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Quarter-Sweep Improving Modified Gauss-Seidel Method for Pricing European Option

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Abstract The aim of this paper is to examine the application of the Quarter-Sweep Improving Modified Gauss-Seidel (QSIMGS) method in evaluating European option which governed by Black-Scholes partial differential equation (PDE). Quarter-sweep Crank-Nicolson approach is applied to approximate the PDE. Then, the generated linear system is solved by using the IMGS method. Some numerical experiments for a family of Gauss-Seidel (GS) methods such as Gauss-Seidel, Modified Gauss-Seidel (MGS) and Improving Modified Gauss-Seidel (IMGS) methods are performed with each full-, half-, and quarter-sweep iterations. Thus, from the numerical results obtained, we can show that the QSIMGS method is the most effective method.

Keywords Quarter-Sweep Improving Modified Gauss-Seidel method; Black-Scholes PDE; Crank-Nicolson scheme.

1 Introduction

Option is a derivative which gives its holder the right but no obligation to trade a certain asset at a certain date with the prescribed price. Financially, that certain date to trade is known as maturity date while the prescribed price for the asset is called the exercise price or strike price. In trading the option, the right to buy the asset is call option while the vice versa is put option. There are two major styles of option namely European and American options, a distinction that is not due to geographical location (Hull [1]). Actually the difference is that European option can be exercised only at maturity whereas American option can be exercised at any time until maturity. The trading of European options is mainly made over-the-counter while those traded on standardized exchanges are American. The pricing of European option is not so complicated than that of American option yet the solution of European option pricing can always adapted to American option pricing. However, our focus in this paper is on the pricing of European option.

To fairly evaluate the price that the holder needs to pay for the privilege of holding the option, Black and Scholes [2] and Merton [3] developed a PDE known as Black-Scholes PDE. Consequently, after the publication of their works, trading option began actively in Chicago Board Options Exchange. Initially, the options were traded informally (Shah [4]). Thus, their works have earned them the 1997 Nobel Prize in Economics. The Black-Scholes PDE is shown as follows

$$\frac{\partial v}{\partial t} = -\frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} - rs\frac{\partial v}{\partial s} + rv \tag{1}$$

where v is the value of the options, t is the time, s is the asset's price, σ is the volatility of the asset's price, and r is the risk free interest rate. The value of the European option at maturity which can also be called as the final time condition for the Black-Scholes PDE is (Higham [5]; Hull [1]; Goto et al. [11])

$$v\left(s,T\right)=\begin{cases}\max\left(s(T)-K,0\right) & \text{for call option}\\ \max\left(K-s\left(T\right),0\right) & \text{for put option}\end{cases} \tag{2}$$

where K is the exercise price and T is the maturity time. And the boundary conditions for Black-Scholes PDE are (Higham [5])

$$v\left(0,t\right)=0 \text{ and } v\left(s_{max},t\right)=s_{max} \tag{3}$$

$$v\left(0,t\right)=Ke^{-r(T-t)} \text{ and } v\left(s_{max},t\right)=0 \tag{4}$$

where European call and put options are denoted by Eqs. (3) and (4) respectively. The variable, s_{max} is the maximum asset price whereby it is sufficiently large.

In order to discretize the PDE in Eq. (1) into a linear system, quarter-sweep Crank-Nicolson approach will be applied. Actually, quarter-sweep approach was initiated by Othman and Abdullah [6] in order to accelerate the execution time by reducing the computational operations without altering the accuracy. Several studies on quarter-sweep approaches had shown that the quarter-sweep approach is superior to full-, and half-sweep approaches (Sulaiman et al. [7, 8]; Koh and Sulaiman [9, 10]). Besides that, the Improving Modified Gauss-Seidel (IMGS) method is another interesting issue to be observed. Kohno et al. [11] improved the Modified Gauss-Seidel (MGS) method by Gunawardena et al. [12] to propose the IMGS method with the preconditioner$Q = I + R(\alpha)$. In this paper, we investigate the effectiveness of QSIMGS method in solving European option pricing problem. So, some numerical experiments are performed in the family of Gauss-Seidel (GS) methods consisting of full-, half- and quarter-sweep iteration based on GS, MGS and IMGS methods.

