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# Identifying More-for-Less Paradox in the Linear Fractional Transportation Problem Using Objective Matrix

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**Abstract** This paper reports the identification of more-for-less paradox in the linear fractional transportation problem using objective matrix. We characterize the  $m \times n$  objective matrices of linear fractional transportation problem for which there exist supplies and demands such that the transportation paradox arises. At the end numerical examples are given for explaining the theory.

Keywords Linear fractional transportation problem, more-for-less paradox.

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### 1 Introduction

The classical linear fractional transportation problem (LFTP) is a mathematical model which has a special mathematical structure. This special mathematical structure can be confirmed by the mathematical formulation of a large number of similar problems. Therefore, it is frequently referred as a particular form of a mathematical model rather than the physical approach in which the problem originates. The more-for-less (MFL) paradox of a LFTP, where one wants to minimize the total distribution costs, maximize the total profit, increases the requirements of one of the demand points and capacity of one of the supply points, may result in a lower optimal solution even though the total requirements have increased. The MFL analysis is useful in decisions of increasing a warehouse stocking level or plant production capacity. In turn this could prompt a decision for enhancing advertising efforts to increase demand in certain markets. The procedure permits computation of the maximum allowable additional units and distribution of these extra units in a systematic approach. In this discussion, the MFL paradox does not include other managerial issues, such as the possibility of the additional holding cost or the possibility of losing alternative opportunities.

Mathematical formulation of classical LFTP [1] is

(

P1) Minimize
$$Z = \frac{N(X)}{D(X)}$$
$$= \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}}{\sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij}}$$

ubject to 
$$\sum_{j=1}^{m} x_{ij} = a_i$$
  
 $\sum_{i=1}^{n} x_{ij} = b_j$   
 $x_{ij}, a_i, b_j \ge 0, \quad i = 1, 2..., m, \quad j = 1, 2..., n$   
 $(x_{ij}) = X \subset S$   
 $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$ 

Let  $P = (c_{ij}, d_{ij})$  and vectors  $(c_{ij})$  and  $(d_{ij})$  lie in  $\mathbb{R}^{m \times n}$  and X is a vector of mn decision variables,  $a_i$  being the availability at  $i^{th}$  supply and  $b_j$  the requirement at  $j^{th}$  demand point. Here  $x_{ij}$  denotes the quantity shipped from source i to sink j. Let X denotes the set of all transportation plans  $x = (x_{ij})$  that fulfill the transportation constraints above. D(x) > 0 for all  $x \in S$ , where S is a polyhedral compact set of feasible points. Further, let LFTP(P, a, b) denote the optimal objective value specified by P, a and b.

The classical LFTP is specified by an  $m \times n$  matrix  $P = (c_{ij}, d_{ij})$ , an *m*-dimensional vector  $a = (a_i)$ , and an *n*-dimensional vector  $b = (b_j)$ ; all numbers  $a_i, b_j c_{ij}$  and  $d_{ij}$  are nonnegative real numbers. This data has the following meaning: there are *m* sources and *n* sinks; at the *i*th source there is a supply of  $a_i$  units, and at the *j*th sink there is a demand of  $b_j$  units. It is assumed that  $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ , i.e. the total supply equals the total demand. The total cost for transporting one unit from the *i*th source to the *j*th sink is  $p_{ij}$ . The main aim is to find a transportation plan that satisfies all the demands and minimizes the overall transportation cost.

Deineko *et al.* [2] gives an exact characterization for linear transportation problem cost matrices that are immune against the transportation paradox. The paradoxical situation in LFTP was discovered by Verma and Puri [1]. In this paper the classical cost minimization transportation problem with fractional objective function was used and a sufficient condition to identify the paradoxical situation in LFTP was developed in the presence of a paradoxical solution. This approach also provides a complete paradoxical range of flow. The main drawback in the approach is that it does not involve any procedure for finding the increment required in supply/demand to reach a better optimal solution without losing the initial shipping route, though for a given flow it can find out the route.

To describe the transportation paradox more precisely, we introduce the following notations: let two vectors  $v = (v_i)$  and  $v' = (v'_i)$  are of equal dimensions, the vector v' dominates the vector v if and only if  $v'_i \ge v_i$  holds for all i; we denote this by  $v' \ge v$ . Let  $a' \ge a \ge 0$ and  $b' \ge b \ge 0$  be two other supply and demand vectors. Then this situation constitutes a MFL paradox in LFTP if and only if LFTP(P, a, b) >LFTP(P, a', b').

