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On the Stability of a Four Species Syn Eco-System with Commensal Prey-Predator Pair with Prey-Predator Pair of Hosts-VI (Normal Steady State)

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Abstract The present paper is devoted to an investigation on a Four Species (S_1, S_2, S_3, S_4) Syn Eco-System with Commensal Prey-Predator pair with Prey-Predator pair of Hosts (Normal Steady state). The System comprises of a Prey (S_1), a Predator (S_2) that survives upon S_1, two Hosts S_3 and S_4 for which S_1, S_2 are Commensal respectively i.e., S_3 and S_4 benefit S_1 and S_2 respectively, without getting effected either positively or adversely. Further S_3 is Prey for S_4 and S_4 is Predator for S_3. The pair (S_1, S_2) may be referred as 1$^{\text{st}}$ level Prey-Predator and the pair (S_3, S_4) the 2$^{\text{nd}}$ level Prey-Predator. The model equations of the system constitute a set of four first order non-linear ordinary differential coupled equations. In all, there are sixteen equilibrium points. Criteria for the asymptotic stability of one of these sixteen equilibrium points : Normal Steady State is established. The system would be stable if all the characteristic roots are negative, in case they are real, and have negative real parts, in case they are complex. The linearized equations for the perturbations over the equilibrium points are analyzed to establish the criteria for stability and the trajectories are illustrated.

Keywords Commensal; Eco-System; Equillibrium point; Host; Prey; Predator; Quasi-linearization; Stable; Trajectories.

2010 Mathematics Subject Classification 92D25; 92D40.

1 Introduction

Research in the area of theoretical Ecology was initiated in 1925 by Lotka [1] and in 1931 by Volterra [2]. Since then many Mathematicians and Ecologists contributed to the growth of this area of knowledge reported in the treatises of May [3], Smith [4], Kushing [5], Kapur [6] etc. The ecological interactions can be broadly classified as Prey-Predator, Commensalism, Competition, Neutralism, Mutualism and so on. Srinivas [7] studied competitive eco-systems of two species and three species with limited and unlimited resources. Later Lakshminarayan and Pattabhi Ramacharyulu [8] studied Prey-Predator ecological models with partial cover for the Prey and alternate food for the Predator.

Recently, Archana Reddy [9] and Bhaskara Rama Sharma [10] investigated diverse problems related to two species competitive systems with time delay, employing analytical and numerical techniques. Further Phani Kumar and Pattabhi Ramacharyulu [11] studied Three Species Ecosystem Consisting of a Prey, Predator and a Host Commensal to the Prey. The present authors Hari Prasad and Pattabhi Ramacharyulu [12–14] discussed on the stability of a four species: A Prey-Predator-Host-Commensal Syn Eco-System.

A Schematic Sketch of the system under investigation is shown here under Figure 1.

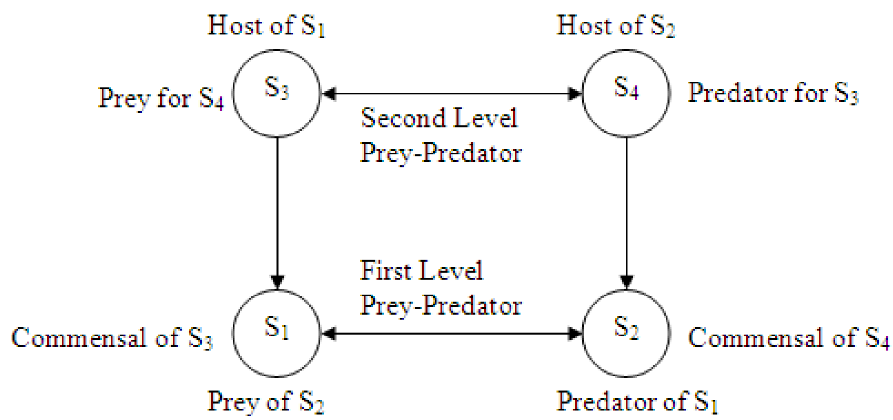

Figure 1: Schematic Sketch of the Syn Eco - System

2 Basic Equations

The model equations for a four species syn eco-system is given by the following system of first order non-linear ordinary differential equations employing the following notation.

Notation

S_1: Prey for S_2 and commensal for S_3.

