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Two Results on Infinite Transversal Theory

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Abstract By the assistance of an independence space on a set S with finite rank, this paper presents a way to determine a family of subsets of S to possess an independent transversal. Afterwards, it provides another result on infinite transversal relative to partial transversals of a family of subsets of S. The two results are in the field of transversal theory.

Keywords transversal; independence space; finite rank; finite partial transversal.

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1 Introduction and Preliminaries

The infinite transversal theory still attracts many researchers attention [1–7]. In addition, there are still some open problems relative to infinite transversals not be answered such as the problems provided by Welsh [1].

Ingleton [2] points out that if $\mathcal{A} = (A_i : i \in I)$ is a family of subsets of a set S, then the partial transversals of $\mathcal{A}$ are the "independent sets" in the sense of Whitney [3] of an "independence structure" (or "matroid" when S is finite). Ingleton [2] discusses some properties of transversal independence structures under the above ideas. Additionally, Bonin *et al.* [4] deal with some characterizations of transversal and transversal matroids.

All these analysis express that infinite transversal theory is still valuable to research on and we should better study on transversal theory with the assistance of matroid theory.

Here, we just deal with two key results about infinite transversal with the assistance of two classes of infinite matroids–pre-independence space and independence spaces. The two classes of infinite matroids are important in infinite matroids [1, 2, 3, 5]. We firstly review some terminology needed later on.

In what follows, S is a set–perhaps–infinite set; $X \subset\subset Y$ represents that X is a finite subset of Y; $|Z|$ denotes the cardinality of a set Z.

Definition 1 [1]

A *pre-independence space* $M_p(S)$ is a set S together with a collection $\mathcal{I}$ of subsets of S (called *independent sets*) such that

(i1) $\mathcal{I} \neq \emptyset$.

(i2) A subset of an independent set is independent.

(i3) If I_1 and I_2 are finite members of $\mathcal{I}$ with $|I_1| < |I_2|$, then there exists x in $I_2 \setminus I_1$ such that $I_1 \cup x \in \mathcal{I}$.

(i4) If $X \subseteq S$ and every finite subset of X is in $\mathcal{I}$, then X is in $\mathcal{I}$.

We shall call a pre-independence space satisfying (i4) an *independence space*.

Definition 2 [1,5]

With a pre-independence space $(S, \mathcal{I})$, we associate a *rank function* ρ taking values from $\{0, 1, 2, \ldots, \infty\}$ and defined for $X \subseteq S$ by

$$\rho(X)\sup\{|Y| : Y \in \mathcal{I}, Y \subset\subset X\}.$$

Similarly to Welsh [1] and Oxley [5], we present the following definitions.

Definition 3 Let I denote an index and $\mathcal{A}$ denote the family $(A_i : i \in I)$ of subsets of S. A set $T \subseteq S$ is a *transversal* of $\mathcal{A}$ if there is a bijection $\pi : T \to I$ such that $x \in A_{\pi(x)}$, $\forall x \in T$. A set $X \subseteq S$ is a *partial transversal* of $\mathcal{A}$ if X is a transversal of some subfamily $(A_i : i \in J)$ of $\mathcal{A}$. $|X|$ is called the *length* of the partial transversal.

For the sake of convenient, if $X = (x_i : i \in J)$ is a partial transversal of $\mathcal{A}$ with finite length, then we call X a *finite partial transversal* of $\mathcal{A}$.

With a pre-independence space $(S, \mathcal{I})$, it is evident to have the following properties.

Lemma 1 Let $(S, \mathcal{I})$ be a pre-independence space with ρ as its rank function. Then

(i) $X \subseteq Y \Rightarrow \rho(X) \leq \rho(Y)$.

(ii) If $X \subseteq S$ and $|X| < \infty$, then $X \in \mathcal{I} \Leftrightarrow \rho(X) = |X|$.
If $X \subseteq S$ and $|X| \not< \infty$, then $X \in \mathcal{I} \Rightarrow \rho(X) = |X|$.

2 Main Results

This section includes two key results on infinite transversal theory.

