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Convergence Theorems of Two-step Iteration Process for Two Asymptotically Quasi-nonexpansive Mappings in the Intermediate Sense

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Abstract In this paper, we establish some weak and strong convergence theorems of modified two-step iteration process for two asymptotically quasi-nonexpansive mappings in the intermediate sense to converge to common fixed points in the setting of real uniformly convex Banach spaces. The results presented in this paper extend, improve and generalize some previous results from the existing literature.

Keywords Asymptotically quasi-nonexpansive mapping in the intermediate sense, modified two-step iteration process, common fixed point, strong convergence, weak convergence, Banach space.

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1 Introduction

Let K be a nonempty subset of a real Banach space E. Let $T\colon K \to K$ be a mapping, then we denote the set of all fixed points of T by $F(T)$. The set of common fixed points of two mappings S and T will be denoted by $F = F(S) \cap F(T)$. A mapping $T\colon K \to K$ is said to be:

(i) nonexpansive if

$$\|Tx - Ty\| \quad \leq \quad \|x - y\|$$

for all $x, y \in K$.

(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \quad \leq \quad \|x - p\|$$

for all $x \in K$ and $p \in F(T)$.

(iii) asymptotically nonexpansive if there exists a sequence $\{k_n\} \in [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - T^n y\| \quad \leq \quad k_n \|x - y\|$$

for all x, $y \in K$ and $n \geq 1$.

(iv) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \in [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - p\| \quad \leq \quad k_n \|x - p\|$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$.

(v) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \quad \leq \quad L\,\|x - y\|$$

for all x, $y \in K$ and $n \geq 1$.

It is clear that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive mapping is uniformly Lipschitzian. Also, if $F(T) \neq \emptyset$, then a nonexpansive mapping is a quasi-nonexpansive mapping and an asymptotically nonexpansive mapping is an asymptotically quasi-nonexpansive mapping but the converse is not true in general.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [6] as a generalization of the class of nonexpansive mappings. Recall also that a mapping $T\colon K \to K$ is said to be asymptotically quasi-nonexpansive in the intermediate sense [21] provided that T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x\in K,\ p\in F(T)} \Big(\|T^n x - p\| - \|x - p\| \Big) \leq 0.$$

From the above definitions, it follows that asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive and asymptotically quasi-nonexpansive mapping in the intermediate sense. But the converse does not hold as the following example:

Example 1 Let $X = \mathbb{R}$ be a normed linear space and $K = [0, 1]$. For each $x \in K$, we define

$$T(x) = \begin{cases} kx, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

where $0 < k < 1$. Then

$$|T^n x - T^n y| = k^n |x - y| \leq |x - y|$$

for all $x, y \in K$ and $n \in \mathbb{N}$.

Thus T is an asymptotically nonexpansive mapping with constant sequence $\{1\}$ and

$$\begin{aligned} \limsup_{n\to\infty}\{|T^n x - T^n y| - |x - y|\} &= \limsup_{n\to\infty}\{k^n\,\|x - y\| - \|x - y\|\} \\ &\leq 0 \end{aligned}$$

because $\lim_{n\to\infty} k^n = 0$ as $0 < k < 1$, for all $x, y \in K$, $n \in \mathbb{N}$ and T is continuous. Hence T is an asymptotically nonexpansive mapping in the intermediate sense.

Example 2 Let $X = \mathbb{R}$, $K = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and $|k| < 1$. For each $x \in K$, define

$$T(x) = \begin{cases} k\,x\,sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then T is an asymptotically nonexpansive mapping in the intermediate sense but it is not asymptotically nonexpansive mapping.

In 1991, Schu [12] introduced the following Mann-type iterative process:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n x_n, \end{aligned} \tag{1}$$

where $T\colon K \to K$ is an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the condition $\delta \leq \alpha_n \leq 1-\delta$ for all $n \geq 1$ for some $\delta > 0$. Hence conclude that the sequence $\{x_n\}$ converges weakly to a fixed point of T.

Since 1972, many authors have studied weak and strong convergence problem of the iterative sequences (with errors) for asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces (see,for example, [6, 7, 9, 10, 11, 12, 13, 19] and references therein).