S_2: Predator surviving upon S_1 and commensal for S_4.

S_3: Host for the commensal (S_1) and Prey for S_4.

S_4: Host of the commensal (S_2) and Predator surviving upon S_4.

$N_i(t)$: The Population strength of S_i at time $t, i = 1, 2, 3, 4$.

t : Time instant

a_i : Natural growth rate of $S_i, i = 1, 2, 3, 4$.

a_{ii} : Self inhibition coefficient of $S_i, i = 1, 2, 3, 4$.

a_{12}, a_{21} : Interaction (Prey-Predator) coefficients of S_1 due to S_2 and S_2 due to S_1.

a_{34}, a_{43}: Interaction (Prey-Predator) coefficients of S_3 due to S_4 and S_4 due to S_3.

a_{13}, a_{24}: Coefficients for commensal for S_1 due to the Host S_3 and S_2 due to the Host S_4

$K_i = \frac{a_i}{a_{ii}}$: Carrying capacities of $S_i, i = 1, 2, 3, 4$.

Further the variables N_1, N_2, N_3, N_4 are non-negative and the model parameters a_1, a_2, $a_3, a_4; a_{11}, a_{22}, a_{33}, a_{44}; a_{12}, a_{21}, a_{13}, a_{24}, a_{34}, a_{43}$ are assumed to be non-negative constants.

The model equations for the growth rates of S_1, S_2, S_3, S_4 are

$$\frac{dN_1}{dt} = a_1 N_1 - a_{11} N_1^2 - a_{12} N_1 N_2 + a_{13} N_1 N_3 \tag{1}$$

$$\frac{dN_2}{dt} = a_2 N_2 - a_{22} N_2^2 + a_{21} N_1 N_2 + a_{24} N_2 N_4 \tag{2}$$

$$\frac{dN_3}{dt} = a_3 N_3 - a_{33} N_3^2 - a_{34} N_3 N_4 \tag{3}$$

$$\frac{dN_4}{dt} = a_4 N_4 - a_{44} N_4^2 + a_{43} N_3 N_4 \tag{4}$$

3 Equilibrium States

The system under investigation has sixteen equilibrium states defined by

$$\frac{dN_i}{dt} = 0, \ i = 1, 2, 3, 4 \tag{5}$$

as given in Table 1.

The present paper deals with the normal steady state only (Sl. No. 16 marked * in the above Table 1). The stability of the other equilibrium states will be presented in the forth coming communications.

4 Stability of the Equilibrium States

Let

$$N = (N_1, N_2, N_3, N_4) = \bar{N} + U \tag{6}$$

where $U = (u_1, u_2, u_3, u_4)$ is a perturbation over the equilibrium state $\bar{N} = \left(\bar{N}_1, \bar{N}_2, \bar{N}_3, \bar{N}_4\right)$.

The basic equations (1), (2), (3), (4) are quasi-linearized to obtain the equations for the perturbed state.

$$\frac{dU}{dt} = AU \tag{7}$$

where

$$A = \begin{bmatrix} a_1 - 2a_{11}\bar{N}_1 - a_{12}\bar{N}_2 + a_{13}\bar{N}_3 & -a_{12}\bar{N}_1 & a_{13}\bar{N}_1 & 0 \\ a_{21}\bar{N}_1 & a_2 - 2a_{22}\bar{N}_2 + a_{21}\bar{N}_1 + a_{24}\bar{N}_4 & 0 & a_{24}\bar{N}_2 \\ 0 & 0 & a_3 - 2a_{33}\bar{N}_3 - a_{34}\bar{N}_3 & -a_{34}\bar{N}_3 \\ 0 & 0 & a_{34}\bar{N}_4 & a_4 - 2a_{44}\bar{N}_4 + a_{43}\bar{N}_3 \end{bmatrix} \tag{8}$$

The characteristic equation for the system is

$$\det\left[A - \lambda I\right] = 0. \tag{9}$$

The equilibrium state is stable, if all the four roots of the equation (9) are negative in case they are real or have negative real parts in case they are complex.

