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Composite Developable Alternative Surfaces

[1]Yeoh Weng Kang and [2]Jamaludin Md Ali
[1,2]School of Mathematical Sciences, Universiti Sains Malaysia
11800 Minden, Pulau Pinang, Malaysia
e-mail: [1]wengkang2005@yahoo.com, [2] jamaluma@cs.usm.my

Abstract Developable alternative patches is introduced as a choice in the design of developable surfaces. This paper extends the previous work by joining consecutive alternative patches subject to various continuity conditions. With a first boundary curve freely specified, $2n + 3$, $n + 4$ and 5 DOF's are available for a second boundary curve of a developable alternative surface containing n patches, when the surface is G^0, G^1 and C^1 continuity, respectively. The composite surfaces are then evaluated by using highlight lines, which effectively in detecting surface irregularities.

Keywords Developable surfaces; degrees of freedom; highlight lines

2010 Mathematics Subject Classification 65D10, 65D17

1 Introduction

A ruled surface is generated by sweeping a straight line through 3-dimensional space [1]. This straight line is referred to as a ruling, or generator of the surface. Developable surfaces are a subset of ruled surfaces, which has a constant tangent plane at all points along any ruling. This surfaces can be rolled out into a plane without tearing or creasing, and are widely utilized in computer aided design and manufacturing [2–6].

Our recent findings [7] of developable alternative patches provide effective solutions compare to Chu and Séquin [8]. This paper extends these findings into geometric design of developable composite alternative surfaces. Adjacent developable patches are joined along their end rulings while maintaining continuity conditions G^0, G^1 or C^1 are then evaluated by using highlight lines, so the quality of the composite surfaces can be determined.

2 Developability Constraints in Alternative Patches

Given any two boundary curves $\mathbf{A}(w)$ and $\mathbf{B}(w)$, a ruled surface patch is defined by connecting each pair of corresponding points (with equal w) with a straight line segment $\mathbf{AB}$ as shown in Figure 1. The line segment $\mathbf{AB}$ is called the ruling at parameter value w. The surface patch is expressed as

$$\mathbf{X}(t, w) = (1 - t)\mathbf{A}(w) + t\mathbf{B}(w), \qquad 0 \leq t \leq 1 \text{ and } 0 \leq w \leq 1$$

where t is the parameter along the rulings.

If the tangent lines of $\mathbf{A}(w)$ and $\mathbf{B}(w)$ at every w and the corresponding ruling remain coplanar, then the surface becomes developable. The co-planarity can be represented as

$$\dot{\mathbf{A}}(w) \times \dot{\mathbf{B}}(w) \bullet [\mathbf{B}(w) - \mathbf{A}(w)] = 0. \tag{1}$$

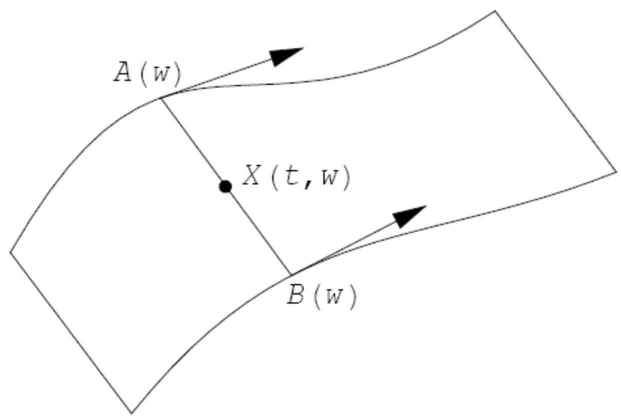

Figure 1: A Ruled Surface

Let both curves $\mathbf{A}(w)$ and $\mathbf{B}(w)$ are alternative cubic curves with control points $\mathbf{A}_0$, $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{B}_0, \mathbf{B}_1, \mathbf{B}_2$ respectively as shown in Figure 2. By using de Casteljau construction, we have

$$\mathbf{A}(w) = (1-w)\mathbf{I} + w\mathbf{K} = (1-w)[(1-w)\mathbf{A}_0 + w\mathbf{A}_1] + w[(1-w)\mathbf{A}_1 + w\mathbf{A}_2],$$

$$\mathbf{B}(w) = (1-w)\mathbf{J} + w\mathbf{L} = (1-w)[(1-w)\mathbf{B}_0 + w\mathbf{B}_1] + w[(1-w)\mathbf{B}_1 + w\mathbf{B}_2].$$