In 2007, Agarwal *et al.* [1] introduced the following iteration process:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n, \ n \geq 1, \end{aligned} \tag{2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$. They showed that this process converge at a rate same as that of Picard iteration and faster than Mann for contractions.

The above process deals with one mapping only. The case of two mappings in iterative processes has also remained under study since Das and Debata [3] gave and studied a two mappings process. Later on, many authors, for example Khan and Takahashi [9], Shahzad and Udomene [14] and Takahashi and Tamura [18] have studied the two mappings case of iterative schemes for different types of mappings.

Ishikawa-type iteration process

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n, \ n \geq 1, \end{aligned} \tag{3}$$

for two mappings has also been studied by many authors including [3, 9, 17].

Recently, Khan *et al.* [8] modified the iteration process (2) to the case of two mappings as follows:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1-\alpha_n)T^n x_n + \alpha_n S^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n, \ n \geq 1, \end{aligned} \tag{4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$. They established weak and strong convergence theorems in the setting of real Banach spaces.

Remark 1

(i) Note that (4) reduces to (2) when $S = T$. Similarly, the process (4) reduces to (1) when $T = I$.

(ii) The process (2) does not reduce to (1) but (4) does. Thus (4) not only covers the results proved by (2) but also by (1) which are not covered by (2).

(iii) The process (4) is independent of (3) neither of them reduces to the other.

In this paper, we prove some weak and strong convergence theorems for two asymptotically quasi-nonexpansive mappings in the intermediate sense using iteration process (4) in the framework of real Banach spaces. The results presented in this paper extend, improve and generalize several known results from the existing literature.

2 Preliminaries

For the sake of convenience, we restate the following concepts.

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is the function $\delta_E(\varepsilon)\colon (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\Big\{1 - \Big\|\frac{1}{2}(x+y)\Big\| : \|x\| = 1,\, \|y\| = 1,\, \varepsilon = \|x-y\|\Big\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

Two mappings $S, T\colon K \to K$, where K is a subset of a normed space E, are said to satisfy the condition (A') [5] if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that either $\|x - Sx\| \geq f(d(x,F))$ or $\|x - Tx\| \geq f(d(x,F))$ for all $x \in K$, where $d(x,F) = \inf\{\|x-p\| : p \in F\}$.

A mapping $T\colon K \to K$ is said to be demiclosed at zero, if for any sequence $\{x_n\}$ in K, the condition x_n converges weakly to $x \in K$ and Tx_n converges strongly to 0 imply $Tx = 0$.

A mapping $T\colon K \to K$ is said to be semi-compact [2] if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^* \in K$ strongly.

A Banach space E has the Kadec-Klee property [15] if for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ it follows that $\|x_n - x\| \to 0$.

Now, we state the following useful lemmas to prove our main results:

Lemma 1 [12] Let E be a uniformly convex Banach space and $0 < \alpha \leq t_n \leq \beta < 1$ for all $n \in \mathbb{N}$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$ hold for some $a \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2 [16] Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two sequences of nonnegative numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$. If one of the following conditions is satisfied:

(i) $\alpha_{n+1} \leq \alpha_n + \beta_n$, $n \geq 1$,

(ii) $\alpha_{n+1} \leq (1+\beta_n)\alpha_n$, $n \geq 1$,

then the limit $\lim_{n\to\infty} \alpha_n$ exists.

Lemma 3 [20] Let $p > 1$ and $R > 1$ be two fixed numbers and E a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g\colon [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda \|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)g(\|x-y\|)$$

for all $x, y \in B_R(0) = \{x \in E : \|x\| \le R\}$, and $\lambda \in [0,1]$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

Lemma 4 [15] Let E be a real reflexive Banach space with its dual E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and p, $q \in W_w(x_n)$ (where $W_w(x_n)$ denotes the set of all weak subsequential limits of $\{x_n\}$). Suppose $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ exists for all $t \in [0,1]$. Then $p = q$.

3 Main Results

In this section, we prove some strong convergence theorems of the iteration scheme (4) for two asymptotically quasi-nonexpansive mappings in the intermediate sense in the framework of real Banach spaces.

