation problem; linear plus linear fractional function; more-for-less paradox.

1 Introduction

The source of the so-called transportation paradox is unclear. Apparently, many researchers have discovered independently from each other the following behavior of the transportation problem: In certain cases of the transportation problem (TP), an increase in the supplies and demands may lead to a decrease in the optimal transportation cost. In other words, by moving bigger amount of goods around, one may save a lot of money. This surely sounds paradoxical [1]. The more-for-less (MFL) paradox in a TP occurs when it is possible to ship more total goods for less (or equal) total cost, while shipping the same amount or more from each origin and to each destination and keeping all the shipping costs non-negative. The occurrence of MFL in TP is not a rare event, and the existing literature has demonstrated the practicality of identifying cases where the paradoxical situation exists [2]. The information of the occurrence of an MFL situation is useful to a manager in deciding which warehouse or plant capacities are to be increased, and which markets should be sought. It could also be a useful tool in analyzing and planning company acquisition, mergers, consolidations and downsizes. The so called MFL paradox in the transportation paradox has been covered from a theoretical stand point by Charnes and Klingman [3], and Charnes et al. [4]. Robb [5] provides an intuitive explanation of the transportation occurrence. Adlakha and Kowalski [6], Adlakha et al. [7, 8] have given an algorithm for solving paradoxical situation in linear transportation problem. They have already discussed MFL paradox in linear transportation problem [6] as well as fixed charge [7] and mixed constraints [8].

The LPLFTPs do exist especially when a compromise between absolute and relative terms is to be maximized [9]. For example, the above problem arises when one wishes to maximize the linear combination of income and profitability. Khurana and Arora [10] presented an algorithm to solve LPLFTP. They considered two special cases of the problem, where the transportation flow is either restricted or enhanced.

In section 2, the mathematical formulation of LPLFTP is discussed, in section 3, finding a better initial basic feasible solution (IBFS) through fractional cost penalty method is presented. Sections 4, 5 and 6 deals with the more-for-less situation and a sufficient condition for paradox in LPLFTP. Numerical examples are presented in sections 3 and 6 respectively for the explanation of algorithms.

2 Mathematical Formulation of LPLFTP

Consider the following LPLFTP problem

$$(P1) \text{ Minimize } Z = \sum_{i=1}^{m}\sum_{j=1}^{n} r_{ij}x_{ij} + \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} s_{ij}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij}x_{ij}}$$

$$\text{subject to } \sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1, 2, ..., m \tag{1}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1, 2, ..., n \tag{2}$$

$$x_{ij} \geq 0, \ i = 1, 2, ..., m; \ j = 1, 2, ..., n \tag{3}$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = F^0 \tag{4}$$

$$a_i > 0, \ b_j > 0, \ i = 1, 2, ..., m; \ j = 1, 2, ..., n$$

where

$\sum_{i=1}^{m}\sum_{j=1}^{n} r_{ij}x_{ij} \geq 0$, $\sum_{i=1}^{m}\sum_{j=1}^{n} s_{ij}x_{ij} \geq 0$, $\sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij}x_{ij} > 0$, $t_{ij} > 0, s_{ij} \geq 0$ and $r_{ij} \geq 0$,

r_{ij}= Per unit capital in transporting quantities from i^{th} supply point to j^{th} destination,

s_{ij}= Per unit depreciation in transporting quantities from i^{th} supply point to j^{th} destination,

t_{ij}= Per unit profit earned in transporting quantities from i^{th} supply point to j^{th} destination,