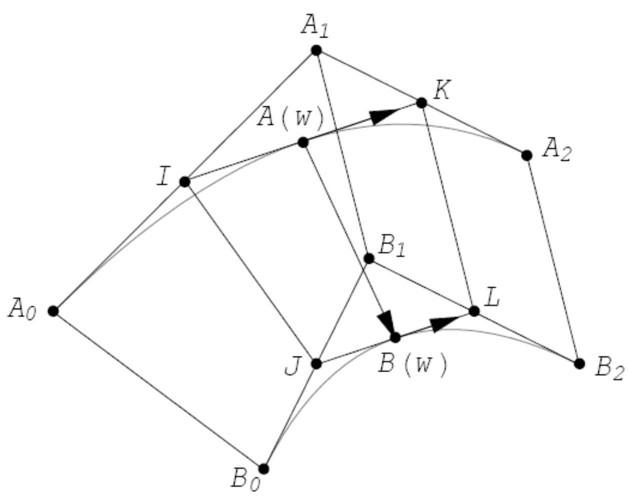

Figure 2: Two Boundary Curves with Six Control Points

Note that for this alternative patch, line segments $\mathbf{IK}$ and $\mathbf{JL}$ lie in the tangent direction at $\mathbf{A}(w)$ and $\mathbf{B}(w)$, respectively. Therefore, the developability condition indicates that

$\mathbf{I}, \mathbf{J}, \mathbf{K}$ and $\mathbf{L}$ lie in the same plane, and equation (1) can be written as

$$\mathbf{IK} \bullet \mathbf{IJ} \times \mathbf{KL} = 0. \tag{2}$$

For solving this equation, let

$$\mathbf{a}_i = \mathbf{A}_i - \mathbf{A}_{i-1} \text{ for } i = 1 \text{ and } 2,$$

$$\mathbf{c}_j = \mathbf{B}_j - \mathbf{A}_j \text{ for } j = 0, 1 \text{ and } 2$$

as shown in Figure 3. By substituting the above equations into (2), we have

$$[(1-w)\mathbf{a}_1 + w\,\mathbf{a}_2] \bullet [(1-w)\mathbf{c}_0 + w\,\mathbf{c}_1] \times [(1-w)\mathbf{c}_1 + w\,\mathbf{c}_2] = 0$$

which gives

$$\mathbf{a}_1 \bullet \mathbf{c}_0 \times \mathbf{c}_1 = 0, \tag{3}$$

$$\mathbf{a}_2 \bullet \mathbf{c}_1 \times \mathbf{c}_2 = 0, \tag{4}$$

$$\mathbf{a}_1 \bullet \mathbf{c}_0 \times \mathbf{c}_2 + \mathbf{a}_2 \bullet \mathbf{c}_0 \times \mathbf{c}_1 = 0, \tag{5}$$

and

$$\mathbf{a}_1 \bullet \mathbf{c}_1 \times \mathbf{c}_2 + \mathbf{a}_2 \bullet \mathbf{c}_0 \times \mathbf{c}_2 = 0. \tag{6}$$

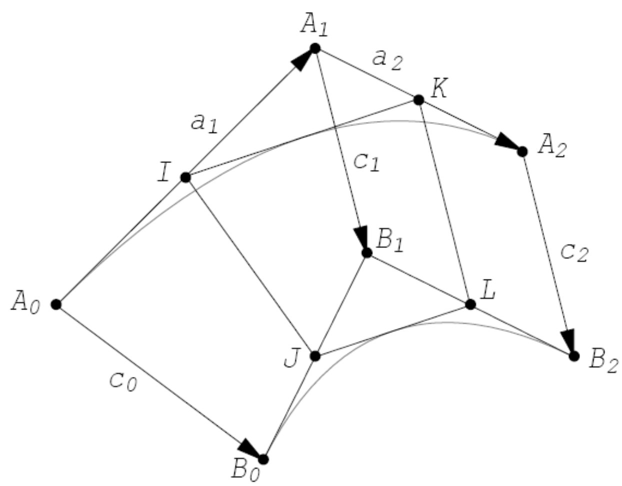

Figure 3: Solving the Developability Condition for Equation (2)

The constraint (3) indicates that the first two pairs of control points $\mathbf{A}_0 - \mathbf{B}_0$ and $\mathbf{A}_1 - \mathbf{B}_1$ lie in the same plane. The last two pairs of control points $\mathbf{A}_1 - \mathbf{B}_1$ and $\mathbf{A}_2 - \mathbf{B}_2$ are also coplanar because of constraint (4). These two constraints will be referred to as the co-planarity condition.