Theorem 1 *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $S, T\colon K \to K$ be two asymptotically quasi-nonexpansive mappings in the intermediate sense such that $F = F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (4). Put*

$$\begin{aligned} G_n &= \max\Big\{ \sup_{x\in K,\ q\in F} \Big(\|T^n x - q\| - \|x-q\| \Big) \vee 0, \\ &\quad \sup_{y\in K,\ q\in F} \Big(\|S^n y - q\| - \|y-q\| \Big) \vee 0,\ \forall n \ge 1 \Big\} \end{aligned} \tag{5}$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. Then $\{x_n\}$ converges to a common fixed point of the mappings S and T if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x,F) = \inf_{p\in F} d(x,p)$.

Proof The necessity is obvious. Thus we only prove the sufficiency. Let $q \in F$. Then from (4) and (5), we have

$$\begin{aligned} \|y_n - q\| &= \|(1-\beta_n)x_n + \beta_n T^n x_n - q\| \\ &\le (1-\beta_n)\|x_n - q\| + \beta_n \|T^n x_n - q\| \\ &\le (1-\beta_n)\|x_n - q\| + \beta_n[\|x_n - q\| + G_n] \\ &= (1-\beta_n)\|x_n - q\| + \beta_n \|x_n - q\| + \beta_n G_n \\ &\le \|x_n - q\| + G_n, \end{aligned} \tag{6}$$

Again using (4), (5) and (6), we obtain

$$\begin{aligned} \|x_{n+1} - q\| &= \|(1-\alpha_n)T^n x_n + \alpha_n S^n y_n - q\| \\ &\le (1-\alpha_n)\|T^n x_n - q\| + \alpha_n \|S^n y_n - q\| \\ &\le (1-\alpha_n)[\|x_n - q\| + G_n] + \alpha_n[\|y_n - q\| + G_n] \\ &\le (1-\alpha_n)\|x_n - q\| + \alpha_n \|y_n - q\| + G_n \\ &\le (1-\alpha_n)\|x_n - q\| + \alpha_n[\|x_n - q\| + G_n] + G_n \\ &\le \|x_n - q\| + 2G_n. \end{aligned} \tag{7}$$

Since by assumption of the theorem $\sum_{n=1}^{\infty} G_n < \infty$, by Lemma 2.2 we know that the limit $\lim_{n\to\infty} \|x_n - q\|$ exists. Also from (7), we obtain

$$d(x_{n+1}, F) \leq d(x_n, F) + 2G_n, \tag{8}$$

for all $n \geq 1$. From Lemma 2 and (8), we know that $\lim_{n\to\infty} d(x_n, F)$ exists. Since $\liminf_{n\to\infty} d(x_n, F) = 0$, we have that $\lim_{n\to\infty} d(x_n, F) = 0$.

Next, we shall prove that $\{x_n\}$ is a Cauchy sequence. Therefore, for any $m, n \geq 1$ and for given $p \in F$, from (7), we have

$$\begin{aligned} \|x_{n+m} - p\| &\leq \|x_{n+m-1} - p\| + 2G_{n+m-1} \\ &\leq \|x_{n+m-2} - p\| + 2(G_{n+m-2} + G_{n+m-1}) \\ &\leq \dots \\ &\leq \|x_n - p\| + 2\sum_{k=n}^{n+m-1} G_k. \end{aligned} \tag{9}$$

Since

$$\lim_{n\to\infty} d(x_n, F) = 0, \qquad \sum_{n=1}^{\infty} G_n < \infty \tag{10}$$

for any given $\varepsilon > 0$, there exists a positive integer n_1 such that

$$d(x_n, F) < \frac{\varepsilon}{8}, \qquad \sum_{k=n}^{\infty} G_k < \frac{\varepsilon}{4} \quad \forall n \geq n_1. \tag{11}$$

Hence, there exists $q \in F$ such that

$$\|x_n - q\| < \frac{\varepsilon}{4} \quad \forall n \geq n_1. \tag{12}$$

Consequently, for any $n \geq n_1$ and $m \geq 1$, from (9), we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - q\| + \|x_n - q\| \\ &\leq \|x_n - q\| + 2\sum_{k=n}^{n+m-1} G_k + \|x_n - q\| \\ &\leq 2\|x_n - q\| + 2\sum_{k=n}^{n+m-1} G_k \\ &< 2\Big(\frac{\varepsilon}{4}\Big) + 2\Big(\frac{\varepsilon}{4}\Big) = \varepsilon. \end{aligned} \tag{13}$$