x_{ij}= Amount transported from the i^{th} origin to the j^{th} destination

a_i= Amount available at the i^{th} origin

b_j= Demand of j^{th} destination

Define the index sets $I = \{1, 2, ..., m\}$, $J = \{1, 2, ..., n\}$ and $K = I \times J$.
Let X= $\{x_{ij}/(i, j) \in K, \ x_{ij}$ satisfying the constraints (1)-(3)$\}$ be a feasible solution of the problem $(P1)$. Denote $I_x = \{(i, j) \in K/x_{ij} > 0, x_{ij} \in X\}$, the set of non-degenerate basic cells. Due to constraint (4) each non-degenerate basic solution will contain $(m + n - 1)$ positive components. Now we consider the dual variables u_i^1, u_i^2, u_i^3 for $i \in I$ and v_j^1, v_j^2, v_j^3 for $j \in J$ such that $u_i^1 + v_j^1 = r_{ij}$, $u_i^2 + v_j^2 = s_{ij}$ and $u_i^3 + v_j^3 = t_{ij}$ for $(i, j) \in I_x$.

Also for non-basic variables, let

$$\begin{aligned} r'_{ij} &= r_{ij} - (u_i^1 + v_j^1) \\ s'_{ij} &= s_{ij} - (u_i^2 + v_j^2) \\ t'_{ij} &= t_{ij} - (u_i^3 + v_j^3)\ for\ (i,j) \in K - I_x \end{aligned} \tag{5}$$

The system (5) has $(m+n-1)$ equations and can be solved independently. Let x_{lj} is a basic variable then arbitrarily set $u_l^1 = 0,\ u_l^2 = 0,\ u_l^3 = 0$ and solve for other dual variables. Having determined the dual variables $u_i^1,\ u_i^2,\ u_i^3$for $i \in I$; $v_j^1,\ v_j^2,\ v_j^3$ for $j \in J$, use these values for determining $r'_{ij},\ s'_{ij},\ t'_{ij}$ for all non-basic variables. Then from [10]

$$V_1 = \sum_{i=1}^{m} a_i u_i^1 + \sum_{j=1}^{n} b_j v_j^1 = \sum_{i=1}^{m}\sum_{j=1}^{n} r_{ij} x_{ij},$$

$$V_2 = \sum_{i=1}^{m} a_i u_i^2 + \sum_{j=1}^{n} b_j v_j^2 = \sum_{i=1}^{m}\sum_{j=1}^{n} s_{ij} x_{ij},$$

$$V_3 = \sum_{i=1}^{m} a_i u_i^3 + \sum_{j=1}^{n} b_j v_j^3 = \sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij} x_{ij},$$

The necessary and sufficient condition for optimality in LPLFTP as given in [10] is given below.

Theorem 1.1. *A basic feasible solution $X* = (x_{ij}^*)$is a local optimum solution of problem (P1) if $R_{ij} = r'_{ij}(V_3)^2 + s'_{ij}V_3 - t'_{ij}V_2 \geq 0\ \forall$ non basic variable x_{ij}.*

The determination of dual variablesu_i^1, u_i^2, u_i^3 for $i \in I$ and v_j^1, v_j^2, v_j^3 for $j \in J$ would facilitate the calculation of R_{ij} for non-basic variables. The solution can be improved if $\exists R_{ij} < 0$ for at least one non-basic variable x_{ij}.

Remark: If there exist one or more $R_{ij} < 0$, then we choose $R_{i_0 j_0} = \min\{R_{ij}/R_{ij} < 0\}$ and introduce the non-basic variable $x_{i_0 j_0}$ into the basis thereby improving the value of Z. The variable which leaves the basis and the value of the new basic variable in the basis can be determined using [10].

3 Heuristic for finding IBFS for LPLFTP

An IBFS to problem $(P1)$ can be obtained by using following method

Step 1. Calculate the value of $r_{ij} + \frac{s_{ij}}{t_{ij}}$ for each cell and prepare the new matrix of fractional values.

Step 2. For a minimization problem calculate the penalty parameter for each row and each column by subtracting the lowest cost of the associated row (column) in question from the next lowest cost in that row/column (if both cost i.e. lowest and next lowest cost are the same, then the penalty is zero). Then an allocation is made to the lowest-cost cell of the row or column with the highest penalty parameter. The procedure continues until all the allocations are made, ignoring rows or columns where the supply from a given source is depleted.

