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Boundary Layer Flow and Heat Transfer Adjacent to a Stretching Vertical Sheet with Prescribed Surface Heat Flux

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Abstract The steady two-dimensional flow adjacent to a vertical, continuously stretching sheet in a viscous and incompressible fluid is studied. It is assumed that the sheet is stretched with a power-law velocity and is subjected to a variable surface heat flux. The governing partial differential equations are reduced to nonlinear ordinary differential equations by a similarity transformation, before being solved numerically by the Keller-box method. Results showed that the heat transfer rate at the surface increases as the velocity exponent parameter, mixed convection parameter and the Prandtl number are increased.

Keywords Similarity solution; heat transfer; numerical solution; stretching sheet.

1 Introduction

Extrusion of a polymer in a melt-spinning process, metals and plastics, the boundary layer along material handling conveyers, the cooling and/or drying of papers and textiles, glass blowing, continuous casting and spinning of fibers are examples of industrial applications, involves flow due to a stretching surface. The two-dimensional steady flow of an incompressible viscous fluid caused by a linearly stretching plate was first discussed by Crane [1]. Since then, many authors have studied various aspects of this problem. For instance, Magyari et al. [2] studied the heat and mass transfer characteristics of the boundary-layer flows induced by continuous surfaces with rapidly decreasing velocities. This problem was then extended by Ali [3] to a vertical surface, where the effect of buoyancy force was taken into consideration. Quite recently, Partha et al. [4] studied the similar problem, by considering exponentially stretching surface. The temperature field in the flow over a linearly stretching surface subject to a variable surface temperature was studied by Grubka and Bobba [5], while Dutta et al. [6] reported the temperature distribution for the uniform surface heat flux condition. Elbashbeshy [7] and Lin and Chen [8] considered the heat transfer characteristics on a stretching horizontal surface subject to a power-law velocity and variable surface heat flux.

Motivated by the above investigations, the present study considers the heat transfer characteristics adjacent to a stretching vertical sheet with a power-law velocity subjected to a variable surface heat flux. This problem is different from the above mentioned investigations where the effect of buoyancy force was not taken into consideration.

2 Problem Formulation

Consider a steady, two-dimensional flow of a viscous and incompressible fluid adjacent to a vertical, continuously stretching sheet placed in the plane $y = 0$ of a Cartesian system of coordinates xy with the x-axis along the sheet, while the y-axis is measured normal to the surface of the sheet. It is assumed that the surface heat flux and the stretching velocity vary in a power-law with the distance from the leading edge, i.e. $q_w(x) = ax^n$ and $u_w(x) = bx^m$ respectively, where a and b are constants and m and n are the exponents. Under these assumptions along with the Boussinesq and boundary-layer approximations, the equations which model the problem under consideration are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta\left(T - T_\infty\right), \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

where u and v are the velocity components along the x-axis and y-axis, respectively. Further, ν, α, β, T, T_∞ and g are the kinematic viscosity, thermal diffusivity, thermal expansion coefficient, fluid temperature, ambient temperature and acceleration due to gravity respectively. The boundary conditions are

$$\begin{array}{l} u = u_w(x), \quad v = 0, \qquad \frac{\partial T}{\partial y} = -\frac{q_w}{k} \quad \text{at} \quad \mathrm{y} = 0, \\ u \to 0, \quad T \to T_\infty \quad \text{as} \quad \mathrm{y} \to \infty, \end{array} \tag{4}$$

where u_w, q_w and k are the velocity of the stretching sheet, the surface heat flux and the thermal conductivity, respectively.

The continuity equation can be satisfied by introducing a stream function ψ such that $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. The momentum and energy equations can be transformed into the corresponding nonlinear ordinary differential equations by the following transformation [9,10]:

$$\eta = \left(\frac{u_w}{\nu x}\right)^{1/2} y, f(\eta) = \frac{\psi}{\left(\nu x u_w\right)^{1/2}}, \theta(\eta) = \frac{k(T - T_\infty)}{q_w}\left(\frac{u_w}{\nu x}\right)^{1/2}, \tag{5}$$

where η is the independent similarity variable. The transformed nonlinear ordinary differential equations are

$$f''' + \frac{m+1}{2}ff'' - mf'^2 + \lambda\theta = 0, \tag{6}$$

$$\frac{1}{\Pr}\theta'' + \frac{m+1}{2}f\theta' - nf'\theta = 0, \tag{7}$$

where primes denote differentiation with respect to η, m is the velocity exponent parameter, n is the temperature exponent parameter, $\Pr = \nu/\alpha$ is the Prandtl number and $\lambda = Gr_x/Re_x^{5/2}$ is the buoyancy or mixed convection parameter with $Gr_x = g\beta q_w x^4/(k\nu^2)$ and $Re_x = u_w x/\nu$ are the local Grashof number and the local Reynolds number, respectively.