5 Stability of the Co-existent State (or) Normal Steady State

(Sl. No. 16 marked * in Table.1)

To discuss the stability of equilibrium point

$$\bar{N}_1 = \frac{a_{22}\alpha_2 - a_{12}\gamma_2}{\beta_1}, \quad \bar{N}_2 = \frac{a_{11}\gamma_2 + a_{21}\alpha_2}{\beta_1}, \ \bar{N}_3 = \frac{\alpha}{\beta}, \ \bar{N}_4 = \frac{\gamma}{\beta}$$

where

$$\alpha = a_3 a_{44} - a_4 a_{34}, \ \beta = a_{33} a_{44} + a_{34} a_{43} > 0, \ \gamma = a_3 a_{43} + a_4 a_{33} > 0 \tag{10}$$

Table 1: Equilibrium States of the Given System

S.No.	Equilibrium State	Equilibrium Point
1	Fully Washed out state	$\overline{N_1}=0, \overline{N_2}=0, \overline{N_3}=0, \overline{N_4}=0$
2	Only the Host (S_4) of S_2 survives	$\overline{N_1}=0, \overline{N_2}=0, \overline{N_3}=0, \overline{N_4}=\frac{a_4}{a_{44}}$
3	Only the Host (S_3)of S_1 survives	$\overline{N_1}=0, \overline{N_2}=0, \overline{N_3}=\frac{a_3}{a_{33}}, \overline{N_4}=0$
4	Only the Predator (S_2) survives	$\overline{N_1}=0, \overline{N_2}=\frac{a_2}{a_{22}}, \overline{N_3}=0, \overline{N_4}=0$
5	Only the Prey (S_1) survives	$\overline{N_1}=\frac{a_1}{a_{11}}, \overline{N_2}=0, \overline{N_3}=0, \overline{N_4}=0$
6	Prey (S_1) and Predator (S_2) washed out	$\overline{N_1}=0, \overline{N_2}=0, \overline{N_3}=\frac{\alpha}{\beta}, \overline{N_4}=\frac{\gamma}{\beta}$ where $\alpha=a_3a_{44}-a_4a_{34}$, $\beta=a_{33}a_{44}+a_{34}a_{43}>0$ $\gamma=a_3a_{43}+a_4a_{33}>0$
7	Prey (S_1) and Host (S_3) of S_1 washed out	$\overline{N_1}=0, \overline{N_2}=\frac{\delta_1}{a_{22}a_{44}}, \overline{N_3}=0, \overline{N_4}=\frac{a_4}{a_{44}}$ where $\delta_1=a_2a_{44}+a_4a_{24}>0$
8	Prey (S_1) and Host (S_4) of S_2 washed out	$\overline{N_1}=0, \overline{N_2}=\frac{a_2}{a_{22}}, \overline{N_3}=\frac{a_3}{a_{33}}, \overline{N_4}=0$
9	Predator (S_2) and Host (S_3) of S_1 washed out	$\overline{N_1}=\frac{a_1}{a_{11}}, \overline{N_2}=0, \overline{N_3}=0, \overline{N_4}=\frac{a_4}{a_{44}}$
10	Predator (S_2) and Host (S_4) of S_2 washed out	$\overline{N_1}=\frac{\delta_2}{a_{11}a_{33}}, \overline{N_2}=0, \overline{N_3}=\frac{a_3}{a_{33}}, \overline{N_4}=0$ where $\delta_2=a_1a_{33}+a_3a_{13}>0$
11	Prey (S_1) and Predator (S_2)survives	$\overline{N_1}=\frac{\alpha_1}{\beta_1}, \overline{N_2}=\frac{\gamma_1}{\beta_1}, \overline{N_3}=0, \overline{N_4}=0$ where $\alpha_1=a_1a_{22}-a_2a_{12}$, $\beta_1=a_{11}a_{22}+a_{12}a_{21}>0$ $\gamma_1=a_1a_{21}+a_2a_{11}>0$
12	Only the Prey (S_1) washed out	$\overline{N_1}=0, \overline{N_2}=\frac{a_2\beta+a_{24}\gamma}{a_{22}\beta}, \overline{N_3}=\frac{\alpha}{\beta}, \overline{N_4}=\frac{\gamma}{\beta}$
13	Only the predator (S_2) washed out	$\overline{N_1}=\frac{a_1\beta+a_{13}\alpha}{a_{11}\beta}, \overline{N_2}=0, \overline{N_3}=\frac{\alpha}{\beta}, \overline{N_4}=\frac{\gamma}{\beta}$
14	Only the Host (S_3) of S_1 washed out	$\overline{N_1}=\frac{a_1a_{22}a_{44}-a_{12}\delta_1}{a_{44}\beta_1}, \overline{N_2}=\frac{a_1a_{21}a_{44}+a_{11}\delta_1}{a_{44}\beta_1},$ $\overline{N_3}=0, \overline{N_4}=\frac{a_4}{a_{44}}$
15	Only the Host (S_4) of S_2 washed out	$\overline{N_1}=\frac{a_{22}\delta_2-a_2a_{12}a_{33}}{a_{33}\beta_1}, \overline{N_2}=\frac{a_{21}\delta_2+a_2a_{11}a_{33}}{a_{33}\beta_1},$ $\overline{N_3}=\frac{a_3}{a_{33}}, \overline{N_4}=0$
16*	**The co-existent state (or) Normal steady state**	$\overline{N_1}=\frac{a_{22}\alpha_2-a_{12}\gamma_2}{\beta_1}, \overline{N_2}=\frac{a_{11}\gamma_2+a_{21}\alpha_2}{\beta_1},$ $\overline{N_3}=\frac{\alpha}{\beta}, \overline{N_4}=\frac{\gamma}{\beta}$ where $\alpha_2=a_1+a_{13}\frac{\alpha}{\beta}$, $\gamma_2=a_2+a_{24}\frac{\gamma}{\beta}>0$