This implies that $\{x_n\}$ is a Cauchy sequence in E and so is convergent since E is complete. Let $\lim_{n\to\infty} x_n = q^*$. Then $q^* \in K$. It remains to show that $q^* \in F$. Let $\varepsilon_1 > 0$ be given. Then there exists a natural number n_2 such that $\|x_n - q^*\| < \frac{\varepsilon_1}{4}$ for all $n \geq n_2$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, there exists a natural number $n_3 \geq n_2$ such that for all $n \geq n_3$

we have $d(x_n, F) < \frac{\varepsilon_1}{5}$ and in particular we have $d(x_{n_3}, F) \leq \frac{\varepsilon_1}{5}$. Therefore, there exists $w^* \in F$ such that $\|x_{n_3} - w^*\| < \frac{\varepsilon_1}{4}$. For any $n \geq n_3$, we have

$$\begin{aligned} \|Sq^* - q^*\| &\leq \|Sq^* - w^*\| + \|w^* - q^*\| \\ &\leq 2\|q^* - w^*\| \\ &\leq 2\Big(\|q^* - x_{n_3}\| + \|x_{n_3} - w^*\|\Big) \\ &< 2\Big(\frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{4}\Big) \\ &< \varepsilon_1. \end{aligned}$$

This implies that $Sq^* = q^*$ and hence $q^* \in F(S)$. Similarly, we can prove that $q^* \in F(T)$. Hence $q^* \in F = F(S) \cap F(T)$. This shows that q^* is a common fixed point of the mappings S and T. Thus $\{x_n\}$ converges strongly to a common fixed point of the mappings S and T. This completes the proof.

Theorem 2 *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $S, T\colon K \to K$ be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings in the intermediate sense such that $F = F(S) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. From arbitrary $x_1 \in K$, let $\{x_n\}$ be the sequence defined by (4). Put*

$$\begin{aligned} G_n &= \max\Big\{\sup_{x\in K,\ q\in F}\Big(\|T^n x - q\| - \|x - q\|\Big) \vee 0, \\ &\qquad \sup_{y\in K,\ q\in F}\Big(\|S^n y - q\| - \|y - q\|\Big) \vee 0,\ \forall n \geq 1\Big\} \end{aligned}$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. Then $\lim_{n\to\infty}\|x_n - Sx_n\| = \lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Proof By Theorem 1, $\lim_{n\to\infty}\|x_n - q\|$ exists for all $q \in F$. Assume that $\lim_{n\to\infty}\|x_n - q\| = r$. If $r = 0$, the conclusion is obvious. Now suppose $r > 0$. We claim $\lim_{n\to\infty}\|x_n - Sx_n\| = \lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Using (4) and Lemma 3, we have

$$\begin{aligned} \|y_n - q\|^2 &= \|(1-\beta_n)T^n x_n + \beta_n x_n - q\|^2 \\ &\leq (1-\beta_n)\|T^n x_n - q\|^2 + \beta_n\|x_n - q\|^2 \\ &\quad -W_2(\beta_n)g(\|T^n x_n - x_n\|) \\ &\leq (1-\beta_n)\|T^n x_n - q\|^2 + \beta_n\|x_n - q\|^2 \\ &\leq (1-\beta_n)[\|x_n - q\| + G_n]^2 + \beta_n\|x_n - q\|^2 \\ &\leq (1-\beta_n)[\|x_n - q\|^2 + \theta_n] + \beta_n\|x_n - q\|^2 \\ &\leq \|x_n - q\|^2 + \theta_n, \end{aligned} \tag{14}$$