Suppose both of the boundary curves have m control points respectively. Then the coordinates of $\mathbf{I}, \mathbf{J}, \mathbf{K}$ and $\mathbf{L}$ are $(m-2)$-degree polynomials in the curve parameter w. Vectors $\mathbf{IJ}, \mathbf{KL}$ and $\mathbf{IK}$ also have coefficients that are $(m-2)$-degree polynomial with $3(m-2)+1 = 3m-5$ coefficients that must vanish for any w. Alternatively stated, there are $3m-5$ coefficients that must vanish for any w.

3 Composite Developable Alternative Surfaces

A composite developable alternative surfaces is constructed by joining consecutive patches along their end rulings. In addition to the developability constraints of each patch, the control points of the surface must satisfy certain constraints to maintain continuities across the patch boundary. The degrees of freedom available for the surface design are thus reduced.

We first examine an alternative patch consisting of two adjecent patches with various continuity conditions. The results are then generalized for a surface consisting of n patches of m pair of control points.

3.1 Counting Degrees of Freedom (DOF)

Suppose the first boundary curve can be freely specified by the designer, there are six control points in 3-dimensional space to be determined for the other curve. Each patch must satisfy its four developability constraints as stated in section 2. Therefore, the number of the remaining degrees of freedom is $6(3) - 4 - 4 = 10$.

Figure 4 shows that the last control point of the first patch must coincide with the first control point of the second patch due to positional continuity (G^0), that is $\mathbf{B}_2 = \mathbf{E}_0$, using up three DOF's. As a result, $10 - 3 = 7$ DOF's are available for the design of the second curve.

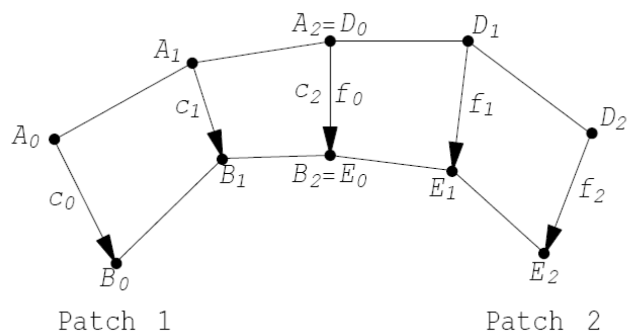

Figure 4: Composite Alternative Surfaces with Two Patches

For gradient continuity (G^1), the tangent vector of the second patch must be collinear with that of the first patch at the end point, which is written as

$$\mathbf{B}_1\mathbf{B}_2 = \mu\mathbf{E}_0\mathbf{E}_1 \tag{7}$$

where μ is the length ratio of the two tangent vectors. Note that $\mathbf{A}_1$, $\mathbf{A}_2 = \mathbf{D}_0$, $\mathbf{D}_1$, $\mathbf{B}_1$, $\mathbf{B}_2 = \mathbf{E}_0$ and $\mathbf{E}_1$ lie in the same plane due to the co-planarity condition. Hence to impose equation (7) only consumes one DOF and thus $10 - 3 - 1 = 6$ DOF's remain for the surface design.

The first derivative continuity (C^1) requires

$$\mathbf{E}_1 - \mathbf{E}_0 = \mathbf{B}_2 - \mathbf{B}_1$$

which uses up only two DOF's. Hence, only $10 - 3 - 2 = 5$ DOF's remain in this case.

As shown in section 2, a developable alternative patch consisting of m pair of control points has $3m - 5$ constraints in specifying the second boundary curve after the first one has been chosen. Thus the composite surface containing n patches must satisfy totally $n(3m - 5)$ equations to ensure its developability.

Any two consecutive patches impose three more constraints on their common boundary for the positional continuity. There are $(n-1)$ such boundaries in the surface, summing up to $3(n - 1)$ constraints that to be satisfy by the surface.

