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A Class of Combinatorial Functions for Eulerian Numbers

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Abstract In this note, we introduce a new class of functions pertaining to binomial coefficients $\binom{n}{m}$ and Eulerian numbers $\left\langle {n \atop m} \right\rangle$, which arise in the recent study of descents in permutations. Given positive integers a and b, let $f_i(x) = 2^{-i}\binom{a+b}{i}\left\langle {i \atop x} \right\rangle$ and $f(x) = \sum_i f_i(x)$. Based on the generating function methods, some identities involving f_i and f are provided.

Keywords Eulerian number; Binomial coefficient; Generating function.

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1 Introduction

Given positive integers n and k, Eulerian number [1], denoted by $\left\langle {n \atop k} \right\rangle$, is the number of permutations $\pi_1\pi_2\cdots\pi_n$ of $\{1, 2, \cdots, n\}$ that have exactly k ascents, namely, k places where $\pi_i < \pi_{i+1}$. An easy observation shows that $\left\langle {n \atop k} \right\rangle = 0$ for $k \geq n$ since there can be at most $n-1$ ascents. In combinatorics Eulerian numbers can be viewed as the coefficients of the Eulerian polynomials [2]

$$A_n(x) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^{n-k}. \tag{1}$$

For example, $A_0(x) = A_1(x) = 1$ and $A_2(x) = 1 + x$.

Eulerian numbers as well as some more familiar Bernoulli numbers, harmonic numbers, Fibonacci numbers, Stirling numbers, Catalan numbers and binomial coefficients, to name only a few, have a number of applications in combinatorics and number theory (see e.g. [2,3]). Since the recurrence for Eulerian numbers is more complicated than for many other special numbers, and they increase very rapidly, combinatorial identities for Eulerian numbers receive less research attention.

Recently, the authors of [4] propose a symmetrical Eulerian identity which appears to be elegant, and provide several different proofs for it. This kind of identities arises in the study of descents in permutations which have a restriction on their largest drop [5]. An interesting inequality concerning Eulerian numbers is introduced in [6]. The work [7] treats the chromatic polynomials of incomparability graphs of posets by virtue of these Eulerian identities. The argument involved there leads us to a new class of functions of Eulerian numbers. Given positive integers a and b, we define $f_i(x) = 2^{-i}\binom{a+b}{i}\left\langle {i \atop x} \right\rangle$ and the summation $f(x) = \sum_{i=x+1}^{a+b} f_i(x)$. We remark that we use the convention throughout this paper that the Eulerian number $\left\langle {0 \atop 0} \right\rangle = 0$. With our notations, the result in [4] can be rewritten in the following form

Theorem 1 *[4] For positive integers a and b,*

$$\sum_{i=a}^{a+b} 2^i f_i(a-1) = \sum_{i=b}^{a+b} 2^i f_i(b-1). \tag{2}$$

In this paper, we provide another identity satisfied by these functions utilizing the instrumental generating function methods, which has numerous applications [8].

Theorem 2 *Let $a \geq 2$ and $b \geq 1$ be integers. We have*

$$f(b-1) - f(a-2) = \frac{f_b(b-1)}{2^a} - \frac{f_{a-1}(a-2)}{2^{b+1}}. \tag{3}$$

The resulting identity (3) is interesting even for the special case of $a = b$. It states that for any integer $a \geq 2$,

$$f(a-1) - f(a-2) = \frac{f_a(a-1)}{2^a} - \frac{f_{a-1}(a-2)}{2^{a+1}}. \tag{4}$$

2 Proof of Theorem 2

In this section, we present a proof of Theorem 2 based on the generating function of Eulerian numbers (see e.g. [2] pp.351).

The generating function for Eulerian numbers is given by

$$E(w,z) = \frac{e^z - e^{wz}}{e^{wz} - we^z} = \sum_{n=0}^{\infty}\sum_{i=0}^{\infty} \left\langle {n \atop i} \right\rangle w^i \frac{z^n}{n!}, \tag{5}$$

which is slightly different from Eq. (7.56) in [2] since they use the convention $\left\langle {0 \atop 0} \right\rangle = 1$. Thereby, we obtain

$$(e^{2wz} - w^2 e^{2z})E(w,z) = (e^{wz} + we^z)(e^{wz} - we^z)E(w,z) = (e^{wz} + we^z)(e^z - e^{wz}). \tag{6}$$

