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Image Compression in Computer Graphics Using Fractal and Wavelet Analysis

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Abstract Research in computer graphics has recently seen considerable activity centered around the use of wavelets and fractals. Fractal-based image coding is based on the theory of iterated function systems (IFS) and is used for image compression. It is very important to give a powerful computer method that can process up to code and decode quickly and high quality (both together). In this article we show how to use the wavelet analysis and fractal geometry in computer graphics and image compression and also how to use them to break down one dimensional function.

Keywords Wavelet analysis; image compression; fractal; fix point

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1 Introduction

Compression and decompression technology of digital image has become an important aspect in the storing and transferring of digital image in information society. Sometimes in order to send an image (2D or 3D) we have to decrease its resolution, so the image information is lost and after sending it by a device encryption techniques, the receiver should decode it again with the same quality as previous. There are many articles on this topic (see [1-4]), but the part of mathematics that is used in this topic, is wavelet analysis and fractal geometry. The problem has been studied previously for the two-dimensional, but speed up the process of compression and security was always important. In this paper we will examine wavelet scheme in computer graphics and we will review the description of fractal compression with different ways. Our primary objective is to meet our demands in computer graphics and it is possible by the direct perception of the necessary mathematical foundation in wavelet.

2 Wavelet Transform

Wavelets are mathematical functions that cut up data into different frequency components, and then study each component with a resolution matched to its scale. This theory was introduced by Goupillaud *et al.* [5]. Wavelets are a set of functions that are obtained by transferring and changing scale of a special function called the mother wavelet. By transferring and changing scale of a function, it is possible to analyze a signal at different scales, and thus makes the field of time - space scales. Small wavelet time makes possible to recognize signals in the range of small (wide frequency) and extensive wavelet in time (focused on the frequency) makes possible to review long-term behavior of the signal and provides a useful tool for analyzing non-stationary signals offer. In this way for unlimited period, Fourier transform is unsuccessful because of its time signals bases. Indeed, all of its local frequencies information are lost. The short time Fourier transform that was introduced by Gabor, resolves this problem by applying the window on the basis of Fourier transform. However, due to having a fixed resolution in time-frequency domain, information for having a better time accuracy in the frequency domain is lost. Similar to Fourier analysis, we can consider several wavelet transforms. For example, when the time and scale are continuous then we will have the continuous wavelet transform (CWT) and the discrete scale axis, has a set of basis functions and any signal with finite energy can be extended in terms of them. If they are discrete in the time-scale axes, we will have a discrete wavelet transformation which can be realized by a filter bank. The main feature of wavelet is the possibility of fine tuning on individual parts signal. For example, a software approximation of a signal may come to seem static, but with a closer view (in smaller scale) of the different behavior will be observed [2].

2.1 Continuous Wavelet Transform

Continuous wavelet transform of a signal is defined as follows:

$$X(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} x(t)\psi^*(\frac{t-b}{a})dt =< x(t), \psi_{ab}(t) > .$$

From the above definition, in the same Fourier mode there is not only one dependence parameter but there are two parameters scale a and the transfer b. Thus, larger values of a correspond to long-term behavior of the signal and smaller amounts of a for short-term cognitive behaviors of a signal are used. Assume the wavelet $\psi(t)$ satisfies the following condition:

$$\int_\omega \frac{|\psi(\omega)|^2}{\omega} d\omega < \infty.$$

It means that the loss of high frequencies is sufficient and its average is zero (simply it is a pass through function). The signal rebuilding relationship is calculated on the continuous conversion as

$$x(t) = \int_a \int_b X(a,b)\psi_{ab}(t)\frac{da}{a^2}db.$$

Remark 1 It is clear that if the parameters a and b are continuous, then the above set functions has high redundancy and it can not establish an orthogonal basis. Naturally the question comes whether the discrete choice of parameters a and b can be established an orthogonal basis for $L^2(\mathbb{R})$. In continuous mode and changing the binary scales, a multi-scale analysis is the set of closed subsets such that

$$\bigcap_m V_m = \{0\}, \quad \bigcup_m V_m = L^2(\mathbb{R}),$$

$$\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots, \quad f(t) \in V_m \Leftrightarrow f(2t) \in V_{m-1}.$$

There exists a function $\varphi(t)$ belonging to V_0 such that φ_{mn} construct an orthogonal basis for each V_m. One can show that if the complement set of V_m on V_{m-1} denoted by W_m, then the space of $L^2(\mathbb{R})$ is covered by direct sum of all such W_m.