For a maximization problem begin by finding the variable $x_{i_1 j_1}$ which corresponds to the highest profit (highest value of$r_{ij} + \frac{s_{ij}}{t_{ij}}$). Assign $x_{i_1 j_1}$ its largest possible value, i.e. $x_{i_1 j_1} = \min(b_{i_1}, a_{j_1})$. Mark (cross out) row i_1 and column j_1 and reduce the corresponding supply and demand by the value of $x_{i_1 j_1}$. Repeat the procedure using only those cells that do not lie in the crossed out rows and columns. Continue this process until there is only one cell in the transportation tableau that can be chosen.

The heuristic presented is based on the fractional cost concept and hence will be referred to as the "Fractional Cost Penalty Method" (FCPM) (Joshi and Gupta [11]). The FCPM uses $r_{ij} + \frac{s_{ij}}{t_{ij}}$ whereas Vogel's Approximation Method (VAM) uses only one of the three values r_{ij},s_{ij} or t_{ij} giving this heuristic a computational advantage over VAM. We demonstrate this heuristic by a numerical example.

Numerical Example 1

Take the following minimization problem. Its optimal IBFS is$X_{13} = 7, X_{21} = 2, X_{23} = 8, X_{31} = 3$,$X_{32} = 8$with$Z^0 = 147 + \frac{78}{217} = 147.36$. LPLFTP is given in Table 1. In Table 3 the optimal solution of the LPLFTP is presented.

Table 1: Problem Formulation

	B1	B2	B3	Supply
A1	7 5 (r_{11}) (s_{11}) (t_{11}) 4	1 8 5	9 4 6	7
A2	2 7 5	9 4 7	5 2 9	10
A3	8 4 3	2 1 9	11 8 2	11
demand	5	8	15	28

Calculate the ratio of $r_{ij} + \frac{s_{ij}}{t_{ij}}$ and using ratio find out the penalties corresponding to the rows and columns that are given in Table 2. The values in the brackets are the IBFS whereas the values outside the brackets are$r_{ij} + \frac{s_{ij}}{t_{ij}}$.

The IBFS of the minimization problem is $X_{13} = 7, X_{21} = 2, X_{23} = 8, X_{31} = 3$, $X_{32} = 8$ with all other $X_{ij} = 0$ and the objective function $Z^0 = 147 + \frac{78}{217} = 147.36$ by the method FCPM presented in this paper whereas VAM method gives $X_{12} = 7, X_{21} = 5$, $X_{23} = 5, X_{32} = 1, X_{33} = 10$ with $Z = 154 + \frac{182}{134} = 155.36$, implying the superiority of the method FCPM for the minimization problem. For finding the optimal solution from IBFS obtained, the procedure of [10] is used. It can be observed that the optimal solution is obtained in lesser iterations (in fact IBFS is the optimal solution in this particular problem), if FCPM is used as compare to VAM (in this particular problem it uses two more iterations).

Table 2: Penalty Matrix using FCPM

	B1	B2	B3	Supply	Penalties	
					(1)	(2)
A1	8.25	2.6	(7) 9.76	7	5.65	1.51
A2	(2) 3.4	9.58	(8) 5.22	10	1.82	1.82
A3	(3) 8.58	(8) 2.11	15	11	6.47	6.42
Demand	5	8	15	28		
Penalties	4.85	.49	4.54			

Table 3: Final Optimal Table

	B1	B2	B3	Supply
A1	7 5 4	1 8 5	9 4 7 6	7
A2	2 7 2 5	9 4 7	5 2 8 9	10
A3	8 4 3 7	2 1 8 9	11 8 2	11
demand	5	8	15	28