We first calculate the left-hand side of (6). Employing (5), we have

$$\begin{aligned} e^{2wz}E(w,z) &= \sum_{k=0}^{\infty} \frac{(2wz)^k}{k!} \sum_{n=0}^{\infty}\sum_{i=0}^{\infty} \left\langle {n \atop i} \right\rangle w^i \frac{z^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{(2wz)^k}{k!} \sum_{n'=k}^{\infty}\sum_{i'=k}^{\infty} \left\langle {n'-k \atop i'-k} \right\rangle w^{i'-k} \frac{z^{n'-k}}{(n'-k)!} \\ &= \sum_{k=0}^{\infty}\sum_{n'=k}^{\infty}\sum_{i'=k}^{\infty} \frac{2^k}{k!} \left\langle {n'-k \atop i'-k} \right\rangle w^{i'} \frac{z^{n'}}{(n'-k)!} \\ &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\sum_{i=k}^{\infty} 2^k \binom{n}{k} \left\langle {n-k \atop i-k} \right\rangle w^i \frac{z^n}{n!}, \end{aligned} \tag{7}$$

and

$$\begin{aligned}
w^2 e^{2z} E(w,z) &= w^2 \sum_{k=0}^{\infty} \frac{(2z)^k}{k!} \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \left\langle {n \atop i} \right\rangle w^i \frac{z^n}{n!} \\
&= w^2 \sum_{k=0}^{\infty} \frac{(2z)^k}{k!} \sum_{n'=k}^{\infty} \sum_{i'=2}^{\infty} \left\langle {n'-k \atop i'-2} \right\rangle w^{i'-2} \frac{z^{n'-k}}{(n'-k)!} \\
&= \sum_{k=0}^{\infty} \sum_{n'=k}^{\infty} \sum_{i'=2}^{\infty} \frac{2^k}{k!} \left\langle {n'-k \atop i'-2} \right\rangle w^{i'} \frac{z^{n'}}{(n'-k)!} \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \sum_{i=2}^{\infty} 2^k \binom{n}{k} \left\langle {n-k \atop i-2} \right\rangle w^i \frac{z^n}{n!}. \qquad (8)
\end{aligned}$$

Next, we expand the right-hand side of (6) and get

$$\begin{aligned}
(e^{wz} + we^z)(e^z - e^{wz}) &= e^{(w+1)z} - e^{2wz} + we^{2z} - we^{(w+1)z} \\
&= \sum_{k=0}^{\infty} \frac{(w+1)^k z^k}{k!} - \sum_{k=0}^{\infty} \frac{(2wz)^k}{k!} \\
&\quad + w \sum_{k=0}^{\infty} \frac{(2z)^k}{k!} - w \sum_{k=0}^{\infty} \frac{(w+1)^k z^k}{k!}. \qquad (9)
\end{aligned}$$

Expanding the corresponding sums in (7)–(9) and identifying coefficients of $w^i z^n$ in (6), we obtain for $n \geq 2$ and $i \geq 2$,

$$\sum_{k=0}^{i} 2^k \binom{n}{k} \left\langle {n-k \atop i-k} \right\rangle - \sum_{k=0}^{n-i+1} 2^k \binom{n}{k} \left\langle {n-k \atop i-2} \right\rangle = \binom{n}{i} - \binom{n}{i-1}. \qquad (10)$$

Recall that we have

$$\left\langle {n \atop m} \right\rangle = \left\langle {n \atop n-m-1} \right\rangle \qquad (11)$$

for integers $n \geq 0$ and $m \geq 0$. Setting $n = a+b$ and $i = a$ in (10), we have

$$\begin{aligned}
\sum_{k=0}^{a} 2^k \binom{a+b}{a+b-k} \left\langle {a+b-k \atop b-1} \right\rangle - \sum_{k=0}^{b+1} 2^k \binom{a+b}{a+b-k} \left\langle {a+b-k \atop a-2} \right\rangle \\
= \binom{a+b}{b} - \binom{a+b}{a-1} \qquad (12)
\end{aligned}$$

by using (11) and a basic property of binomial coefficients. Setting $i = a+b-k$ in the first summation of (12) and $i = a+b-k+1$ in the second summation of (12), we obtain

$$\sum_{k=b}^{a+b} 2^{-i} \binom{a+b}{i} \left\langle {i \atop b-1} \right\rangle - \sum_{i=a}^{a+b} 2^{-(i-1)} \binom{a+b}{i-1} \left\langle {i-1 \atop a-2} \right\rangle = \frac{\binom{a+b}{b} - \binom{a+b}{a-1}}{2^{a+b}} \qquad (13)$$

by rearranging the terms.

Now, the right-hand side of (13) is exactly $f(b-1) - f(a-2)$ by our definitions. The left-hand side of (13) may be rewritten as

$$\frac{2^{-b}\binom{a+b}{b}\left\langle{b \atop b-1}\right\rangle}{2^a} - \frac{2^{-(a-1)}\binom{a+b}{a-1}\left\langle{a-1 \atop a-2}\right\rangle}{2^{b+1}} \tag{14}$$

which is $f_b(b-1)/2^a - f_{a-1}(a-2)/2^{b+1}$ as in Eq. (3). The proof of Theorem 2 is thus completed.

3 Conclusion

We defined a class of functions of Eulerian numbers in this note. Some useful Eulerian identities can be expressed in terms of these functions. Our proof is based on a generating function method which is shown to be useful in many different fields (see e.g. [**?**] for an application in network theory). One can easily imagine extensions of these functions that incorporate multiple variables. Other properties such as convexity and recursive formula are worth further investigation.

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