Let $\mathcal{A} = (A_i : i \in I)$ be a family of subsets of S. If $|A_i| < \infty$ for every $(i \in I)$, then there are some transversal results for $\mathcal{A}$ (see Welsh [1, p.390, Theorem 2 & 5]). If $|A_i| \not< \infty$ for some $(i \in I)$, then Mao [6] presents the necessary and sufficient conditions for $\mathcal{A}$ to have a transversal using Hall's transversal theory. How about the expression of the above relative results using matroid language? The answer will be useful to discuss infinite matroid theory and also transversal theory as that in Welsh [1, pp.389-392]. The following theorem deals with a transversal result for $\mathcal{A}$ using matroid language.

Theorem 1 *Let $(S, \mathcal{I})$ be an independence space with rank function ρ, $\rho(S) < \infty$ and $\mathcal{A} = (A_i : i \in I)$ be a family of subsets of S. Then the following statements are equivalent.*

(i) *$\mathcal{A}$ has an independent transversal, i.e. $\mathcal{A}$ has a transversal T and T is independent in $(S, \mathcal{I})$.*

(ii) *There is $\mathcal{B} = (B_i : i \in I)$ satisfying $\rho(A(J)) \geq \rho(B(J)) \geq |J|$ for any $J \subset\subset I$ and $\mathcal{I} \ni B_i \subset\subset A_i, (i \in I)$, where $A(J) = \cup_{i \in J} A_i$ and $B(J) = \cup_{i \in J} B_i$.*

Proof

(1)⇒(2)

Suppose $\mathcal{A}$ has an independent transversal $(x_i : i \in I)$. Considering $(x_i : i \in I) \in \mathcal{I}$ with $\rho(S) < \infty$ and Lemma 1, we may indicate $\rho(x_i : i \in I) \leq \rho(S) < \infty$. In addition, $(S, \mathcal{I})$ has finite rank. Let $(B_i : i \in I) = (x_i : i \in I)$. In light of (i2) and Lemma 1, we may easily gain $\rho(B(J)) = |J|$ and $\rho(A(J)) \geq \rho(B(J))$, for any $J \subset\subset I$.

Moreover, $\rho(A(J)) \geq \rho(B(J)) \geq |J|$.

(2)⇒(1)

Since $(S, \mathcal{I})$ has finite rank, $B_i \in \mathcal{I}, (i \in I)$ and Lemma 1, it causes $\rho(B_i) = |B_i| < \rho(S) < \infty, (i \in I)$. That is to say, $\mathcal{B}$ is a family of finite subsets of S. According to [1, p.390, Theorems 2 & 5], we may express that $\mathcal{B}$ has an independent transversal T. Evidently, T is also an independent transversal of $\mathcal{A}$. □

We present an example to explain the effect of Theorem 1.

Example 1 Let $S = \{1, 2, \ldots\}$, n be a given positive integral number and $\mathcal{I} = \{X : |X| \leq n\}$. Posit $\mathcal{A} = (A_i : i \in I)$ to be a family of subsets of S with $|I| = n$. There is $A_i \in \mathcal{A}$ satisfying $|A_i| \not< \infty$.

The first is to prove $(S, \mathcal{I})$ to be an independence space.

(i1), (i2) and (i3) are evidently satisfied by $(S, \mathcal{I})$.

If $X \subseteq S$ and $|X| > n$, then there exists $Y \subset\subset X$ satisfying $|Y| = n + 1$, and so, $Y \notin \mathcal{I}$. That is to say, (i4) holds in $(S, \mathcal{I})$.

Thus, we carry out $(S, \mathcal{I})$ to be an independence space.

Using Theorem 1 and $\rho(S) = n$, where ρ is the rank function of $(S, \mathcal{I})$, we conclude that (2) in Theorem 1 is true for $\mathcal{A}$, and so, $\mathcal{A}$ has an independent transversal in view of Theorem 1.

We may point out that in Example 1, Welsh [1, p.390, Theorem 2] fails to determine an independent transversal of $\mathcal{A}$ because not all of A_i $(i \in I)$ are finite in Example 1. However, Theorem 1 solves this problem.

For a given independence space $(S, \mathcal{I})$ with rank function ρ, there are only two cases for $(S, \mathcal{I})$: one is $\rho(S) < \infty$ and the other is $\rho(S) \not< \infty$.

If $\rho(S) < \infty$, i.e. $(S, \mathcal{I})$ has finite rank, then Theorem 1 gives a way to decide whether or not a family $\mathcal{A}$ of subsets of S has an independent transversal.