The organization of the paper is as follows. In section 2, the discretization process of the Black-Scholes PDE using quarter-sweep CN approach is presented. It proceeds with section 3 which shows the formulation of the family of GS methods. Numerical results are then displayed in section 4. Finally, concluding remarks are given as well as suggestion of future works.

2 Quarter-Sweep Crank-Nicolson Approximation Equations

In this section, the discretization of the Black-Scholes PDE in Eq. (1) is considered. Figure 1 illustrates the finite grid networks in order to form the full-, half- and quarter-sweep approximation equations for (1). According to Figure 1, the full-, half- and quarter-sweep iterative methods will compute approximate values onto the solid node points only, until the convergence criterion is achieved. Then, the remaining points will be obtained by using direct method, see Othman and Abdullah [6], Sulaiman et al. [7, 8] and Koh and Sulaiman [9, 10].

Before deriving the full-, half- and quarter-sweep finite difference approximation equation for problem (1), assume that the solution domain (1) can be uniformly divided into $m = 2^h, h \geq 2$and L time steps in the s and t directions. Hence, the solution domain (1) of the problem is covered by a mesh of grid-lines

$$\left.\begin{array}{ll} s_i = s_0 + i\Delta s, & i = 0,1,2,\cdots,m \\ t_j = t_0 + j\Delta t, & j = 0,1,2,\cdots,L \end{array}\right\}$$

Figure 1: a, b and c Show the Node Points for the Full-, Half-, and Quarter-sweep Cases Respectively

where, subintervals in the s and t directions are represented by Δs and Δt respectively, which are uniform and defined as

$$\left.\begin{array}{l}\Delta s = \frac{s_{\max} - s_0}{m},\ m = n+1 \\ \Delta t = \frac{T - t_0}{L}\end{array}\right\}$$

By using CN scheme to discretize problem (1), the approximation equation can be developed as follows,

$$\begin{aligned}&\frac{v_{i,j+1} - v_{i,j}}{\Delta t} \\ &= -\sigma^2 (s_0 + ip\Delta s)^2 \left(\frac{v_{i-p,j} - 2v_{i,j} + v_{i+p,j} + v_{i-p,j+1} - 2v_{i,j+1} + v_{i+p,j+1}}{4 (p\Delta s)^2} \right) \\ &- r (s_0 + ip\Delta s) \left(\frac{v_{i+p,j} - v_{i-p,j} + v_{i+p,j+1} - v_{i-p,j+1}}{4p\Delta s} \right) + r \left(\frac{v_{i,j} + v_{i,j+1}}{2} \right). \qquad (5)\end{aligned}$$

Then Eq. (5) can be simplified as

$$c_i v_{i-p,j} + a_i v_{i,j} + b_i v_{i+p,j} = f_{i,j+1} \qquad (6)$$

where

$$c_i = \frac{1}{2p\Delta s} \left(\frac{\beta_i}{p\Delta s} - \frac{\lambda_i}{2} \right), a_i = \frac{1}{\Delta t} + \frac{r}{2} - \frac{\beta_i}{(p\Delta s)^2}, b_i = \frac{1}{2p\Delta s} \left(\frac{\beta_i}{p\Delta s} + \frac{\lambda_i}{2} \right),$$

$$f_{i,j+1} = -c_i v_{i-p,j+1} + \left(\frac{2}{\Delta t} - a_i \right) v_{i,j+1} - b_i v_{i+p,j+1}, \beta_i = -\frac{1}{2}\sigma^2 (s_0 + i\Delta s), \lambda_i = -r (s_0 + i\Delta s).$$