# 2 More-for-Less Analysis in LFTP

The MFL phenomenon is based on relaxing the equality constraints for a given LFTP. Consider the problem with relaxed constraints as

 $\mathbf{S}$ 

(P2) Minimize 
$$Z = \frac{N(X)}{D(X)} = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}}{\sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij}}$$
  
where  $\sum_{j=1}^{m} x_{ij} \ge a_i$   
 $\sum_{i=1}^{n} x_{ij} \ge b_j$   
 $x_{ij}, a_i, b_j \ge 0, \quad i = 1, 2..., m, \quad j = 1, 2..., n$ 

# 3 Definitions

- **Paradoxical flow-** Let  $Z^0$  and  $Z^P$  be the objective functions corresponding to the flows  $F^0$  and  $F^P$  respectively. Then it is a paradoxical flow if on increasing the flow from  $F^0$  to  $F^P$  the value of the objective function decreases steadily from  $Z^0$  to  $Z^P$ .
- **Paradoxical solution-** A solution  $X^p$  of (P2) yielding the objective function-flow pair  $(Z^p, F^p)$  is called a 'Paradoxical solution', if for any other feasible solution of (P2) yielding a flow pair (Z, F), we have:

 $(Z,F) > (Z^p,F^p)$ or $Z = Z^p$ , but  $F < F^p$  or  $F = F^p$ , but  $Z > Z^p$ 

**Dual variables-** For a basic feasible solution of problem (P1), dual variables  $u_i, v_j, u'_i, v'_j$  where (i, j) is basic a cell, are given by

$$u_i + v_j = c_{ij}, \ u'_i + v'_j = d_{ij}.$$

Let the optimal feasible solution of (P1) yields the value  $Z^o = \frac{N^0}{D^0}$  of the objective function  $\frac{N(x)}{D(x)}$ . The condition for the paradoxical situation in the LFTP is given by following theorem [1].

#### 4 Sufficient condition for the paradoxical situation in LFTP

**Theorem 1** If there exists a cell (i, j) in the table corresponding to an optimal solution  $X^0$  of the problem (P1) such that

$$D^{0}(u_{i}+v_{j})-N^{0}(u_{i}^{'}+v_{j}^{'})<0, and \lambda>0$$

and the basis corresponding to a basic feasible solution of the problem (P1) with  $a_i$  replaced by  $a_i + \lambda$  and  $b_j$  by  $b_j + \lambda$  is same as that corresponding to  $X^0$  then there exists a paradoxical solution. **Proof** See Verma and Puri [1].

A total cost matrix P is called immune against the LFTP paradox if regardless of the choice of the supplies and demands the paradox do not arise. In other words, for all the supply vectors a and a' with  $a' \ge a$  and for all the demand vectors b and b' with  $b' \ge b$  an immune matrix P satisfies LFTP $(P, a, b) \le$  LFTP(P, a', b'). In this note, an exact and simple characterization of all the cost matrices that are immune against the paradox is given. Note that the computational effort to solve a LFTP is higher than the effort needed for the recognition of immune matrices.

#### 5 The Characterization of Immune Matrices

Consider some fixed  $m \times n \cos t$  matrix  $P = (c_{ij}, d_{ij})$ . Then four integers q, r, s, t with  $1 \le q$ ,  $s \le m$  and  $1 \le r$ ,  $t \le n$  (where  $q \ne s \operatorname{and} r \ne t$ ) form a bad quadruple if

$$\frac{c_{qt} + c_{sr}}{d_{qt} + d_{sr}} < \frac{c_{qr}}{d_{qr}} \tag{1}$$

**Lemma 1** If there exists a bad quadruple for the cost matrix  $P = (c_{ij}, d_{ij})$ , then P is not immune against the LFTP paradox.

**Proof:** Consider the supply vector a that has supply 1 at source q and supply 0 at every other source and the demand vector b that has demand 1 at sink r and demand 0 at every other sink. Let the supply vector a' result from a by increasing the supply at source s to 1, and let the demand vector b' result from b by increasing the demand at sink t to 1. Clearly,  $a' \ge a$  and  $b' \ge b$ . Then one can send one unit directly from source q to sink t, and another unit directly from source s to sink r. This yields

$$LFTP(P, a', b') = \frac{c_{qt} + c_{sr}}{d_{qt} + d_{sr}}.$$

By inequality (1) we have LFTP(P, a', b') < LFTP(P, a, b).