If $\rho(S) \not< \infty$. [1, p.390, Theorem 2 & 5] presents a way to consider $\mathcal{A}$ to have an independent transversal if $\mathcal{A}$ is a family of finite subsets of S. But, if not all of A_i $(i \in I)$ are finite, there are no way to decide $\mathcal{A}$ to have a transversal.

If we consider under what conditions, $\mathcal{A}$ a family of subsets of S will have an independent transversal, then by Theorem 1, we may deal with a large number of independence spaces defined on S.

In transversal theory, there are two important parts: one is to consider its transversal for a family subsets $\mathcal{A}$ of S and another is to consider the partial transversal of $\mathcal{A}$. One of the famous result in infinite transversal theory is given by Welsh [1, p.390, Theorem 3]. It proves that the collection of partial transversals of $\mathcal{A}$ is a pre-independence space on S.

Additionally, we may describe by Definition 3 that a partial transversal T of $\mathcal{A}$ satisfies $T = \cup_{p\in P} T_p$, where T_p is a finite partial transversal of $\mathcal{A}, (p \in P)$. However, if T_q is a finite partial transversal of $\mathcal{A}, (q \in Q)$, then $\cup_{q\in Q} T_q$ is perhaps not a partial transversal of $\mathcal{A}$.

Therefore, the collection of finite partial transversal of $\mathcal{A}$ is more general and more valuable up to the above sense. However, Welsh [1, p.390, Theorem 3] does not consider with the collection of finite partial transversal. Here, we may also consider the collection of finite partial transversal.

Theorem 2 *Let $\mathcal{A} = (A_i : i \in I)$ be a family of subsets of a set S. Then the collection $\mathcal{I}$ of finite partial transversals of $\mathcal{A}$ is a pre-independence space on S. If $(S, \mathcal{I})$ has finite rank, then these finite partial transversals form an independence space.*

Proof We divide the proof in two steps:

Step 1:

We will check that (i1)-(i3) hold for $(S, \mathcal{I})$.

$\emptyset \in \mathcal{I}$ is obvious. Thus, (i1) is evidently.

Let $X \subseteq Y \in \mathcal{I}$. We may indicate that there exists $(A_p : p \in P) \subseteq \mathcal{A}$ such that Y is a transversal of $(A_p : p \in P)$ and $|Y| < \infty$. So, $|P| = |Y| < \infty$ is true. Furthermore, this follows $Y = X \cup \{y_1, y_2, \ldots, y_t\}$ and $|P| = t + |X|$, where $\{y_1, \ldots, y_t\} = Y \setminus X$. Hence, X is a transversal of $(A_q : q \in Q) \subseteq (A_p : p \in P)$ and $|X| < |Y| < \infty$. Namely, $X \in \mathcal{I}$ holds. Thus, (i2) holds.

Let $X, Y \in \mathcal{I}$ and $|X| < |Y| < \infty$.

These supposition reveal that there are $(A_h : h \in H)$ and $(A_r : r \in R)$ as two subsets of $\mathcal{A}$ such that X, Y is a transversal of $(A_h : h \in H)$ and $(A_r : r \in R)$ respectively. Additionally, there are $|R| = |Y|$ and $|X| = |H|$. Therefore, we may suppose $X = (x_1, \ldots, x_t, x_{t+1}, \ldots, x_h), X \cap Y = (x_{t+1}, \ldots, x_h)$ and $Y = (y_1, \ldots, y_t, x_{t+1}, \ldots, x_h, y_{h+1}, \ldots, y_r)$.

If $X \subset Y$.

We may obtain $Y = (x_1, \ldots, x_t, x_{t+1}, \ldots, x_h, y_{h+1}, \ldots, y_r)$. $|H| < |R|$ shows us that $X \cup y_r$ is a transversal of $(A_h : h \in H) \cup (A_r) = (A_g : g \in G)$ where $H \cup r = G$.

If $X \not\subset Y$.