4 More-for-Less Paradox in LPLFTP

The MFL (or nothing) paradox in the LPLFTP occurs when it is possible to ship more total goods for less (equal) total cost even if at least the same amount is shipped from each origin to destinations and all shipping costs are non negative. The MFL paradox is based on relaxing the equality constraint for a given LPLFTP.

$$(P2)\ \text{Minimize } Z = \sum_{i=1}^{m}\sum_{j=1}^{n} r_{ij}x_{ij} + \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} s_{ij}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij}x_{ij}}$$

$$\text{subject to } \sum_{j=1}^{n} x_{ij} \geq a_i, \qquad i = 1, 2, \ldots\ldots, m$$

$$\sum_{i=1}^{m} x_{ij} \geq b_j, \qquad j = 1, 2, \ldots\ldots, n$$

$$x_{ij} \geq 0,\ i = 1, 2, \ldots\ldots, m;\ j = 1, 2, \ldots\ldots, n$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = F^0$$

$$a_i > 0,\ b_j > 0,\ i = 1, 2, ..., m;\ j = 1, 2, ..., n$$

where

$$\sum_{i=1}^{m}\sum_{j=1}^{n} r_{ij}x_{ij} \geq 0,\ \sum_{i=1}^{m}\sum_{j=1}^{n} s_{ij}x_{ij} \geq 0,\ \sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij}x_{ij} > 0\ , t_{ij} > 0, s_{ij} \geq 0 \text{ and } r_{ij} \geq 0$$

Clearly an optimal feasible solution of $(P1)$ is a feasible solution of $(P2)$, yielding the objective function flow pair(Z^0, F^0).

Definition (Paradoxical solution):
A solution X^p of $(P2)$ yielding the objective function-flow pair (Z^p, F^p) is called a 'Paradoxical solution', if for any other feasible solution of $(P2)$ yielding a flow pair (Z, F), we have

$$(Z, F) > (Z^p, F^p)$$

or

$$Z = Z^p, \text{ but } F < F^p$$

or

$$F = F^p\ , \text{ but } \ Z > Z^p$$

Let the optimal feasible solution of $(P1)$yield a value $Z^o = R^0 + \frac{S^0}{T^0}$ of the objective function$R(X) + \frac{S(x)}{T(x)}$. Then the condition for a paradoxical situation in a LPLFTP is given by the following theorem.

5 Sufficient Condition for Paradoxical Solution in LPLFTP

Theorem 1.2: If there exist a cell (i, j)in the table corresponding to the optimal solution $X^0$of the problem$(P1)$, such that

$$\left[\lambda\left[T^{0^2} + \lambda T^0(u_i^3 + v_j^3)\right](u_i^1 + v_j^1) + \lambda T^0(u_i^2 + v_j^2) - \lambda S^0(u_i^3 + v_j^3)\right] < 0$$

and the basis corresponding to a basic feasible solution of the problem$(P1)$with $a_i$replaced by $a_i + \lambda$and $b_j$by $b_j + \lambda$is the same as that corresponding to X^0, then there exist a paradoxical situation.

Proof: At an optimal basic feasible solution of$(P1)$.

$$Z^0 = R^0 + \frac{S^0}{T^0} = \sum_{i\in I}\sum_{j\in J} r_{ij}x_{ij} + \frac{\sum_{i\in I}\sum_{j\in J} s_{ij}x_{ij}}{\sum_{i\in I}\sum_{j\in J} t_{ij}x_{ij}}$$

$$= \sum_{i\in I}\sum_{j\in J}(u_i^1 + v_j^1)x_{ij} + \frac{\sum_{i\in I}\sum_{j\in J}(u_i^2 + v_j^2)x_{ij}}{\sum_{i\in I}\sum_{j\in J}(u_i^3 + v_j^3)x_{ij}}$$

$$(x_{ij} = 0,\ \forall \text{ non} - \text{basic cells } (i, j))$$

$$= \sum_{i\in I} a_i u_i^1 + \sum_{j\in J} b_j v_j^1 + \frac{\sum_{i\in I} a_i u_i^2 + \sum_{j\in J} b_j v_j^2}{\sum_{i\in I} a_i u_i^3 + \sum_{j\in J} b_j v_j^3}$$