The second boundary curve has $n(m)$ control points, contributing $3nm$ degrees of freedom. For satisfying positional continuity, the number of DOF's thus becomes $3nm-n(3m-5) - 3(n - 1) = 2n + 3$.

To ensure the gradient continuity across each boundary gives additional $(n - 1)$ constraints. The degrees of freedom for the surface design are further reduced to $(2n + 3) - (n - 1) = n + 4$.

As discussed before, the first derivative continuity impose two constraints on each common boundary. Since there has $(n - 1)$ of such boundaries, the remaining DOF's are computed as $(2n+3)-2(n-1) = 5$. Table 1 summarizes the corresponding DOF's for G^0, G^1 and C^1.

Table 1: Available Degrees of Freedom for Various Continuity Conditions

Continuity	Number of Degrees of Freedom
G^0	$2n + 3$
G^1	$n + 4$
C^1	5

4 Highlight Lines

A highlight line [9] is created by an assumed linear light source idealized by a straight line with an infinite extension, as shown in Figure 5. The imprint of the light source on the surface is the collection of all the surface points for which the extended surface normal passed through the light source.

From Figure 5, assume the light source $\mathbf{L}(t)$ is given by

$$\mathbf{L}(t) = \mathbf{G} + \mathbf{H}t$$

where $\mathbf{G}$ is a point on $\mathbf{L}(t)$ and $\mathbf{H}$ is a vector defining the direction of $\mathbf{L}(t)$. Let $\mathbf{Q}$ be a point on the surface, while $\mathbf{N}$ be the directional vector of the corresponding surface normal. The extended surface normal $\mathbf{M}(s)$ at $\mathbf{Q}$ is a line passing through $\mathbf{Q}$ in the direction of $\mathbf{N}$, and it can be difined as

$$\mathbf{M}(s) = \mathbf{Q} + \mathbf{N}s.$$

If lines $\mathbf{L}(t)$ and $\mathbf{M}(s)$ intersect, that is, if the perpendicular distance d between both lines given by [10]

$$d = \frac{|(\mathbf{H} \times \mathbf{N}) \bullet (\mathbf{G} - \mathbf{Q})|}{\|(\mathbf{H} \times \mathbf{N})\|}$$

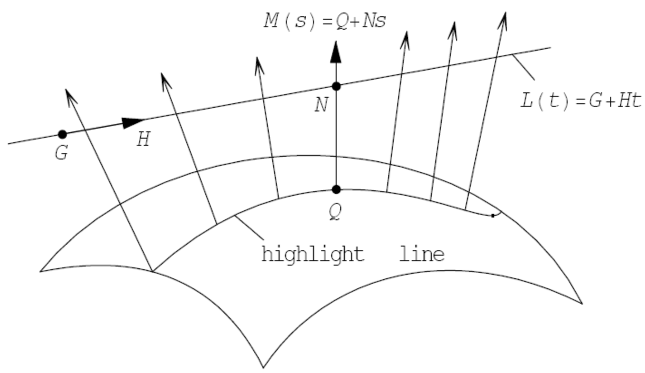

Figure 5: Definition of Highlight Line

is zero, then point $\mathbf{Q}$ belongs to the highlight line.

For evaluating the quality of a given surface, the user can inspect the entire surface by translating or rotating the light source, and thereby sweeping the highlight line over the surface. Besides, a single light source can be replaced by an array of parallel light sources. This creates a family of highlight line covering the surface as shown in Figure 6. Therefore, the sensitivity of the highlight lines allows for the detection of the small surface irregularities effectively.

5 Examples and Discussion

An alternative surface comprised of two developable alternative patches with each one having three pair of control points is used as a test example to verify the derived results. The first and second boundary curves are referred to as the **A**-curve and **B**-curve respectively in the following examples.

There are various ways [7] to use in the surface design, each of which has different computational requirement in solving the constrained control points. This paper will not address the advantages of one particular design method over another. Instead, the focus is to demonstrate some feasible steps for designing the surfaces with limited DOF's and satisfy the required continuity conditions.