where $\theta_n = 2\|x_n - q\| G_n + G_n^2$ with $\sum_{n=1}^{\infty}\theta_n < \infty$. Again using (4), (14) and Lemma 3, we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|(1-\alpha_n)T^nx_n+\alpha_nS^ny_n-q\|^2 \\
&\leq (1-\alpha_n)\|T^nx_n-q\|^2+\alpha_n\|S^ny_n-q\|^2 \\
&\quad -W_2(\alpha_n)g(\|T^nx_n-S^ny_n\|) \\
&\leq (1-\alpha_n)[\|x_n-q\|+G_n]^2+\alpha_n[\|y_n-q\|+G_n]^2 \\
&\quad -W_2(\alpha_n)g(\|T^nx_n-S^ny_n\|) \\
&\leq (1-\alpha_n)[\|x_n-q\|^2+\theta_n]+\alpha_n[\|y_n-q\|^2+\lambda_n] \\
&\quad -W_2(\alpha_n)g(\|T^nx_n-S^ny_n\|) \\
&\leq (1-\alpha_n)[\|x_n-q\|^2+\theta_n]+\alpha_n[\|x_n-q\|^2+\theta_n] \\
&\quad +\alpha_n\lambda_n-W_2(\alpha_n)g(\|T^nx_n-S^ny_n\|) \\
&\leq \|x_n-q\|^2+\theta_n+\lambda_n \\
&\quad -W_2(\alpha_n)g(\|T^nx_n-S^ny_n\|) \qquad (15)
\end{aligned}$$

where $\lambda_n = 2\|y_n-q\|G_n+G_n^2$ with $\sum_{n=1}^{\infty}\lambda_n<\infty$.

Observe that $W_2(\alpha_n)\geq\delta^2$ and $\sum_{n=1}^{\infty}\beta_n<\infty$. Now (15) implies that

$$\begin{aligned}
\delta^2\sum_{n=1}^{\infty}g(\|T^nx_n-S^ny_n\|) &< \|x_1-q\|^2+\sum_{n=1}^{\infty}\theta_n \\
&\quad +\sum_{n=1}^{\infty}\lambda_n<\infty. \qquad (16)
\end{aligned}$$

Therefore, we have $\lim_{n\to\infty}g(\|T^nx_n-S^ny_n\|)=0$. Since g is strictly increasing and continuous at 0, it follows that

$$\lim_{n\to\infty}\|T^nx_n-S^ny_n\|=0. \qquad (17)$$

Now taking limsup on both the sides of (6), we obtain

$$\limsup_{n\to\infty}\|y_n-q\| \leq r. \qquad (18)$$

Since T is asymptotically quasi-nonexpansive mapping in the intermediate sense, we can get that

$$\|T^nx_n-q\| \leq \|x_n-q\|+G_n. \qquad (19)$$

for all $n\geq 1$. Taking limsup on both the sides of (19), we obtain

$$\limsup_{n\to\infty}\|T^nx_n-q\| \leq r. \qquad (20)$$

Now

$$\begin{aligned}
\|x_{n+1}-q\| &= \|(1-\alpha_n)T^nx_n+\alpha_nS^ny_n-q\| \\
&= \|(T^nx_n-q)+\alpha_n(S^ny_n-T^nx_n)\| \\
&\leq \|T^nx_n-q\|+\alpha_n\|S^ny_n-T^nx_n\|
\end{aligned}$$

yields that

$$r \leq \liminf_{n\to\infty} \|T^n x_n - q\|. \tag{21}$$

So that (20) gives $\lim_{n\to\infty} \|T^n x_n - q\| = r$.

On the other hand, since S is asymptotically quasi-nonexpansive mapping in the intermediate sense, we have

$$\begin{aligned}\|T^n x_n - q\| &\leq \|T^n x_n - S^n y_n\| + \|S^n y_n - q\| \\ &\leq \|T^n x_n - S^n y_n\| + \|y_n - q\| + G_n,\end{aligned}$$

so we have

$$r \leq \liminf_{n\to\infty} \|y_n - q\|. \tag{22}$$

By using (18) and (22), we obtain

$$\lim_{n\to\infty} \|y_n - q\| = r. \tag{23}$$

Thus $r = \lim_{n\to\infty} \|y_n - q\| = \lim_{n\to\infty} \|(1-\beta_n)(x_n - q) + \beta_n(T^n x_n - q)\|$ gives by Lemma 1 that