It can be shown that λ is independent of x if $n = (5m-3)/2$. Thus, in the presence of buoyancy force, similarity is achieved under this limitation. For $n = (5m-3)/2$, equation (7) becomes

$$\frac{1}{\Pr}\theta'' + \frac{m+1}{2}f\theta' - \frac{5m-3}{2}f'\theta = 0. \tag{8}$$

The transformed boundary conditions are:

$$\begin{aligned} &f(0) = 0, \quad f'(0) = 1, \quad \theta'(0) = -1, \\ &f'(\infty) \to 0, \quad \theta(\infty) \to 0. \end{aligned} \tag{9}$$

Further, $\lambda > 0$ and $\lambda < 0$ correspond to assisting (aiding) and opposing flows, respectively. It is worth mentioning that for $\lambda = 0$, equations (6) and (8) are decoupled and this case corresponds to the forced convection flow past a stretching sheet.

The physical quantities of interest are the skin friction coefficient C_f and the local Nusselt number Nu_x, which are defined as

$$C_f = \frac{\tau_w}{\rho u_w^2/2}, \quad Nu_x = \frac{xq_w}{k(T_w - T_\infty)}, \tag{10}$$

where the wall shear stress τ_w and surface heat flux q_w are given by

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}, \quad q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0}, \tag{11}$$

with μ being the dynamic viscosity. Using the non-dimensional variables (5), we obtain

$$\frac{1}{2}C_f Re_x^{1/2} = f''(0), \quad Nu_x/Re_x^{1/2} = 1/\theta(0). \tag{12}$$

We notice that in the absence of buoyancy force, the analytical solution of equation (6) for $m = 1$ was reported by Crane [1] as

$$f(\eta) = 1 - e^{-\eta}, \tag{13}$$

and the solution for the thermal field is

$$\theta(\eta) = \frac{1}{\Pr}e^{-\eta\Pr}\frac{M\left(\Pr - n, \Pr + 1, -\Pr e^{-\eta}\right)}{M\left(\Pr - n, \Pr, -\Pr\right)}, \tag{14}$$

where $M(a,b,z)$ denotes the confluent hypergeometric function (see Abramowitz and Stegun [11]) with

$$\begin{aligned} M(a,b,z) &= 1 + \sum_{k=1}^{\infty}\frac{a_k}{b_k}\frac{z^k}{k!}, \\ a_k &= a(a+1)(a+2)\cdots(a+k-1), \\ b_k &= b(b+1)(b+2)\cdots(b+k-1). \end{aligned}$$

Further, from equations (13) and (14), the skin friction coefficient $f''(0)$ and the surface temperature $\theta(0)$ can be shown to be given by

$$\begin{aligned} f''(0) &= -1, \\ \theta(0) &= \frac{1}{\Pr}\frac{M(\Pr - n, \Pr + 1, -\Pr)}{M(\Pr - n, \Pr, -\Pr)}. \end{aligned} \tag{15}$$

3 Results and discussion

The nonlinear ordinary differential equations (6) and (8) subjected to (9) have been solved numerically using a finite-difference scheme known as the Keller-box method [12], for some values of velocity exponent parameter m, buoyancy parameter λ and Prandtl number Pr. Comparison of the values of $\theta(0)$ with those obtained by Elbashbeshy [7] and Liu [13] for several values of Pr when $m = 1$ in the case of force convection flow ($\lambda = 0$) is listed in Table 1. It is observed that the results show a very good agreement.