2.2 Discrete Wavelet Transform

Wavelet expansion of a function only can be achieved by discrete axis scale. Discrete wavelet transform is defined by making discrete time axis. If for this purpose in relation to continuous transform, change t with n, $a^{-k}n$ would not be an integer so we are following the problem in the frequency domain. Equivalent to the analog domain filters that are defined by $H(j\Omega) = a^{\frac{k}{2}} H(j\Omega a)$, with Conversion to the discrete field obtains, we will have:

$$H_k(z) = H(z^{2^i}).$$

Whenever $H(z)$ considered as a high pass band, so $H_k(z)$ will be multi-band and not pass through therefore is necessary closed it as series to a low pass filter $H(z)$. In this case the output of each level is as follows:

$$H(z), G(z)H(z^2), G(z)G(z^2)H(z^4), ...$$

2.3 A matrix formula for reform

Our studies of multi-scale analysis will focus on wavelets defined on a bounded domain, although we will also refer to wavelets on the unbounded real line wherever appropriate. In the bounded case, each interval V^j has a finite basis, allowing us to use matrix notation in more topics. It is often convenient to put the different scaling functions $\phi_i^j(x)$ for a given level j, together into a single row matrix,

$$\Phi^j(x) = [\phi_0^j(x), ..., \phi_{m^j-1}^j(x)],$$

where m^j is the dimension of V^j. We can do the same for the wavelets

$$\Psi^j(x) = [\psi_0^j(x), ..., \psi_{n^j-1}^j(x)],$$

where n^j is the dimension of W^j. Since W^j is the orthogonal complement of V^j in V^{j+1}, the dimensions of these spaces satisfy $m^{j+1} = m^j + n^j$.The conditions that the subinterval V^j be nested is equivalent to requiring that the scaling functions be refinable. That is, for all $j = 1, 2, ...$ there must be a matrix of the form P^j such that

$$\Phi^{j-1} = \Phi^j(x)P^j. \tag{1}$$

In other words, each scaling at level $j-1$ must be expressible as a linear combination of finer scaling functions at level j. Note that since V^j and V^{j-1} have dimensions m^j and m^{j-1} respectively, P^j is an $m^j \times m^{j-1}$ matrix (Its height is greater than its width). As the wavelet space W^{j-1} is by definition also a subspace of V^j, we can write the wavelets $\Psi^{j-1}(x)$ as linear combinations of the scaling functions $\Phi^j(x)$. This means, there is an $m^j \times n^{j-1}$ matrix of constants Q^j satisfying

$$\Psi^{j-1} = \Phi^j(x)Q^j. \tag{2}$$

Example 1 In Harr base in a specified level j, there exist $m^j = 2^j$ scaling functions and $n^j = 2^j$ wavelets. Therefore, there must be refinement matrices describing how the two scaling functions in V^j and the two wavelets in W^j can be made from the four scaling functions in V^2:

$$P^2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, Q^2 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

In the case of wavelets constructed on the unbounded real line, the columns of P^j, are shifted versions of one another, as are the columns of Q^j. Thus, each column defines the matrix. So P^j and Q^j are completely identified by sequences $(..., p_{-1}, p_0, p_1, ...)$ and $(..., q_{-1}, q_0, q_1, ...)$ respectively, that are also related to the j. Equations (1) and (2) are often expressed as

$$\Phi(x) = \sum_i p_i \phi(2x - i) \quad , \quad \Psi(x) = \sum_i q_i \phi(2x - i).$$