**Lemma 2** If the cost matrix  $P = (c_{ij}, d_{ij})$  is not immune against the LFTP paradox, then there exists some bad quadruple for P.

**Proof:** By the assumptions of the Lemma 1, there exist two supply vectors a and a' with  $a' \ge a$  and two demand vectors b and b' with  $b' \ge b$  such that LFTP(P, a, b) > LFTP(P, a', b'). Denote the corresponding optimal transportation plans by  $x = (x_{ij})$  and  $x' = (x'_{ij})$ . It is convenient to translate this situation into a bipartite multigraph G where one vertex class is formed by the sources and the other vertex class is formed by the sinks. A nonzero value  $x_{ij}$  yields a black edge (backward arc) with weight  $x_{ij}$  between source i and sink j and a nonzero value  $x'_{ij}$  yields a corresponding red edge (forward arc) with weight  $x'_{ij}$  between the source i and sink j. The cost of a (red or black) edge between source i and sink j is  $p_{ij}$ . Some readers might prefer to use a flow interpretation and regard (x' - x) as flow in the residual graph with respect to x.

It is well known from the flow theory [3] that the multigraph G can be decompose into a finite number  $T_1, ..., T_k$  of simple paths and a finite number  $L_1, ..., L_l$  of simple cycles that satisfy the following properties: every cycle has an even number of edges alternatively consisting of red and black edges, every path starts in a source and ends in a sink, starts with a red edge and ends with a red edge alternatively consists of red and black edges. There exist nonnegative real  $\xi$ -values [3]  $\xi(T)$  and  $\xi(L)$  for every path T and for every cycle L such that

- (i) For every black edge [i, j] the value  $x_{ij}$  equals the sum of the  $\xi$ -values of all paths and cycles containing the edge [i, j] and
- (ii) For every red edge [i, j] the value  $x'_{ij}$  equals the sum of the  $\xi$ -values of all paths and cycles containing the edge [i, j].

For every path T and for every cycle L define the numerator c(T), c(L) and d(T), d(L) as the sum of the costs of all black edges in T and L respectively. Define the costs c'(T), c'(L)and d'(T), d'(L) as the sum of the costs of all red edges in T and L, respectively. Then clearly

$$LFTP(P, a, b) = \frac{\sum_{i=1}^{k} c(T_i)\xi(T_i) + \sum_{j=1}^{l} c(L_j)\xi(L_j)}{\sum_{i=1}^{k} d(T_i)\xi(T_i) + \sum_{j=1}^{l} d(L_j)\xi(L_j)}$$
(2)

and

$$LFTP(P, a', b') = \frac{\sum_{i=1}^{k} c'(T_i)\xi(T_i) + \sum_{j=1}^{l} c'(L_j)\xi(L_j)}{\sum_{i=1}^{k} d'(T_i)\xi(T_i) + \sum_{j=1}^{l} d'(L_j)\xi(L_j)}$$
(3)

Since LFTP(P, a, b) > LFTP(P, a', b') holds, from (2) and (3) that there exists a cycle L among  $(L_j, j = 1, ..., l)$  with c(L) > c'(L), d(L) > d'(L) or there exists a path T among  $(T_j, i = 1, ..., k)$  with

c(T) > c'(T), d(T) > d'(T).

If c(L) > c'(L), d(L) > d'(L) holds for some cycle L then it can be shown that  $x = (x_{ij})$  is not an optimal transportation plan. Consider a new transportation plan  $y = (y_{ij})$  for P, a, bwhere y results from x by decreasing all values  $x_{ij}$  along black edges of L by some  $\varepsilon > 0$  and increasing all values  $x_{ij}$  along red edges of L by the same amount  $\varepsilon$ . This new transportation plan y is also feasible for P, a, b but its objective value is by  $z(L)\varepsilon - z'(L)\varepsilon > 0$  smaller than the objective value of plan x; that clearly is a contradiction. (An alternative way of z(L) > z'(L) cannot arise, be making use of the negative cycle theorem for minimum cost flow problems [3].)