From Eq. (6), the value of p which corresponds to 1, 2 and 4 indicates the full-, half- and quarter-sweep cases respectively. Then, we can rewrite (6) in a matrix form for each time layer,j as

$$A \underset{\sim}{v} = \underset{\sim}{f} \qquad (7)$$

where

$$A = \begin{bmatrix} a_{1p} & b_{1p} & & \\ c_{2p} & a_{2p} & b_{2p} & \\ & \ddots & \ddots & \ddots \\ & & c_{m-p} & a_{m-p} \end{bmatrix}_{\left(\left(\frac{m}{p}\right)-1\right)\times\left(\left(\frac{m}{p}\right)-1\right)},$$

$$\underset{\sim}{v} = \begin{bmatrix} v_{1p,j} & v_{2p,j} & \cdots & v_{m-p,j} \end{bmatrix}^T,$$

$$\underset{\sim}{f} = \begin{bmatrix} f_{1p,j+1} & f_{2p,j+1} & \ldots & f_{m-p,j+1} \end{bmatrix}^T.$$

3 Formulation of the family of Gauss-Seidel methods

As mentioned previously to solve the linear system in Eq. (7), we consider the standard GS, MGS (Gunawardena et al., [12]; Koh & Sulaiman [8]) and IMGS (Kohno et al., [11]) methods. So as to develop and implement a family of GS algorithms, multiply both sides of Eq. (7) with preconditioner such as

$$QA\underset{\sim}{v} = Q\underset{\sim}{f} \tag{8}$$

where

$$Q = I + R(\alpha),$$

$$R(\alpha) = \begin{bmatrix} 0 & -\alpha b_{1p} & 0 & \cdots & 0 \\ 0 & 0 & -\alpha b_{2p} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha b_{m-2p} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{\left(\left(\frac{m}{p}\right)-1\right)\times\left(\left(\frac{m}{p}\right)-1\right)},$$

I = Identity matrix.
As we know, A is a tridiagonal matrix of order$\left(\left(\frac{m}{p}\right)-1\right)$ therefore, I is an identity matrix for the preconditioner, Q of order$\left(\left(\frac{m}{p}\right)-1\right)$. If$\alpha = 0$, it is the GS method, whereas if$\alpha = 1$, it will become MGS method (Gunawardena *et al.*, [12]; Koh and Sulaiman, [8]) and for IMGS (Kohno et al., [11]), $\alpha$is selected adequately. Based on Eq. (8), the linear system can be rewritten as:

$$A*\underset{\sim}{v} = \underset{\sim}{f}* \tag{9}$$

where

$$A* = QA,$$

$$\underset{\sim}{f}* = Q\underset{\sim}{f}.$$

Generally, the solution of linear system in Eq. (9) can be computed by implementing Algorithm 1 as follows.

Algorithm 1:

(i) Initializing all the parameters. Set k = 0.

(ii) For $i = 1p, 2p, \cdots, m-p$, calculate

$$v_i^{(k+1)} = \frac{1}{A*_{ii}} \left(f*_i - \sum_{j=1}^{i-1} A*_{ij}\, v_j^{(k+1)} - \sum_{j=i+1}^{i-n} A*_{ij}\, v_j^{(k)} \right)$$

(iii) Convergence test.

If the error tolerance $\left| v_i^{(k+1)} - v_i^{(k)} \right| < \varepsilon = 10^{-10}$ is satisfied, the value option at that time is $v_i^{(k+1)}$ and the algorithm end.

Else, set k = k+1 and go to step ii.

4 Numerical Results

Numerical experiments are performed to examine the effectiveness of the family of GS methods. The criteria concerned in these experiments include the number of iterations, computational time and maximum absolute error. The matrix sizes tested are 512, 1024, 2048, 4096, 8192 and 16384. As for the time steps, we have 100 time steps. The parameters are $T = 0.5$ (year), $K = 10.0$, $r = 0.05$, $\sigma = 0.2$, and $s \in [1\mathrm{E}-6, 30]$(Goto et al. [13]). We consider European put option in these experiments. The exact solution used to evaluate the accuracy of the numerical solutions is given by (Black & Scholes, 1973)

$$v = Ke^{-rt} \left(N\left(-d_2\right)\right) - sN\left(-d_1\right) \tag{10}$$

where

$$d_1 = \frac{\ln\left(\frac{s}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln\left(\frac{s}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t},$$

N is the cumulative normal distribution.

The numerical results of the analysis and the summary of the performance of the iterative methods compare to the classical FSGS methods are tabulated in Table 1 and Table 2 respectively.