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{24}$$

Now

$$\|y_n - x_n\| = \beta_n \|T^n x_n - x_n\|.$$

Hence by (24), we obtain

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{25}$$

Also note that

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|(1-\alpha_n)T^n x_n + \alpha_n S^n y_n - x_n\| \\ &\leq \|T^n x_n - x_n\| + \alpha_n \|T^n x_n - S^n y_n\| \\ &\to 0 \text{ as } n \to \infty, \end{aligned} \tag{26}$$

so that

$$\begin{aligned}\|x_{n+1} - y_n\| &\leq \|x_{n+1} - x_n\| + \|y_n - x_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{27}$$

Furthermore, from

$$\begin{aligned}\|x_{n+1} - S^n y_n\| &\leq \|x_{n+1} - x_n\| + \|x_n - T^n x_n\| \\ &\quad + \|T^n x_n - S^n y_n\|\end{aligned}$$

using (17), (24) and (26), we find that

$$\lim_{n\to\infty} \|x_{n+1} - S^n y_n\| = 0. \tag{28}$$

Then

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &\leq \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| + \left\|T^{n+1}x_{n+1} - T^{n+1}x_n\right\| \\
&\quad + \left\|T^{n+1}x_n - Tx_{n+1}\right\| \\
&\leq \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| + L\left\|x_{n+1} - x_n\right\| \\
&\quad + L\left\|T^n x_n - x_{n+1}\right\| \\
&= \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| + L\left\|x_{n+1} - x_n\right\| \\
&\quad + L\alpha_n\left\|T^n x_n - S^n y_n\right\|
\end{aligned}$$

yields

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{29}$$

Now

$$\begin{aligned}
\|x_n - S^n x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - S^n y_n\| \\
&\quad + \|S^n y_n - S^n x_n\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - S^n y_n\| \\
&\quad + L\|y_n - x_n\| \\
&\quad \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus

$$\begin{aligned}
\|x_{n+1} - Sx_{n+1}\| &\leq \left\|x_{n+1} - S^{n+1}x_{n+1}\right\| + \left\|S^{n+1}x_{n+1} - Sx_{n+1}\right\| \\
&\leq \left\|x_{n+1} - S^{n+1}x_{n+1}\right\| + L\left\|S^n x_{n+1} - x_{n+1}\right\| \\
&\leq \left\|x_{n+1} - S^{n+1}x_{n+1}\right\| + L\Big(\left\|S^n x_{n+1} - S^n y_n\right\| \\
&\quad + \left\|S^n y_n - x_{n+1}\right\|\Big) \\
&\leq \left\|x_{n+1} - S^{n+1}x_{n+1}\right\| + L^2\left\|x_{n+1} - y_n\right\| \\
&\quad + L\left\|S^n y_n - x_{n+1}\right\|
\end{aligned}$$

implies

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{30}$$

This completes the proof.

Theorem 3 *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $S, T\colon K \to K$ be two uniformly L-Lipschitzian asymptotically* *quasi-nonexpansive mappings in the intermediate sense such that* $F = F(S) \cap F(T) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $[\delta, 1-\delta]$ *for some* $\delta \in (0,1)$. *From arbitrary* $x_1 \in K$, *let* $\{x_n\}$ *be the sequence defined by* (4). *Put*

$$G_n = \max\Big\{ \sup_{x\in K,\ q\in F} \Big(\|T^n x - q\| - \|x - q\| \Big) \vee 0, \sup_{y\in K,\ q\in F} \Big(\|S^n y - q\| - \|y - q\| \Big) \vee 0,\ \forall n \geq 1 \Big\}$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. *If at least one of the mappings* S *and* T *is semi-compact, then the sequence* $\{x_n\}$ *converges strongly to a common fixed point of* S *and* T.

Proof Without loss of generality, we may assume that T is semi-compact. This with (29) means that there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\} \to x^* \in K$. Since S and T are continuous, then from (29) and (30), we find

$$\|x^* - Tx^*\| = \lim_{n_k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0$$

and

$$\|x^* - Sx^*\| = \lim_{n_k\to\infty} \|x_{n_k} - Sx_{n_k}\| = 0.$$

This shows that $x^* \in F = F(S) \cap F(T)$. According to Theorem 1 the limit $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Then

$$\lim_{n\to\infty} \|x_n - x^*\| = \lim_{n_k\to\infty} \|x_{n_k} - x^*\| = 0,$$

which means that $\{x_n\}$ converges to $x^* \in F$. Thus the sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings S and T. This completes the proof.