$$\beta_1 = a_{11}a_{22} + a_{12}a_{21},\ \alpha_2 = a_1 + a_{13}\frac{\alpha}{\beta},\ \gamma_2 = a_2 + a_{24}\frac{\gamma}{\beta} > 0 \tag{11}$$

This would exist only when

$$a_{22}\alpha_2 > a_{12}\gamma_2, a_3a_{44} > a_4a_{34} \tag{12}$$

Let us consider small deviations $u_1(t),\ u_2(t),\ u_3(t),\ u_4(t)$ from the steady state, that is

$$N_i(t) = \bar{N}_i + u_i(t), i = 1,2,3,4 \tag{13}$$

where $u_i(t)$ is a small perturbations in the species S_i.

Substituting (13) in the equations (1), (2), (3), (4) and neglecting products and higher powers of $u_1,\ u_2,\ u_3,\ u_4$

We get

$$\frac{du_1}{dt} = -a_{11}\bar{N}_1u_1 - a_{12}\bar{N}_1u_2 + a_{13}\bar{N}_1u_3 \quad ; \quad \frac{du_2}{dt} = a_{21}\bar{N}_2u_1 - a_{22}\bar{N}_2u_2 + a_{24}\bar{N}_2u_4 \tag{14}$$

$$\frac{du_3}{dt} = a_{33}\bar{N}_3u_3 - a_{34}\bar{N}_3u_4 \quad ; \quad \frac{du_4}{dt} = a_{43}\bar{N}_4u_3 - a_{44}\bar{N}_4u_4 \tag{15}$$

The characteristic equation of which is

$$\left(\lambda^2 + A_1\lambda + B_1\right)\left(\lambda^2 + A_2\lambda + B_2\right) = 0 \tag{16}$$

where

$$A_1 = a_{11}\bar{N}_1 + a_{22}\bar{N}_2, \quad B_1 = (a_{11}a_{22} - a_{12}a_{21})\bar{N}_1\bar{N}_2 \tag{17}$$

$$A_2 = a_{33}\bar{N}_3 + a_{44}\bar{N}_4, \quad B_2 = (a_{33}a_{44} + a_{34}a_{43})\bar{N}_3\bar{N}_4 \tag{18}$$

Let λ_1, λ_2 and λ_3, λ_4 be the zeros of the quadratic polynomials $\lambda^2 + A_1\lambda + B_1$ and $\lambda^2 + A_2\lambda + B_2$ on the above equation (16) respectively.