No matter to write $(A_r : r \in R)$ as $(A_{t+1}, \ldots, A_h, A_{h+1}, \ldots, A_r, A'_1, \ldots, A'_t)$, where $y_j \in A_j\,(j = h+1, \ldots, r), y_\alpha \in A'_\alpha, A'_\alpha = A_\alpha$ $(\alpha = 1, \ldots, t)$, and $x_\gamma \in A_\gamma(\gamma = t+1, \ldots, h)$. We may assure firmly that there is $\beta \in \{h+1, \ldots, r\} \cup \{1, \ldots, t\}$ satisfying $y_\beta \notin X = (x_i : i = 1, \ldots, h)$. Otherwise, $Y \subseteq X$ holds, and so, $|Y| \leq |X|$ holds. This carries out a contradiction to $|X| < |Y|$.

Namely, $X \cup y_\beta$ is a transversal of $(A_h : h \in H) \cup A_\beta$, and further, $X \cup y_\beta \in \mathcal{I}$.

Summing up, (i3) is satisfied by $\mathcal{I}$.

Consequently, $(S, \mathcal{I})$ is a pre-independence space.

Step 2:

Assume $\rho(S) < \infty$. We will check the satisfaction of (i4) for $(S, \mathcal{I})$.

Suppose $X \subseteq S$ and $Y \in \mathcal{I}$ for any $Y \subset\subset X$.

In light of Definition 1, $\rho(S) < \infty$ will cause that there is a maximal element B of $\mathcal{I}$ satisfying $\rho(S) = \rho(B)$. Thus, by Lemma 1, $\rho(S) = |B| < \infty$ holds.

Additionally, for any $A \in \mathcal{I}$, there is $|A| \leq |B|$ according to Lemma 1.

Because by Definition 1 and the above discussion, there exists a maximal independent subset I_X of X satisfying $\rho(X) = \rho(I_X) = |I_X|$.

If $X \in \mathcal{I}$, (i4) holds.

If $X \notin \mathcal{I}$. Let $D = x \cup I_X$ where $x \in X \setminus I_X$. Then $D \notin \mathcal{I}$ is found out in view of the maximality of I_X in X. In addition, $\rho(I_X) \leq \rho(S) < \infty$ holds according to Lemma 1. That is, X has a finite subset D not in $\mathcal{I}$. This follows a contradiction to the supposition.

In other words, (i4) holds.

Therefore, $(S, \mathcal{I})$ is an independence space.$\Box$

The following example shows that $\rho(S) < \infty$ is necessary in Theorem 2.

Example 2 Let $S = \{1, 2, 3, \ldots\}$ and $\mathcal{A} = (A_n = (1, 2, \ldots, n) : n = 1, 2, \ldots)$. Let $\mathcal{I}$ be the family of finite partial transversals of $\mathcal{A}$. We see that for any finite subset $B = (b_1, b_2, \ldots, b_t) \subseteq S$, where $b_1 < b_2 < \ldots < b_t$ and $t < \infty$. We may select $A_{b_1} = (1, 2, \ldots, b_1), A_{b_2} = (1, 2, \ldots, b_1, \ldots, b_2), \ldots, A_{b_t} = (1, 2, \ldots, b_1, \ldots, b_2, \ldots, b_t)$.

Then B is a transversal of $(A_{b_1}, A_{b_2}, \ldots, A_{b_t}) \subseteq \mathcal{A}$.

By the above discussion, for any $Y \subset\subset S$, there is $Y \in \mathcal{I}$. However, $|S| \not< \infty$ follows $S \notin \mathcal{I}$. In other words to say, (i4) is not satisfied for $(S, \mathcal{I})$.

In view of Definition 1, $(S, \mathcal{I})$ is not an independence space.

Additionally, we notice that $\rho(S) = sup\{|T| : T \subset\subset S, T \in \mathcal{I}\} = |S| \not< \infty$. Namely, $(S, \mathcal{I})$ does not have finite rank.

3 Conclusion

With the assistance of matroid theory, we present a result for a family $\mathcal{A} = (A_i : i \in I)$ of subsets of S to have an independent transversal. In addition, we also find out a result relative to partial transversal for $\mathcal{A}$. We hope that the two theorems presented in this paper are not only complement to infinite transversal theory and infinite matroids especially independence space and pre-independence space, but also gives some contribution in solving the problems relative to infinite transversal provided in Welsh [1, pp.391-392], and also, to deal with some other problems in infinite transversal theory.

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