Applying Theorem 1, we obtain strong convergence of the process (4) under the condition (A') as follows:

Theorem 4 *Let* E *be a real uniformly convex Banach space and* K *be a nonempty closed convex subset of* E. *Let* $S, T\colon K \to K$ *be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings in the intermediate sense such that* $F = F(S) \cap F(T) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $[\delta, 1-\delta]$ *for some* $\delta \in (0,1)$. *From arbitrary* $x_1 \in K$, *let* $\{x_n\}$ *be the sequence defined by* (4). *Put*

$$G_n = \max\Big\{ \sup_{x\in K,\ q\in F} \Big(\|T^n x - q\| - \|x - q\| \Big) \vee 0, \sup_{y\in K,\ q\in F} \Big(\|S^n y - q\| - \|y - q\| \Big) \vee 0,\ \forall n \geq 1 \Big\}$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. *Let* S *and* T *satisfy the condition* (A'), *then the sequence* $\{x_n\}$ *converges strongly to a common fixed point of* S *and* T.

Proof We proved in Theorem 2 that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0, \ \lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{31}$$

From the condition (A') and (31), either

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - Sx_n\| = 0,$$

or

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Hence

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

Since $f\colon [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$, therefore we have

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

It follows, as in the proof of Theorem 1, that $\{x_n\}$ converges strongly to a common fixed point of the mappings S, T. This completes the proof.

4 Weak Convergence Theorem

In this section, we prove a weak convergence theorem of the iteration process (4) in the framework of real uniformly convex Banach spaces.

Lemma 5 Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $S, T\colon K \to K$ be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings in the intermediate sense such that $F = F(S) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. From arbitrary $x_1 \in K$, let $\{x_n\}$ be the sequence defined by (4). Put

$$\begin{aligned} G_n &= \max\Big\{ \sup_{x\in K,\ q\in F} \Big(\|T^n x - q\| - \|x - q\| \Big) \vee 0, \\ &\qquad \sup_{y\in K,\ q\in F} \Big(\|S^n y - q\| - \|y - q\| \Big) \vee 0,\ \forall n \geq 1 \Big\} \end{aligned}$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. Then $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ exists for all p, $q \in F$ and $t \in [0,1]$.

Proof By Theorem 1, we know that $\{x_n\}$ is bounded. Letting

$$a_n(t) = \|tx_n + (1-t)p - q\|$$

for all $t \in [0,1]$. Then $\lim_{n\to\infty} a_n(0) = \|p - q\|$ and $\lim_{n\to\infty} a_n(1) = \|x_n - q\|$ exists by Theorem 1. It, therefore, remains to prove the Lemma 5 for $t \in (0,1)$. For all $x \in K$, we define the mapping $W_n\colon K \to K$ by

$$W_n x = (1-\alpha_n)T^n x + \alpha_n S^n((1-\beta_n)x + \beta_n T^n x).$$

Then

$$\|W_n x - W_n y\| \quad \leq \quad \|x - y\| + 2G_n, \tag{32}$$

for all $x, y \in K$. Setting

$$S_{n,\,m} = W_{n+m-1} W_{n+m-2} \dots W_n, \quad m \geq 1 \tag{33}$$

and

$$b_{n,\,m} \quad = \quad \|S_{n,\,m}(tx_n + (1-t)p) - (tS_{n,\,m}x_n + (1-t)S_{n,m}q)\|\,. \tag{34}$$

From (32) and (33), we have

$$\|S_{n,\,m}x - S_{n,\,m}y\| \quad \leq \quad \|x - y\| + 2\sum_{i=n}^{n+m-1} G_i \tag{35}$$

for all $x, y \in K$, where $S_{n,\,m}x_n = x_{n+m}$ and $S_{n,\,m}p = p$ for all $p \in F$. Thus

$$\begin{aligned} a_{n+m}(t) &= \|tx_{n+m} + (1-t)p - q\| \\ &\leq b_{n,\,m} + \|S_{n,\,m}(tx_n + (1-t)p) - q\| \\ &\leq b_{n,\,m} + a_n(t) + 2\sum_{i=n}^{n+m-1} G_i. \end{aligned} \tag{36}$$