Case (A) When the roots λ_1, λ_2 and λ_3, λ_4 noted to be negative

Hence the co-existent state is **stable** and the equations (14), (15) yield the solutions,

$$u_1 = \left[\frac{\left(a_{12}\bar{N}_1u_{20} - \mu_3 - \mu_4\right) - (\mu_1 + \mu_2 - u_{10})\left(a_{11}\bar{N}_1 + \lambda_2\right)}{\lambda_2 - \lambda_1}\right]e^{\lambda_1 t}$$
$$+ \left[\frac{\left(a_{12}\bar{N}_1u_{20} - \mu_3 - \mu_4\right) - (\mu_1 + \mu_2 - u_{10})\left(a_{11}\bar{N}_1 + \lambda_1\right)}{\lambda_1 - \lambda_2}\right]e^{\lambda_2 t} + \mu_1 e^{\lambda_3 t} + \mu_2 e^{\lambda_4 t} \tag{19}$$

$$u_2 = \left[\frac{\left(a_{12}\bar{N}_1u_{20} - \mu_3 - \mu_4\right) - (\mu_1 + \mu_2 - u_{10})\left(a_{11}\bar{N}_1 + \lambda_2\right)}{(\lambda_1 - \lambda_2)\,a_{12}\bar{N}_1}\left(a_{11}\bar{N}_1 + \lambda_1\right)\right]e^{\lambda_1 t}$$
$$+ \left[\frac{\left(a_{12}\bar{N}_1u_{20} - \mu_3 - \mu_4\right) - (\mu_1 + \mu_2 - u_{10})\left(a_{11}\bar{N}_1 + \lambda_1\right)}{(\lambda_2 - \lambda_1)\,a_{12}\bar{N}_1}\left(a_{11}\bar{N}_1 + \lambda_2\right)\right]e^{\lambda_2 t}$$
$$+\mu_3 e^{\lambda_3 t} + \mu_4 e^{\lambda_4 t} \tag{20}$$

$$u_3 = \left[\frac{a_{34}\bar{N}_3 u_{40} + \left(a_{33}\bar{N}_3 + \lambda_4\right) u_{30}}{\lambda_4 - \lambda_3}\right] e^{\lambda_3 t} + \left[\frac{a_{34}\bar{N}_3 u_{40} + \left(a_{33}\bar{N}_3 + \lambda_3\right) u_{30}}{\lambda_3 - \lambda_4}\right] e^{\lambda_4 t} \tag{21}$$

$$u_4 = \left[\frac{a_{34}\bar{N}_3 u_{40} + \left(a_{33}\bar{N}_3 + \lambda_4\right) u_{30}}{\left(\lambda_3 - \lambda_4\right) a_{34}\bar{N}_3} \left(a_{33}\bar{N}_3 + \lambda_3\right)\right] e^{\lambda_3 t} + \left[\frac{a_{34}\bar{N}_3 u_{40} + \left(a_{33}\bar{N}_3 + \lambda_3\right) u_{30}}{\left(\lambda_4 - \lambda_3\right) a_{34}\bar{N}_3} \left(a_{33}\bar{N}_3 + \lambda_4\right)\right] e^{\lambda_4 t} \tag{22}$$

where

$$\eta_1 = a_{13}\bar{N}_3 \left(\lambda_1 + a_{22}\bar{N}_2\right) \left[\frac{a_{34}\bar{N}_3 u_{40} + \left(a_{33}\bar{N}_3 + \lambda_4\right) u_{30}}{\lambda_4 - \lambda_3}\right] \tag{23}$$

$$\eta_2 = a_{13}\bar{N}_3 \left(a_{22}\bar{N}_2 + \lambda_2\right) \left[\frac{a_{34}\bar{N}_3 u_{40} + \left(a_{33}\bar{N}_3 + \lambda_3\right) u_{30}}{\lambda_3 - \lambda_4}\right] \tag{24}$$

$$\bar{\alpha}_1 = a_{13}\bar{N}_1 + a_{22}\bar{N}_2, \quad \bar{\beta}_1 = \left(a_{11}a_{22} + a_{12}a_{21}\right) \bar{N}_1\bar{N}_2 \tag{25}$$

$$\mu_1 = \frac{\left(\lambda_3 - \lambda_4\right) a_{34}\bar{N}_3\eta_1 - \left(a_{12}a_{24}\bar{N}_1\bar{N}_2\right) \left[a_{34}\bar{N}_3 u_{40} + \left(a_{33}\bar{N}_3 + \lambda_4\right) u_{30}\right] \left(a_{33}\bar{N}_3 + \lambda_3\right)}{\left(\lambda_3^2 + \bar{\alpha}_1\lambda_3 + \bar{\beta}_1\right) \left(\lambda_3 - \lambda_4\right) a_{34}\bar{N}_3} \tag{26}$$