Suppose a_i is replaced by $a_i + \lambda$ and $b_j$by $b_j + \lambda$ for some $\lambda > 0$, and then consider the problem $(P2)$ as

$$
\begin{aligned}
(P3)\ \text{Minimize } & Z = R(X) + \frac{S(X)}{T(X)} \\
\text{subject to } & \sum_{j \in J} x_{ij} = \hat{a}_i, \qquad i \in I \\
& \sum_{i \in I} x_{ij} = \hat{b}_j, \qquad j \in J \\
& x_{ij} \geq 0 \\
\text{where } & \hat{a}_k = a_k \quad k \in I - \{i\} \text{ and } \hat{a}_i = a_i + \lambda \\
& \hat{b}_l = b_l \quad l \in J - \{j\} \text{ and } \hat{b}_j = b_j + \lambda \\
& \text{for } \lambda > 0
\end{aligned}
$$

Clearly, every feasible solution of $(P3)$is a feasible solution of$(P1)$.

The basis corresponding to optimal basic feasible solution of $(P1)$will yield a basic feasible solution for$(P3)$, for which value of the objective function $\hat{Z}$would be given as

$$
\begin{aligned}
\hat{Z} &= \sum_{k \neq i} a_k u_k^1 + \sum_{l \neq j} b_l v_l^1 + (a_i + \lambda) u_i^1 + (b_j + \lambda) v_j^1 \\
&\qquad + \frac{\sum_{k \neq i} a_k u_k^2 + \sum_{l \neq j} b_l v_l^2 + (a_i + \lambda) u_i^2 + (b_j + \lambda) v_j^2}{\sum_{k \neq i} a_k u_k^3 + \sum_{l \neq j} b_l v_l^3 + (a_i + \lambda) u_i^3 + (b_j + \lambda) v_j^3} \\
&= \sum_{k \in I} a_k u_k^1 + \sum_{l \in J} b_l v_l^1 + \lambda (u_i^1 + v_j^1) + \frac{\sum_{k \in I} a_k u_k^2 + \sum_{l \in J} b_l v_l^2 + \lambda (u_i^2 + v_j^2)}{\sum_{k \in I} a_k u_k^3 + \sum_{l \in J} b_l v_l^3 + \lambda (u_i^3 + v_j^3)} \\
&= R^0 + \lambda (u_i^1 + v_j^1) + \frac{S^0 + \lambda (u_i^2 + v_j^2)}{T^0 + \lambda (u_i^3 + v_j^3)} \\
\hat{Z} - Z^0 &= R^0 + \lambda (u_i^1 + v_j^1) + \frac{S^0 + \lambda (u_i^2 + v_j^2)}{T^0 + \lambda (u_i^3 + v_j^3)} - R^0 - \frac{S^0}{T^0} \\
\hat{Z} - Z^0 &= \frac{\left[\lambda \left[{T^0}^2 + \lambda T^0 (u_i^3 + v_j^3)\right] (u_i^1 + v_j^1) + \lambda T^0 (u_i^2 + v_j^2) - \lambda S^0 (u_i^3 + v_j^3)\right]}{T^0 \left[T^0 + \lambda (u_i^3 + v_j^3)\right]} < 0 \\
&\Rightarrow \hat{Z} < Z^0
\end{aligned}
$$

Thus new flow $\hat{F} = \sum_{k \in I} a_k + \lambda = \sum_{l \in J} b_l + \lambda = F^0 + \lambda$. Thus the value of the objective function has fallen below the optimal value of $(P1)$ and flow has increased.