5.1 G^0 Continuity

Suppose the **A**-curve is constructed with $\mathbf{A}_0 = (-15, -15, 10)$, $\mathbf{A}_1 = (-10, -5, 20)$, $\mathbf{A}_2 = (0, 0, 22)$, $\mathbf{D}_0 = (0, 0, 22)$, $\mathbf{D}_1 = (5, 5, 22)$ and $\mathbf{D}_2 = (16, 8, 18)$. The user is allowed to specify $\mathbf{B}_0 = (-10, -25, 0)$ and direction of vector $\mathbf{c}_2 = (10, -5, -6)$, consuming three and two DOF's respectively. The remaining two DOF's are used to place the control point $\mathbf{E}_1$ of the second patch at $(18, -2, 14)$. Figure 7 illustrates the frame of composite developable surfaces solved from the system with $\mathbf{B}_1 = \left(-\frac{10}{3}, -\frac{35}{3}, \frac{40}{3}\right)$, $\mathbf{B}_2 = (10, -5, 16)$ and $\mathbf{E}_2 = \left(\frac{163}{5}, -\frac{23}{5}, \frac{38}{5}\right)$.

There are various choices that can be used to form the alternative cubic curves as

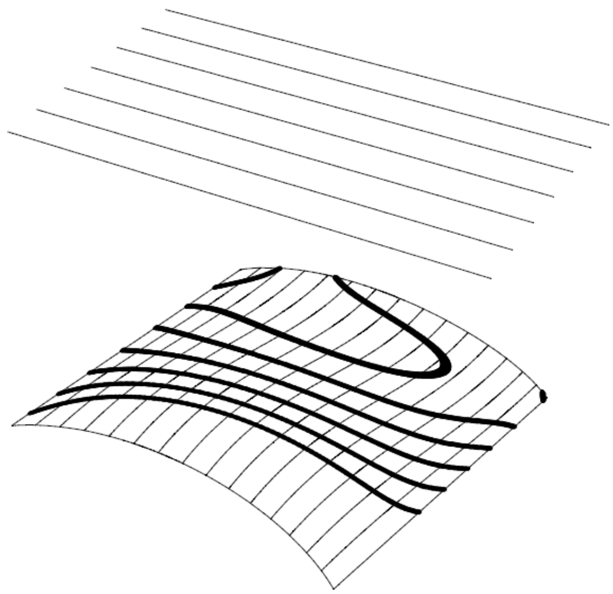

Figure 6: Family of Highlight Lines Created from Array of Parallel Light Sources

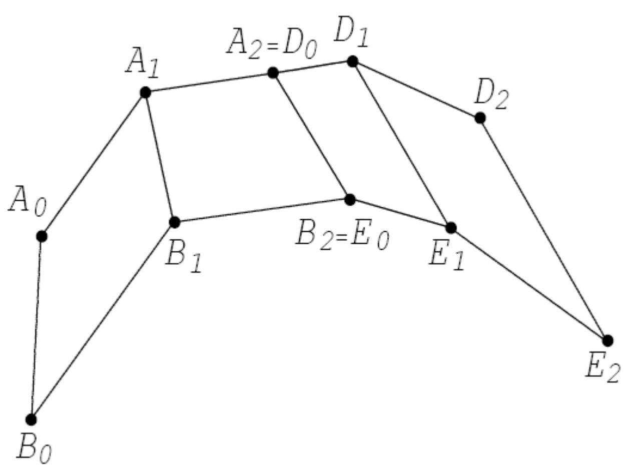

Figure 7: Frame of G^0 Continuity

boundary curves, [7]. However, all the boundary curves in this section are formed by using the same basis functions, so the comparison between surfaces with different continuity conditions can be done effectively.

Figure 8 shows one resultant developable surfaces base on the control points of Figure 7. The surface is then evaluated by a family of highlight lines which crosses the common boundary of the two patches, as illustrated in Figure 9.