$$\mu_2 = \frac{\left(\lambda_4 - \lambda_3\right) a_{34}\bar{N}_3\eta_2 - \left(a_{12}a_{24}\bar{N}_1\bar{N}_2\right) \left[a_{34}\bar{N}_3 u_{40} + \left(a_{33}\bar{N}_3 + \lambda_3\right) u_{30}\right] \left(a_{33}\bar{N}_3 + \lambda_4\right)}{\left(\lambda_4^2 + \bar{\alpha}_1\lambda_4 + \bar{\beta}_1\right) \left(\lambda_4 - \lambda_3\right) a_{34}\bar{N}_3} \tag{27}$$

$$\mu_3 = a_{13}\bar{N}_1 \left[\frac{a_{34}\bar{N}_3 u_{40} + \left(a_{33}\bar{N}_3 + \lambda_4\right) u_{30}}{\lambda_4 - \lambda_3}\right] - \left(a_{11}\bar{N}_1 + \lambda_3\right) \mu_1 \tag{28}$$

$$\mu_4 = a_{13}\bar{N}_1 \left[\frac{a_{34}\bar{N}_3 u_{40} + \left(a_{33}\bar{N}_3 + \lambda_3\right) u_{30}}{\lambda_3 - \lambda_4}\right] - \left(a_{11}\bar{N}_1 + \lambda_4\right) \mu_2 \tag{29}$$

and $u_{10},\ u_{20},\ u_{30},\ u_{40}$ are the initial values of $u_1,\ u_2,\ u_3,\ u_4$ respectively.

There would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates u_1, u_2, u_3, u_4 and the initial values of the perturbations $u_{10}\left(t\right), u_{20}\left(t\right), u_{30}\left(t\right), u_{40}\left(t\right)$ of the species S_1, S_2, S_3, S_4. Of these 576 situations some typical variations are illustrated through respective solution curves that would facilitate to make some reasonable observations. And the solution curves are illustrated in Figure 2 and Figure 3 and the conclusions are presented here.

Case (i) If $u_{10} < u_{20} < u_{30} < u_{40}$ and $a_2 < a_1 < a_3 < a_4$

In this case the natural birth rates of the predator (S_2), prey (S_1), host (S_3) of S_1 and the host (S_4) of S_2 are in ascending order. Initially the predator (S_2) dominates over the Prey (S_1) till the time instant t_{12}^* and thereafter the dominance is reversed. The time t_{12}^* may be called dominance time of S_2 over S_1.

Case (ii) If $u_{20} < u_{40} < u_{10} < u_{30}$ and $a_3 < a_2 < a_4 < a_1$

In this case host (S_3) of S_1 has the least natural birth rate. Initially the host (S_3) of S_1 dominates over the prey (S_1), host (S_4) of S_2, predator (S_2) till the time instant $t^*_{13},\ t^*_{43},\ t^*_{23}$ respectively and thereafter the dominance is reversed.

Case (B) When $\lambda_1,\ \lambda_2$ the roots noted to be negative

The trajectories in this case are same as in case(A).

Case (a) If $\left(a_{33}\bar{N}_3 - a_{44}\bar{N}_4\right)^2 < 4a_{34}a_{43}\bar{N}_3\bar{N}_4$, the roots $\lambda_3,\ \lambda_4$ are complex. Hence the coexistent state is **stable** and this is illustrated in Figure 4.

Case (b) If $\left(a_{33}\bar{N}_3 - a_{44}\bar{N}_4\right)^2 > 4a_{34}a_{43}\bar{N}_3\bar{N}_4$, one root ($\lambda_3$) negative while the other root (λ_4) is positive. Hence the coexistent state is **unstable** and the solution curves are illustrated in Figures (5), (6).

Case (i) If $u_{40} < u_{30} < u_{10} < u_{20}$ and $a_1 < a_3 < a_4 < a_2$

In this case the natural birth rates of the prey (S_1), host (S_3) of S_1, host (S_4) of S_2 and the predator (S_2) are in ascending order. Initially the prey (S_1) dominates over the host (S_4) of S_2, host (S_3) of S_1 till the time instant $t^*_{41},\ t^*_{31}$ respectively and there after the dominance is reversed. Also the host (S_3) of S_1 dominates over the host (S_4) of S_2 till the time instant t^*_{43} and the dominance gets reversed thereafter.