Table 1: Values of the Surface Temperature $\theta(0)$ for different values of λ, m and Pr

λ	m	Pr	Numerical solution	Series solution, equation (15)	Elbashbeshy [7]	Liu [13]
0	0.6	1	1.8900			
0		0.72	2.3703			
0	1		1.2367	1.236657472	1.2253	
		1	1	1	1	
		10	0.2688	0.268768515	0.2688	
		6.7	0.3333	0.333303061		0.333303
1	1	1	0.9240			
2			0.8842			

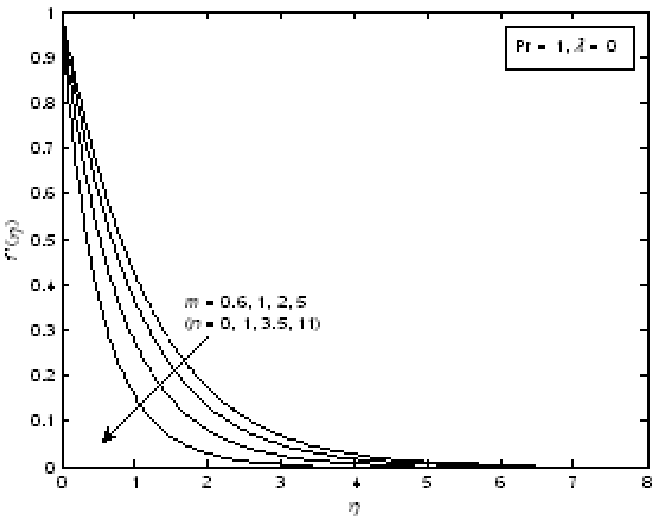

Figure 1: Velocity Profiles $f'(\eta)$ for some values of m (and n)

Figures 1 and 2 show the velocity and temperature distributions for some values of velocity exponent parameter m (and temperature exponent parameter n) with fixed values

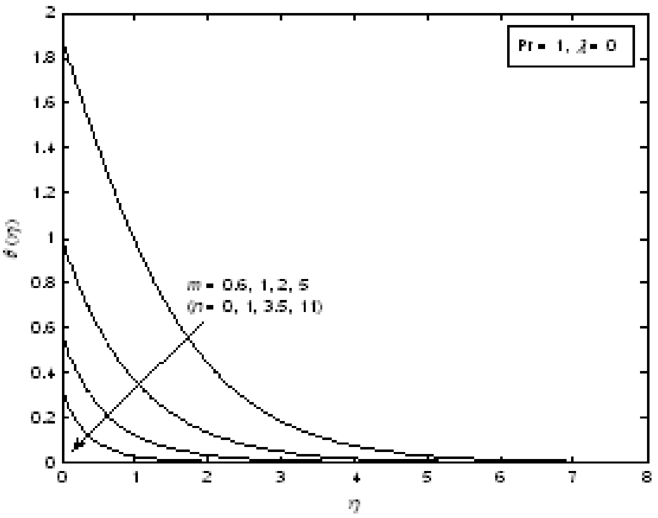

Figure 2: Temperature Profiles θ (η) for some values of m (and n)

of Pr and λ, respectively. It is seen that the boundary layer thickness of both velocity and temperature profiles decrease as the value of m increases, and in consequence increase the skin friction coefficient $f''(0)$ (in absolute sense) and the local Nusselt number $1/\theta(0)$. The effects of buoyancy parameter λ on velocity and temperature distributions are presented respectively in Figures 3 and 4 when the other parameters are fixed to unity. It is observed that the mixed convection parameter results in a diverse behavior of velocity and temperature profiles in the boundary layer.

The samples of velocity and temperature profiles for various values of Prandtl number (Pr) are displayed in Figures 5 and 6, respectively. Both the graphs demonstrate that the increase of Pr results in a decrease of the velocity and temperature boundary layer thicknesses. Thus, both the skin friction coefficient $|f''(0)|$ and the local Nusselt number $1/\theta(0)$ increase as Pr increases. Figures 1 – 6 show that the far field boundary conditions are satisfied asymptotically, which support the validity of the numerical results obtained.

Finally, the numerical results for the skin friction coefficient $f''(0)$ and the local Nusselt number (heat transfer rate at the surface) $1/\theta(0)$ for various values of velocity exponent parameter m when $\Pr = 1$ are presented in Figures 7 and 8, respectively. The results presented in these figures are in agreement with the velocity and temperature profiles presented in Figures 1 and 2. As the value of the buoyancy parameter λ increases, the velocity gradient at the surface $f''(0)$ changes its signs from negative to positive (for $m = 0.6$ and $m = 1$). Physically positive values of $f''(0)$ mean the fluid exert a drag force on the sheet, and negative values mean the opposite. This is not surprising since in the absence of buoyancy force ($\lambda = 0$), the formation of the boundary layer depends solely on the stretching sheet (the sheet exerts a drag force on the fluid). Different behaviors are observed for the temperature profiles, where it is observed that for all values of parameters considered, the values of the local Nusselt number $1/\theta(0)$ are always positive, which mean the heat is transferred from