Thus for flow $F^0 + \lambda$, there exist one feasible solution with value of objective function less than Z^0. This implies that for flow $F^0 + \lambda$, optimal value of the objective function will be strictly less than Z^0. This implies that 'Paradoxical Solution' exists.

6 Heuristic for finding Paradoxical Solution in LPLFTP

First of all find the optimal solution of given LPLFTP. After it the process of calculating shadow price matrices for $R(X), S(X)$ and $D(X)$ is exactly the same as evaluating cells using the modified distribution method (MODI). Prepare the matrix M

$$\left(i.e.\ M = \left[\lambda\left[T^{0^2} + \lambda T^0(u_i^3 + v_j^3)\right](u_i^1 + v_j^1) + \lambda T^0(u_i^2 + v_j^2) - \lambda S^0(u_i^3 + v_j^3)\right]\right)$$

using optimal solution and shadow price matrix. Suppose an unloaded cell (q, r) appear in Table 4 and its corresponding value in matrix M with negative value M_{qr}as shown in Table 5 Cell(q, r) is not loaded so there must be at least one other loaded cell in row q and in column r each (in Table 4), lets take (k, r) and (q, p), to satisfy demand and supply requirement. The value X_{kr} and X_{qp} represent the optimal load at those locations respectively (in Table 4), the superscript * represents the location with the negative value in matrix M as indicated within brackets (in Table 5).

Table 4: Allotments (Basic Cells)

	Column r	Column p
Row k	r_{kr} s_{kr} X_{kr} t_{kr}	-----
Row q	r_{qr} s_{qr} t_{qr}	r_{qp} s_{qp} X_{qp} t_{qp}

Table 5: Identifying Negative Values in M

	Column r	Column p
Row k	M_{kr}	–
Row q	$(M_{qr})*$	M_{qp}

Now notice (in Table 4) that if cell (k, r) (or cell (q, p)) is only loaded cell in that column (or row), than by eliminating column r (or row q) we eliminate the negative value from matrix M (in Table 5) and associated loaded cell without affecting the values of the other negative values of matrix M. This means that the remaining LPLFTP has no MFL solution (without changing the initial transportation route). Therefore, the load causing the MFL phenomena is located in the column r (on the row q). The same analysis holds if more than one negative entries in matrix M of an optimal solution. Hence to explore the possibility of existence of an MFL situation, the first step is to prepare the matrix M using an optimal solution of a LPLFTP.

Search for mfl Solution

As already mentioned, the mfl phenomenon is based on relaxing the equality constraints for a given LFtp. Since $\sum\sum x_{ij}$ in the mfl is greater than the corresponding sum in nominal LPLFtp due to constraint relaxation, the gain resulting from moving load from a cells with a lowest cost coefficient d_{ij} *or* c_{ij} (minimize or maximize) to a cell with a smaller/larger cost coefficient must offset the extra cost resulting from the additional load.

A given LPLFtp is called perfect if the optimal solution has the $\max(m, n)$ loaded cells. In the MFL phenomenon, sometimes moving load from a larger cost coefficient cell to a smaller cost coefficient cell (vice versa) reduces the size of the basis, causing the LFTP to move closer to perfection, and this gives rise to the problem of degeneracy. Since the optimal shipping route is kept the same, the problem of degeneracy is resolved by loading zero shipments to the cells which were initially in the basis.

Table 6: Shifting Allotments

	Column r	Column?
Row k	$+\delta$	$-\delta$
Row?	$-\delta$	

Table 7: Assigning Allotments

	Column r	Column?
Row k	Max $(a_k\ , b_r\ )$	0
Row?	0	

Assume that there is a cell (k, r)which yields some gain (decrease of the total cost) during the decomposition process (refer Table 6 and Table 7). While increasing the load at cell (k, r) by δ, the same value is subtracted from other cell(s) in row k and column r, and then the rest of the load is distributed optimally in compliance with the equality/inequality constraints. Following this procedure, there will be an instance when the load in either row k or column r will be exhausted and only one load-maximum of a_k or b_r would remain. Notice that during this process, the equality constraint is relaxed, as the final load of $\max(a_k, b_r)$ is greater than one of the values a_k or b_r. Since this assignment of maximum load at cell (k, r) satisfies the column and row constraints simultaneously, we take a step towards decomposition of the LPLFtp. The proposed algorithm for finding an mfl solution of LPLFTP is as follows:

MFL Procedure

This procedure is similar to the procedure given in [5] with some modification for LPLFTP. In this procedure $r_{ij} + \frac{s_{ij}}{t_{ij}}$ [in Step 6] is used to pick out the cell (k, r)unlike the cost coefficient in [5].

Step 1. Solve $(P2)$ using any method and find the optimal solution.

Step 2. Create $M = \left[\lambda\left[T^{0^2} + \lambda T^0(u_i^3 + v_j^3)\right](u_i^1 + v_j^1) + \lambda T^0(u_i^2 + v_j^2) - \lambda S^0(u_i^3 + v_j^3)\right]$ matrix using optimal solution and shadow prices corresponding to R^0,S^0 and T^0.

Step 3. Identify negative entries in the matrix M and related columns and rows.

Step 4. Pick out the rows and columns of the matrix M with most negative entries.

Step5. Pick out rows/columns with single loading cell among those identified in Step 4. If the loading cells are more than one among these rows/columns then pass on to the next number of negative entries in the rows/columns and pick the rows/columns with single loading cell among them. Continue the process until a row/column with a single loading cell is obtained.

Step 6. Pick out the single loading cell (k, r) with the lowest/highest value of $r_{ij} + \frac{s_{ij}}{t_{ij}}$ (for minimization /maximization) among those identified in Step 5.

Step 7. Assign $X_{kr} = \max(a_k, b_r)$.

Step 8. Delete k^{th} row and r^{th} column from the cost matrix and the related matrix (M).

Step 9. If the reduced matrix M contains any negative entry, go to Step 3.

Step 10. Solve the reduced problem as a regular unbalanced LPLFTP.

Remark: In our method we increase the total demand or supply corresponding to the value of $\max(a_k, b_r)$ (not going beyond the value of $\max(a_k, b_r)$) to the cell (k, r) (where (k, r) is any single loading cell identified in most entries of negative values corresponding to row or column).

Numerical Example 2

Step 1. The optimal solution of numerical example 1 is $X_{13} = 7, X_{21} = 2, X_{23} = 8, X_{31} = 3$, $X_{32} = 8$ and all other $X_{ij} = 0$ with $Z^0 = 147 + \frac{78}{217} = 147.36$ and flow 28. Table 8 represents the optimal solution.

Step 2. Create $M = \left[\lambda\left[T^{0^2} + \lambda T^0(u_i^3 + v_j^3)\right](u_i^1 + v_j^1) + \lambda T^0(u_i^2 + v_j^2) - \lambda S^0(u_i^3 + v_j^3)\right]$ matrix using above optimal solution and shadow prices corresponding to R^0,S^0 andT^0. Table 9, 10 and 11 represents the shadow prices with respects to r_{ij}, s_{ij} *and* t_{ij}.

Step 3. Identify negative entries in the Table 12 corresponding to the columns and rows.

Step 4. Row A_2 and column B_2 each have four negative entries in matrix M.

Step 5. Column B_2 have single loading cell.

Step 6. Choose column B_2 as it has single loading cell.

Step 7. Assign$X_{32} = \max(a_3, b_2) = 11$.