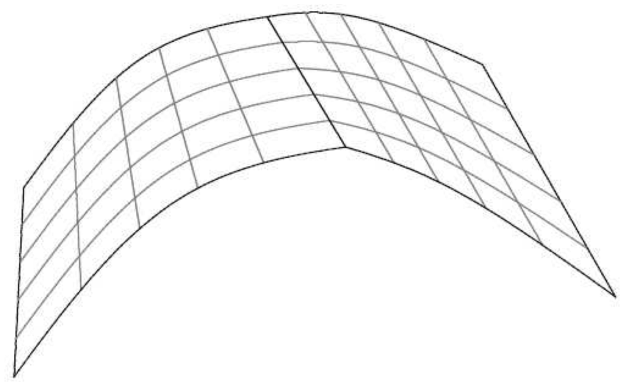

Figure 8: Design Example of G^0 Continuity

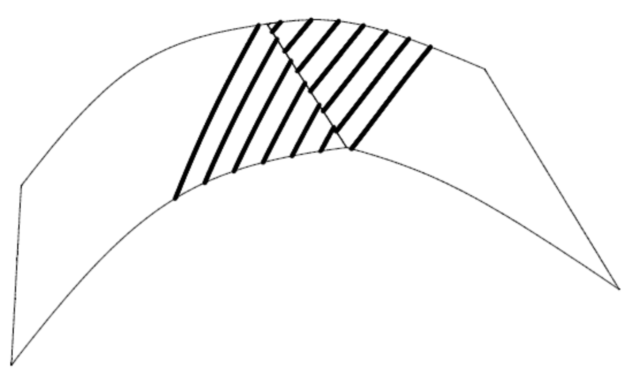

Figure 9: Composite Surface of Figure 8 with a Family of Highlight Lines

From Figure 9, it is obvious that all highlight lines forming disconnected curves while crossing the common boundary. Therefore, the composite surface with G^0 continuity is not satisfactory from the viewpoint of smoothness.

5.2 G^1 Continuity

Suppose the **A**-curve is constructed with $\mathbf{A}_0 = (-15, -15, 10)$, $\mathbf{A}_1 = (-10, -5, 20)$, $\mathbf{A}_2 = (0, 0, 22)$, $\mathbf{D}_0 = (0, 0, 22)$, $\mathbf{D}_1 = \left(\frac{10}{2}, \frac{5}{2}, \frac{46}{2}\right)$, and $\mathbf{D}_2 = (15, 5, 19)$. To place $\mathbf{B}_0 = (-10, -25, 0)$ and direction of vector $\mathbf{c}_2 = (10, -5, -6)$ for the first patch consumes three and two DOF's respectively. Positions for $\mathbf{B}_1 = \left(-\frac{10}{3}, -\frac{35}{3}, \frac{40}{3}\right)$ and $\mathbf{B}_2 = (10, -5, 16)$ are fully determined after these five DOF's have been used up. The last DOF is used to locate the position of $\mathbf{E}_1$ at $\left(\frac{50}{3}, -\frac{5}{3}, \frac{52}{3}\right)$ along the $\mathbf{B}_1\mathbf{B}_2$ direction. Figure 10 shows the resulting surface with $\mathbf{E}_2 = \left(\frac{90}{3}, \frac{5}{3}, \frac{36}{3}\right)$, while Figure 11 illustrates the smoothness of the surface by using a family of highlight lines.