By using Theorem 2.3 [4], we have

$$\begin{aligned} b_{n,\,m} &\leq \phi^{-1}(\|x_n - p\| - \|S_{n,\,m}x_n - S_{n,\,m}p\|) \\ &\leq \phi^{-1}(\|x_n - p\| - \|x_{n+m} - p + p - S_{n,\,m}p\|) \\ &\leq \phi^{-1}\Big(\|x_n - p\| - (\|x_{n+m} - p\| - \|S_{n,\,m}p - p\|)\Big), \end{aligned} \tag{37}$$

and so the sequence $\{b_{n,\,m}\}$ converges to 0 as $n \to \infty$ for all $m \geq 1$. Thus, fixing n and letting $m \to \infty$ in (37), we have

$$\begin{aligned} \limsup_{m\to\infty} a_{n+m}(t) \quad &\leq \quad \phi^{-1}\Big(\|x_n - p\| - \Big(\lim_{m\to\infty}\|x_m - p\| - \|S_{n,\,m}p - p\|\Big)\Big) \\ &\quad + a_n(t) + 2\sum_{i=n}^{n+m-1} G_i, \end{aligned} \tag{38}$$

and again letting $n \to \infty$, we obtain

$$\limsup_{n\to\infty} a_n(t) \quad \leq \quad \phi^{-1}(0) + \liminf_{n\to\infty} a_n(t) + 0 = \liminf_{n\to\infty} a_n(t).$$

This shows that $\lim_{n\to\infty} a_n(t)$ exists, that is,

$$\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$$

exists for all $t \in [0, 1]$. This completes the proof.

Theorem 5 *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec-Klee property and K be a nonempty closed convex subset of E. Let $S, T\colon K \to K$ be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings in the intermediate sense such that $F = F(S) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. From arbitrary $x_1 \in K$, let $\{x_n\}$ be the sequence defined by (4). Put*

$$\begin{aligned} G_n &= \max\Big\{ \sup_{x\in K,\ q\in F} \Big(\|T^n x - q\| - \|x-q\| \Big) \vee 0, \\ & \sup_{y\in K,\ q\in F} \Big(\|S^n y - q\| - \|y-q\| \Big) \vee 0,\ \forall n \geq 1 \Big\} \end{aligned}$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. If the mappings $I - S$ and $I - T$, where I denotes the identity mapping, are demiclosed at zero. Then $\{x_n\}$ converges weakly to a common fixed point of the mappings S and T.

Proof By Theorem 1, we know that $\{x_n\}$ is bounded and since E is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $p \in K$. By Theorem 2, we have

$$\lim_{n\to\infty} \left\|x_{n_j} - Sx_{n_j}\right\| = 0,\ \ \lim_{n\to\infty} \left\|x_{n_j} - Tx_{n_j}\right\| = 0.$$

Since the mappings $I - S$ and $I - T$ are demiclosed at zero, therefore $Sp = p$ and $Tp = p$, which means $p \in F$. Now, we show that $\{x_n\}$ converges weakly to p. Suppose $\{x_{n_i}\}$ is another subsequence of $\{x_n\}$ converges weakly to some $q \in K$. By the same method as above, we have $q \in F$ and $p, q \in w_w(x_n)$. By Lemma 5, the limit

$$\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$$

exists for all $t \in [0,1]$ and so $p = q$ by Lemma 4. Thus, the sequence $\{x_n\}$ converges weakly to $p \in F$. This completes the proof.

Remark 2 Theorems of this paper can also be proved with error terms.

Remark 3 Our results extend, improve and generalize many known results given in the existing literature.

5 Conclusion

The results proved in this paper of the iteration scheme (4) for more general class of quasi-nonexpansive, asymptotically nonexpansive and asymptotically quasi-nonexpansive mappings not only covers the results proved by iteration scheme (2) but also by (1) which are not covered by (2). Thus our results are good improvement and extension of some corresponding previous results from the existing literature.

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