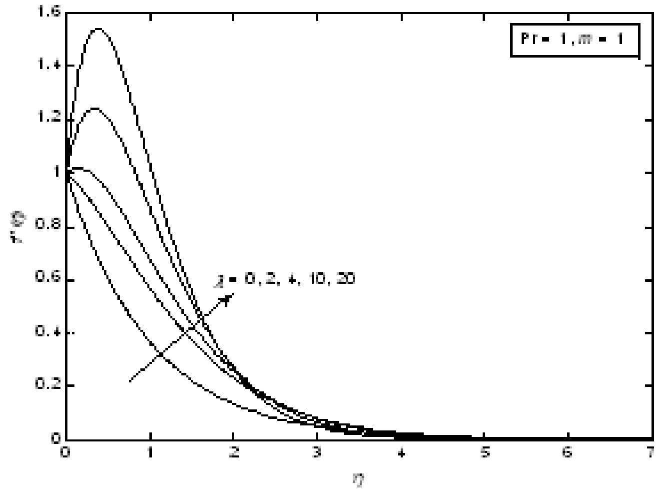

Figure 3: Velocity Profiles $f'(\eta)$ for some values of Buoyancy Parameter λ

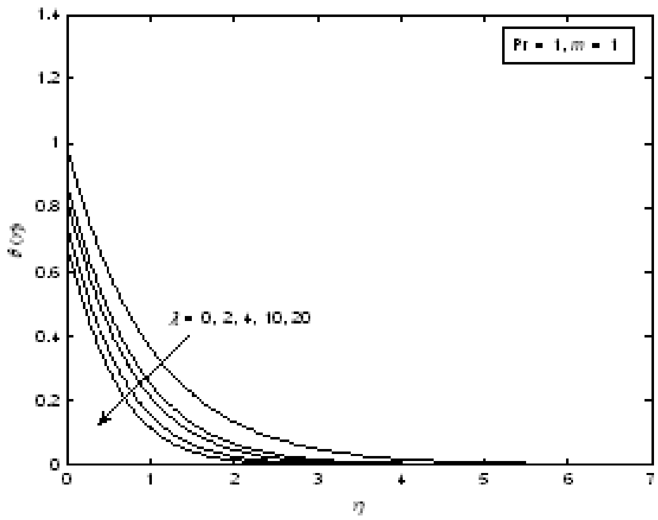

Figure 4: Temperature Profiles $\theta(\eta)$ for some values of Buoyancy Parameter λ