5 Conclusion

In the computational experiments, we have examined the iterative methods with different mesh sizes in terms of number of iterations, computational time and maximum absolute errors. Based on the results for different mesh sizes, the accuracies of all the iterative methods are in good agreement. As we can see from the numerical results, QSIMGS method has the least number of iterations and compute with the fastest time for all mesh sizes. In

Table 1: Comparison of Number of Iterations, Computational Time and Maximum Absolute Errors for the Family of GS Methods

Methods	Number of Iterations						Computational Time (Seconds)						Maximum Absolute Errors					
	Mesh Sizes																	
	512	1024	2048	4096	8192	16384	512	1024	2048	4096	8192	16384	512	1024	2048	4096	8192	16384
FSGS	53	174	613	2181	7739	27261	0.17	0.81	5.52	40.13	290	2151.43	3.34E-5	7.27E-6	7.02E-5	4.56E-4	8.63E-4	1.12E-3
FSMGS	22	65	224	796	2838	10057	0.10	0.49	2.86	20.47	149.49	1130.38	3.34E-5	7.27E-6	7.01E-5	4.55E-4	8.62E-4	1.12E-3
FSIMGS	10 (1.51)	16 (1.731)	47 (1.7635)	101 (1.8567)	201 (1.9222)	394 (1.959)	0.05	0.22	0.86	3.30	13.93	51	3.34E-5	7.27E-6	7.01E-5	4.55E-4	8.62E-4	1.12E-3
HSGS	19	53	174	613	2181	7739	0.05	0.22	1.27	8.71	51.78	382.73	4.91E-4	1.23E-4	3.08E-5	7.02E-5	4.56E-4	8.63E-4
HSMGS	10	22	65	224	796	2838	0.03	0.11	0.68	3.93	24.93	180.4	4.91E-4	1.23E-4	3.08E-5	7.01E-5	4.55E-4	8.62E-4
HSIMGS	7 (1.2)	10 (1.51)	16 (1.731)	47 (1.7635)	101 (1.8567)	201 (1.9222)	0.03	0.08	0.32	1.21	4.57	18.14	4.91E-4	1.23E-4	3.08E-5	7.01E-5	4.55E-4	8.62E-4
QSGS	10	19	53	174	613	2181	0.01	0.05	0.24	1.15	7.13	54.83	1.99E-3	4.91E-4	1.23E-4	3.08E-5	7.02E-5	4.56E-4
QSMGS	6	10	22	65	224	796	0.02	0.08	0.15	0.8	5.98	39.25	1.99E-3	4.91E-4	1.23E-4	3.08E-5	7.01E-5	4.55E-4
QSIMGS	5 (1.07)	7 (1.2)	10 (1.51)	16 (1.731)	47 (1.7635)	101 (1.8567)	0.01	0.03	0.11	0.48	1.79	7.05	1.99E-3	4.91E-4	1.23E-4	3.08E-5	7.01E-5	4.55E-4

fact, it can decrease the number of iterations about 90.57-99.62% compare to FSGS method. In terms of computational time, QSIMGS speeds up approximately 94.11-99.67% faster than FSGS execution time. In conclusion, QSIMGS method is the most effective method among the family of GS methods by having less number of iterations and shorter computational time. Nonetheless, it manages to retain the accuracy of the standard GS method. In future works, we can deal with other types of problems like American option pricing (Koh et al., [14]; Hon [15]) or 2 dimensional PDE option pricing (Jeong et al., [16]) problems by applying the QSIMGS method.

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Table 2: Percentages reduction of number of iterations and computational time of the tested iterative methods relative to FSGS method

Methods	No. of iterations reduced (%)	Computational time reduced (%)
FSMGS	58.49-63.50	39.51-48.99
FSIMGS	81.13-98.55	70.59-97.63
HSGS	64.15-71.89	70.59-82.21
HSMGS	81.13-89.73	82.35-91.61
HSIMGS	86.79-99.26	82.35-99.16
QSGS	81.13-92.08	93.83-97.45
QSMGS	88.68-97.11	88.24-98.18
QSIMGS	90.57-99.63	94.12-99.67

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