Case (ii) If $u_{10} < u_{40} < u_{30} < u_{20}$ and $a_2 < a_1 < a_4 < a_3$

In this case predator (S_2) has the least natural birth rate. Initially it is dominated over by the host (S_3) of S_1, host (S_4) of S_2, prey (S_1) till the time instant $t^*_{32},\ t^*_{42},\ t^*_{12}$ respectively and thereafter the dominance is reversed.

Case (C) When the root (λ_1) is negative while the other root (λ_2) is positive. Hence the coexistent state is **unstable** and the trajectories in this case are same as in case(A).

Case (a) If $\left(a_{33}\bar{N}_3 - a_{44}\bar{N}_4\right)^2 < 4a_{34}a_{43}\bar{N}_3\bar{N}_4$, the roots $\lambda_3,\ \lambda_4$ are complex. This is illustrated in Figure 7.

Case (b) If $\left(a_{33}\bar{N}_3 - a_{44}\bar{N}_4\right)^2 > 4a_{34}a_{43}\bar{N}_3\bar{N}_4$, one root ($\lambda_3$) is negative while the other root (λ_4) is positive and the solution curves are illustrated in Figure 8 and Figure 9.

Case (i) If $u_{20} < u_{30} < u_{40} < u_{10}$ and $a_2 < a_3 < a_4 < a_1$

In this case predator (S_2) has the least natural birth rate and the prey (S_1) dominates the host (S_4) of S_2, host (S_3) of S_1, predator (S_2) in natural growth rate as well as in its population strength.

Case (ii) If $u_{40} < u_{20} < u_{10} < u_{30}$ and $a_1 < a_3 < a_4 < a_2$

In this case prey (S_1) has the least natural birth rate. Initially it is dominated over by the predator (S_2), host (S_4) of S_2 till the time instant $t^*_{21},\ t^*_{41}$ respectively and thereafter the dominance is reversed. Also the host (S_3) of S_1 dominates over the predator (S_2), host (S_4) of S_2 till the time instant $t^*_{23},\ t^*_{43}$ respectively and the dominance gets reversed thereafter.

5.1 Trajectories of Perturbations

The trajectories in the $u_3 - u_4$ plane given by

$$\left[u_4^{(a-1)(v_1-v_2)}\right] d_1 = \frac{(u_3 - v_1 u_4)^{av_1 - d}}{(u_3 - v_2 u_4)^{av_2 - d}} \tag{30}$$

where v_1 and v_2 are roots of the quadratic equation

$$av^2 + bv + c = 0 \tag{31}$$

and

$$a = a_{43}\bar{N}_4,\ d = a_{44}\bar{N}_4,\ b = a_{33}\bar{N}_3 - d,\ c = a_{34}\bar{N}_3 \tag{32}$$

and d_1 is an arbitrary constant. This is illustrated in Figure 10.

6 Perturbation Graphs

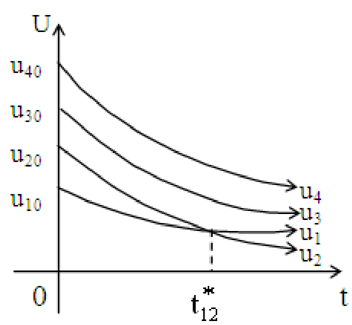

Figure 2: Graph of $u_{10} < u_{20} < u_{30} < u_{40}$ and $a_2 < a_1 < a_3 < a_4$

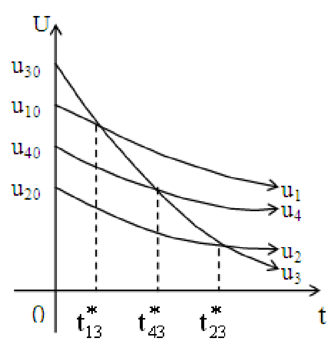

Figure 3: Graph of $u_{20} < u_{40} < u_{10} < u_{30}$ and $a_3 < a_2 < a_4 < a_1$