Note that the above equations can be written as a single equation

$$[\Phi^{j-1}][\Psi^{j-1}] = \Phi^j [P^j | Q^j]. \tag{4}$$

Substituting the matrices from the previous example into equation (4) along with the appropriate basis functions gives

$$[\phi_0^1 \phi_1^1 \Psi_0^1 \Psi_1^1] = [\phi_0^2 \phi_1^2 \phi_2^2 \phi_3^2] \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

All functions in ϕ^{j-1} are perpendicular to all functions in $\Psi^{j-1}(x)$, so we have

$$< \Phi_k^{j-1} | \Psi_l^{j-1} >= 0.$$

For simultaneous arguments about the entire internal products we can define some new notation for the internal matrix product. The matrix whose (k, l) entry is $< \phi_k^{j-1} | \Psi_l^{j-1} >$ is shown by $\lfloor < \Phi^{j-1} | \Psi^{j-1} > \rfloor$. With this condition we can assume that the form of vertical on the wavelet below, we again rewrite $[< \Phi^{j-1} | \Psi^{j-1} >] = 0$.With the substitution in the equation above we get

$$\lfloor < \Phi^{j-1} | \Psi^j > \rfloor Q^j = 0.$$

The set of all possible scales is called empty space of $\lfloor < \Phi^{j-1} | \Psi^j > \rfloor$ and Q^j's columns should create a basis for this distance. There are many bases for a matrix of empty space, that show there exist many different wavelet bases for a given space W^j wavelet.

3 Fractal Geometry

Fractal geometry plays an important role in advanced mathematics especially, chaos theory. Lately, the fractals are known as one of the important tools in computer graphics and video compression [1]. The following algorithm can be used for fractal coding.(A) Image is segmentation to small cells of range $\{R_i\}$. It is better to be square.(B) An image sequence covered the range of cells. Their size may be large.(C) There is an affine function W of the cellular domain to its range for each cell of range.(D) The code is made for coded fractal image, including a list of information that have been obtained for each cell and this information is obtained from mapping the domain takes to the cell. This information can even be the code of the color. See figure 1.

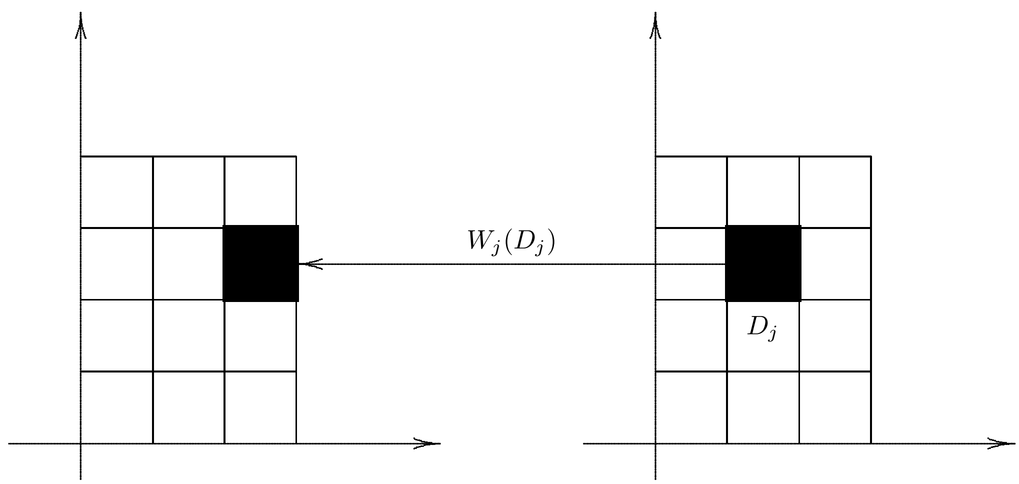

Figure 1: Left Hand is Domain Block and Right Hand is Range Block

4 Conclusion

Fractal image compression is still in its infancy and there are many aspects of this fast developing technology which are still partially unsolved, or untried. In the above we could compare two important methods for image compression. The algorithm that we mentioned, may actually be slower than using wavelets, but when a picture is often referred to, is very useful.

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