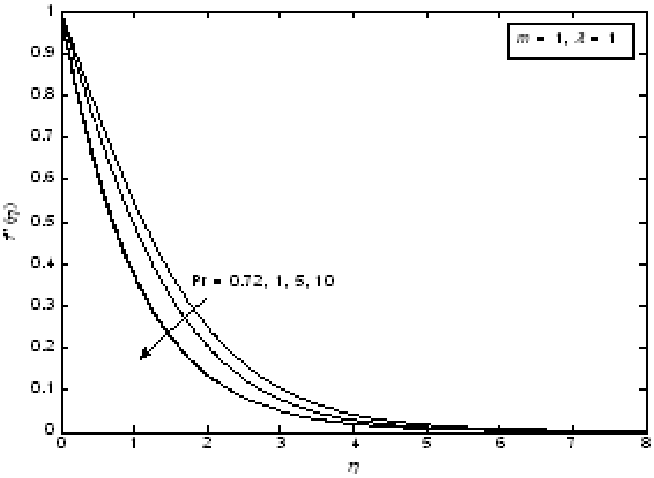

Figure 5: Velocity Profiles $f'(\eta)$ for some values of Prandtl Number, Pr

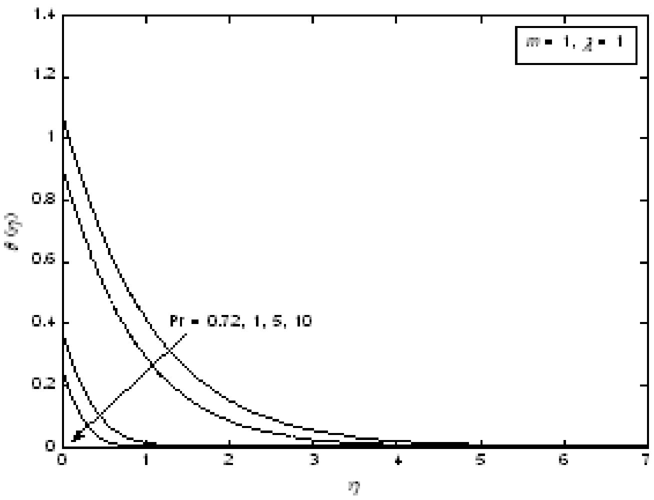

Figure 6: Temperature Profiles θ (η) for some values of Prandtl Number, Pr

the hot sheet to the cool fluid, regardless of the existence of the buoyancy force. Moreover, Figures 7 and 8 show that for the buoyancy opposing flow ($\lambda < 0$), the solution exists only for small negative values of λ. Beyond these values, the boundary layer separates from the surface, and thus no solution is obtained.

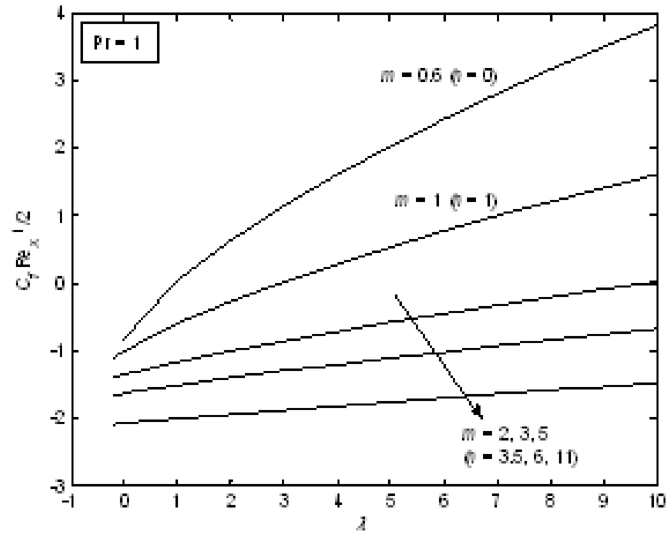

Figure 7: Variation of the Skin Friction Coefficient $C_f \mathrm{Re}_x{}^{1/2}$against λ for some values of m (and n)

4 Conclusions

In this paper, the heat transfer characteristics near a stretching vertical sheet have been studied. The boundary layer equations governing the flow are reduced to ordinary differential equations using a similarity transformation. Using a numerical technique, these equations are then solved to obtain the velocity and temperature distributions as well as the skin friction coefficient and the local Nusselt number for various values of the velocity exponent parameter, buoyancy parameter and Prandtl number. It is found that the heat transfer rate at the surface increases with increasing values of these parameters.

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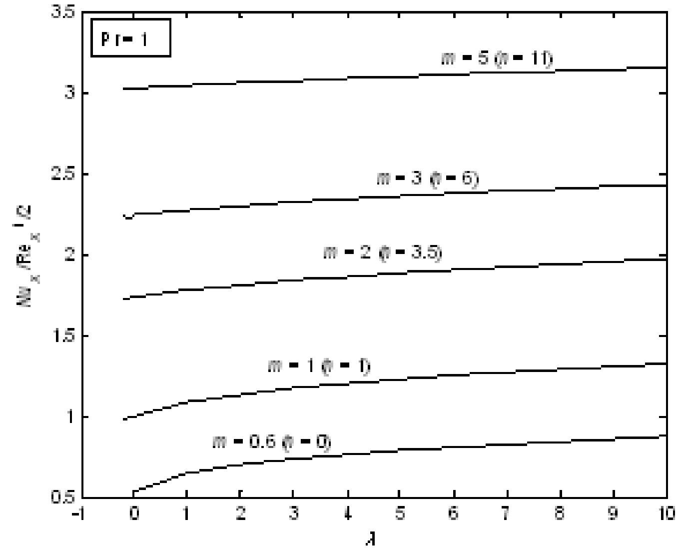

Figure 8: Variation of the Local Nusselt Number $Nu_x/\mathrm{Re}_x{}^{1/2}$against λ for some values of m (and n)

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