Step 8. Delete row A_3 and column B_2 from Table 12.
Since there are no more negative entries as shown in Table 13, solve the reduced cost matrix as a LPLFTP. Hence, the MFL solution to the problem is $X_{13} = 7, X_{21} = 5$ $X_{23} = 8, X_{32} =$

Table 8: Optimal Solution of the LPLFTP

	B1	B2	B3	Supply
A1	7 5 4	1 8 5	9 4 7 6	7
A2	2 7 2 5	9 4 7	5 2 8 9	10
A3	8 4 3 7	2 1 8 9	11 8 2	11
demand	5	8	15	28

Table 9: u_i^1 and v_i^1

6	0	9	**4**
2	-4	5	**0**
8	2	11	**6**
2	**-4**	**5**	

Table 10: u_i^2 and v_i^2

9	6	4	**2**
7	4	2	**0**
4	1	-1	**-3**
7	**4**	**2**	

Table 11: u_i^3 and v_i^3

2	4	6	**-3**
5	7	9	**0**
7	9	11	**2**
5	**7**	**9**	

Table 12: Matrix M

	B1	B2	B3
A1	286935	990	435919
A2	97477	-194110	244942
A3	389186	97599	543161

Table 13: Reduced Matrix M

	B1	B3
A1	286935	435919
A2	97477	244942

11 for flow 31 with $Z = 135 + \frac{90}{238} = 135.38$. We can ship 3 more units in $Z = 135.38$ as compare to 28 units shipped in$Z = 147.36$. So we can ship more goods in lesser price, which surely sounds paradoxical.

7 Conclusion

An efficient heuristic algorithm for solving MFL paradox and algorithm for finding the initial basic feasible solution for LPLFTP is presented in this paper. There is no symmetric method yet exists in the literature to find the best MFL solution for an LPLFTP without losing the optimal shipping route. The approach here permits easy identification of such a situation and computation of the maximal allowable units and distribution of these extra units in a systematic approach. Since an MFL solution is obtained through reducing the number of loaded cells, the gains could be even greater in a LPLFTP.

References

[1] Deineko, V. G., Klinz, B. and Woeginger, G. J. Which Matrices are Immune against the Transportation Paradox? *Discrete Applied Mathematics*, 2003. 130: 495–501.

[2] Adlakha, V. and Kowalski, K. A Quick Sufficient Solution to the More for Less Paradox in Transportation Problem. *Omega*, 1998. 26: 541–547.

[3] Charnes, A. and Klingman, D. The More-for-Less Paradox in Distribution Models. *Cahiers du Centre d'Etudes Recherche Operationnelle*, 1971. 13: 11–22.

[4] Charnes, A., placeS. Duffuaa and Ryan, M. The More- for-Less Paradox in Linear Programming. *European Journal of Operation Research*, 1987. 31: 194–197.

[5] Robb, D. J. The More- for-Less Paradox in Distribution Models: An Intuitive Explanation. *IIE Transactions*, 1990. 22: 377–378.

[6] Adlakha, V. and Kowalski, K. A Heuristic Method for More-for-Less in Distribution Related Problems. *International Journal of Mathematical Education in Science and Technology*, 2001. 32: 61–71.

[7] Adlakha, V., Kowalski, K., Vemuganti, R. R. and Lev, B. More-for-Less Algorithm for Fixed Charge Transportation Problems. *Omega*, 2007. 35: 116–127.

[8] Adlakha, V., Kowalski, K. and Lev, B. Solving Transportation Problem with Mixed Constraints. *International Journal of Management Science and Engineering Management*, 2006. 1: 47–52.

[9] Teterav, A. G. On a Generalization of Linear and Piece-wise Linear Programming. *(Russ.), Ekon, I Mat. Met.*, 1969. 5: 440–447.

[10] Khurana, A. and Arora, S. R. The Sum of a Linear and Linear Fractional Transportation Problem with Restricted and Enhanced Flow. *Journal of Interdisciplinary Mathematics*, 2006. 9: 373–383.

[11] Joshi, V. D. and Gupta, N. A Heuristic for Obtaining a Better Initial Solution for the Linear Fractional Transportation Problem. *Bulletin of Australian Society of Operation Research*, 2009. 28: 45–50.