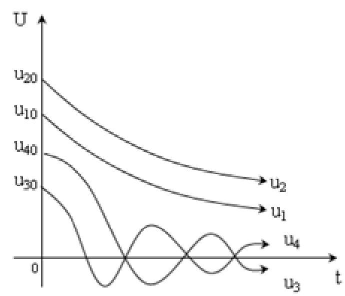

Figure 4: Graph of $u_{30} < u_{40} < u_{10} < u_{20}$ and $a_3 < a_4 < a_1 < a_2$

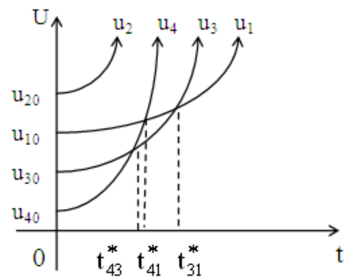

Figure 5: Graph of $u_{40} < u_{30} < u_{10} < u_{20}$ and $a_1 < a_3 < a_4 < a_2$

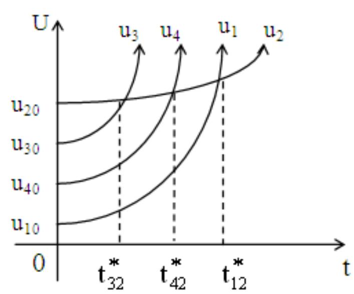

Figure 6: Graph of $u_{10} < u_{40} < u_{30} < u_{20}$ and $a_2 < a_1 < a_4 < a_3$

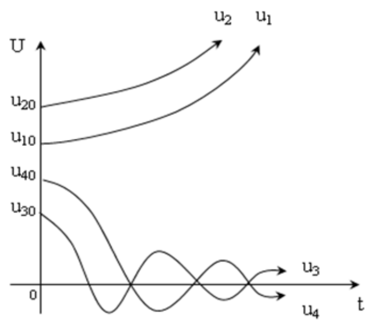

Figure 7: Graph of $u_{30} < u_{40} < u_{10} < u_{20}$ and $a_4 < a_3 < a_1 < a_2$

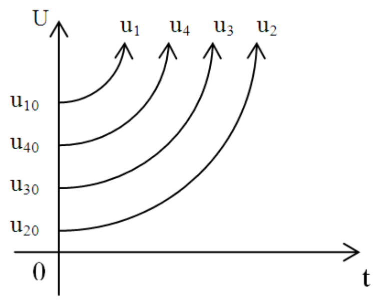

Figure 8: Graph of $u_{20} < u_{30} < u_{40} < u_{10}$ and $a_2 < a_3 < a_4 < a_1$

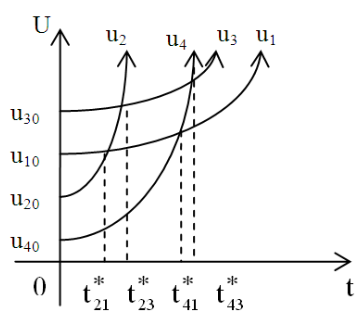

Figure 9: Graph of $u_{40} < u_{20} < u_{10} < u_{30}$ and $a_1 < a_3 < a_4 < a_2$

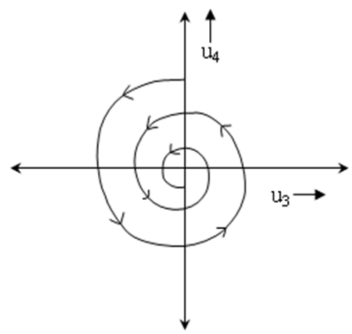

Figure 10: Graph of $\left[u_4^{(a-1)(v_1-v_2)}\right] d_1 = \frac{(u_3-v_1u_4)^{av_1-d}}{(u_3-v_2u_4)^{av_2-d}}$

7 Open Problem

Investigate some relation-chains between the species such as Prey-Predation, Neutralism, Commensalism, Mutualism, Competition and Ammensalism between four species (S_1, S_2, S_3, S_4) with the population relations.

S_1 a Prey to S_2 and Commensal to S_3, S_2 is a Predator living on S_1 and Commensal to S_4, S_3 a Host to S_1, S_4 a Host to S_2 and S_3 a Prey to S_4, S_4 a Predator to S_3.

The present paper deals with the study on stability of Co-existent State only of the above problem. The stability of the other equilibrium states is to be investigated and the perturbation curves, the trajectories of perturbations of the other equilibrium states are to be studied. The numerical solutions for the growth rate equations can also computed employing Runge Kutta fourth order method.

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