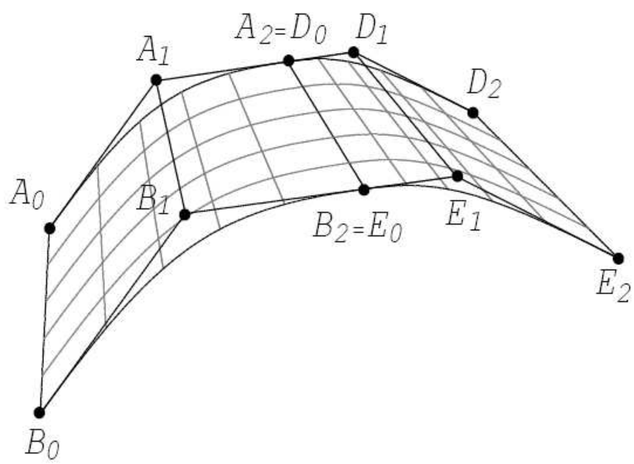

Figure 10: Design Example of G^1 Continuity

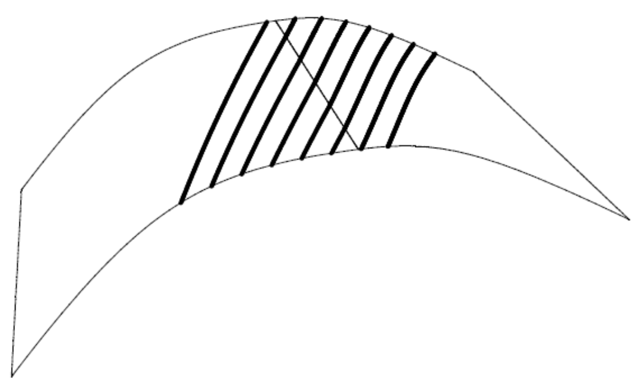

Figure 11: Composite Surface of Figure 10 with a Family of Highlight Lines

In this case all the highlight lines are connected and continuous when crossing the common boundary. This means that the composite surface with G^1 continuity has overcomes the poor quality of G^0 continuity.

5.3 C^1 Continuity

Suppose the **A**-curve is specified as $\mathbf{A}_0 = (-15, -15, 10)$, $\mathbf{A}_1 = (-10, -5, 20)$, $\mathbf{A}_2 = (0, 0, 22)$, $\mathbf{D}_0 = (0, 0, 22)$, $\mathbf{D}_1 = (10, 5, 24)$, and $\mathbf{D}_2 = (20, 8, 20)$. Five DOF's are consumed in specifying position of $\mathbf{B}_0 = (-10, -25, 0)$ and direction of vector $\mathbf{c}_2 = (10, -5, -6)$ for the first patch. $\mathbf{B}_1 = \left(-\frac{10}{3}, -\frac{35}{3}, \frac{40}{3}\right)$ and $\mathbf{B}_2 = (10, -5, 16)$ are then automatically determined by the system. $\mathbf{E}_0 = (10, -5, 16)$ and $\mathbf{E}_1 = \left(\frac{70}{3}, \frac{5}{3}, \frac{56}{3}\right)$ are also fully defined because of C^1 continuity. The remaining control point is computed as $\mathbf{E}_2 = \left(\frac{110}{3}, \frac{17}{3}, \frac{40}{3}\right)$ and Figure 12 shows the resulting surface for this situation. This surface is then evaluated by using a family of highlight lines as illustrated in Figure 13.

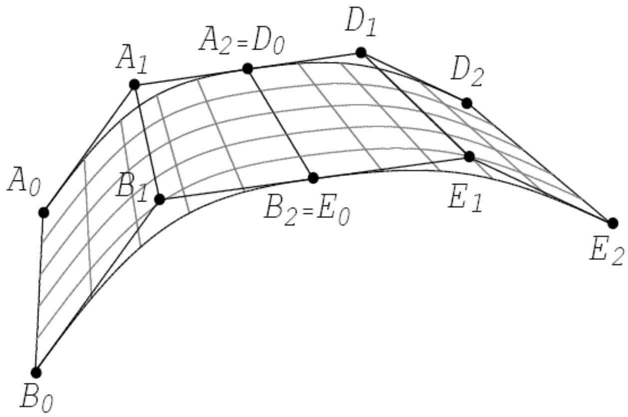

Figure 12: Design Example of C^1 Continuity

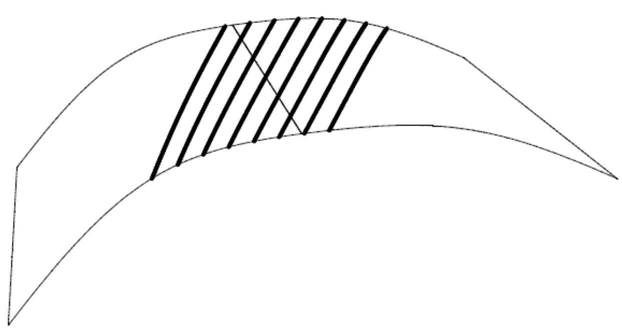

Figure 13: Composite Surface of Figure 12 with a Family of Highlight Lines

From Figure 13, every highlight line forms a connected and also straight line while crossing the common boundary. Therefore, composite with C^1 continuity provides the best quality or smoothness that is needed for surface design.

6 Conclusion

From the discussion above, we know that the smoothness of the composite surfaces is endorsed beginning G^1 continuity. As the conclusion, the composite developable alternative surfaces that can be formed by easy way, provide satisfactory quality surface with low continuity conditions (G^1 or C^1), can be readily extend the applications of developable surfaces in geometric modelling.

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