Consequently, c(T) > c'(T), d(T) > d'(T) must hold for some path T. Since all costs Z are nonnegative, the path T consists of at least three edges. Without loss of generality it can be assumed that T starts in a source and ends in a sink. Let the first vertex of T is a source s, the second vertex is a sink r, the last but one vertex is a source q and the last vertex is a sink t. Consider the transportation plan  $Z = (z_{ij})$  for P, a, b which results from  $x = (x_{ij})$ in the following way: decrease all values  $x_{ij}$  along black edges of T by an  $\varepsilon > 0$ , increase all values  $x_{ij}$  along red edges of T except the first red edge [s, r] and the last red edge [q, t] by the same number  $\varepsilon$  and increase the value  $x_{qr}$  by  $\varepsilon$ . Then the resulting transportation plan *P* is feasible for *P*, *a*, *b*. (Note that if *T* has length 3 then Z = x) Since *x* is an optimal transportation plan, the change in the objective value from *x* to *Z* must be nonnegative. Therefore,

$$0 \leq -\left(\frac{c(T)}{d(T)}\right)\varepsilon + \left(\frac{c'(T)}{d'(T)} - \left(\frac{c_{qt} + c_{sr}}{d_{qt} + d_{sr}}\right)\right) + \frac{c_{qr}}{d_{qr}}\varepsilon$$
$$= \left(\frac{c'(T)}{d'(T)} - \frac{c(T)}{d(T)}\right)\varepsilon + \left(\frac{c_{qr}}{d_{qr}} - \frac{c_{qt} + c_{sr}}{d_{qt} + d_{sr}}\right)\varepsilon < \left(\frac{c_{qr}}{d_{qr}} - \frac{c_{qt} + c_{sr}}{d_{qt} + d_{sr}}\right)\varepsilon$$

Here the last inequality follows from z(T) > z'(T). Since  $\varepsilon > 0$ , this yields

$$\frac{c_{qr}}{d_{qr}} - \frac{c_{qt} + c_{sr}}{d_{qt} + d_{sr}} > 0.$$

Hence, the four numbers q, r, s, t form a bad quadruple.

The concluding theorem based on the above lemmas can be stated as:

**Theorem 2** The cost matrix  $P = (c_{ij}, d_{ij})$  is immune against the LFTP paradox if and only if there exists no bad quadruple for P.

**Proof:** From Lemma 1 and Lemma 2.

### 6 Numerical Examples

The bad quadruple situation by examples from [4–6]. The examples (Table 1 and Table 2) are chosen because its optimal solution is available and provides a point of common comparison.

For the above example

$$\frac{c_{32}}{d_{32}} > \frac{c_{33} + c_{12}}{d_{33} + d_{12}}$$

which means that immunization condition is violated for q = 3, r = 2, s = 1, t = 3. So Z is not immune against the transportation paradox.

For the example

$$\frac{c_{23}}{d_{23}} > \frac{c_{21} + c_{13}}{d_{21} + d_{13}}$$

which means that immunization condition is violated for q = 2, r = 3, s = 1, t = 1. So Z is not immune against the transportation paradox.

## 7 Conclusions

In this paper, the classical linear fractional transportation problem has been considered and occurrence of the so-called more-for-less paradox is studied. This paradox is completely ignored in the text as well as the advance books. The main reason behind this may be that it is considered as a rather odd phenomenon which hardly occurs in any practical situation. The paradox should be given much more attention both in the practical applications as well as in teaching.

	B1		B2		B3		B4		Supply
	5		8		7		6		-
AI		2		1		3		2	5
	6		10		5		5		
A2		1				0		0	3
		1		4		2		3	
	7		15		3		16		
A3									15
		2		1		1		2	
	15		21		8		18		
A4									12
		3		1		1		1	
demand	5		14		10		6		35

Table 1: Problem Formulation for Equality Constraints

Table 2: Problem Formulation for Mix Constraints

	B1		B2		B3		Supply
A1	5		4		2		r
		6		3		4	= 5
A2	6		5		3		> 10
		7		4		2	≥ 10
A3	8		9		4		
		6		5		2	$\leq 9$
demand	=8		$\geq 15$		$\leq 6$		

If the MFL paradox situation is identify in starting position then demand and supply can be manageable according to the condition of market and as the time goes the advertisement policies would be converted to increase the supply and demand at the specific market area. By studying MFL phenomena we have sufficient time to manipulate the strategies for unpredictable growing global market. In addition we hope that a lot of the existing excellent software for LFTP will be extended to include at least a preprocessing routine for deciding whether the cost matrix is immune or not against the paradox. If the cost matrix is not immune, and there are optimal dual variables satisfying theorem 1, an option allowing post processing of the optimal solution should be available. The cost of these additional computations is modest and may provide valuable new insight in the problem from which the data for the actual LFTP originates.

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