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A Comparative Study on Cubic Bezier and Beta-Spline Curves

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Abstract Most researchers today prefer to use Bezier curve in their works rather than Beta-spline curve. There are several reasons and one of them is that Bezier curve is faster to plot than Beta-spline. However Beta-spline also has several advantages and one of it is this curve is built on G^2 continuity condition. This property makes Beta-spline achieves the required smoothness faster than Bezier curve. In this paper, we introduce the method of Beta-spline control points evaluation and make comparison between Bezier and Beta-spline curves in terms of their continuity, and circle approximation.

Keywords Bezier, Beta-spline, Circle approximation, Control points evaluation

2010 Mathematics Subject Classification 65D07

1 Introduction

Bezier curves and surfaces are commonly discussed by most researchers [1-8]. However, only a few used Beta-spline in their research. Beta-spline was introduced in 1981 which is an extension of B-spline [9]. The main advantage of this curve is it is built on G^2 continuity. This is discussed in detail by Rovenski [10] and Liao and Huang [11]. Another advantage is it has shape parameters: bias and tension that can control the shape of the curve locally. Liu [12] explained theoretically and Hadi *et al.* [13-19] showed graphically the way each control point controls the curve.

There are two types of Bezier and Beta-spline: rational and non-rational. This paper discusses non-rational cubic Bezier and Beta-splines. The flow of this paper is given as follows: Section 2 gives the elementary and continuity of Bezier and Beta-spline curves. Section 3 shows the control points evaluation of Bezier and Beta-spline curves. Section 4 discusses the approximation of a circle using Bezier and Beta-spline curves. We end this paper with a conclusion in Section 5 which is the conclusion.

2 Elementary Bezier and Beta-spline Curve

Basically a cubic Bezier curve is defined as

$$P\left(t\right) = B_0\left(t\right)v_0 + B_1\left(t\right)v_1 + B_2\left(t\right)v_2 + B_3\left(t\right)v_3, \tag{1}$$

where

$B_i\left(t\right)$ is a Bernstein's polynomial, $i = 1, 2, 3$
v_i is the control point, $i = 1, 2, 3.$

This equation can also be written in matrix form as,

$$P\left(t\right)=\left[T\right]\left[M\right]\left[V\right] \tag{2}$$

where,

$$\left[T\right]=\left[t^3 t^2 t\ 1\right] \qquad\qquad 0<t<1, \tag{3}$$

$$\left[M\right]=\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

is the basic Bezier Matrix, and

$$\left[V\right]=\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{5}$$

is the matrix of control points.

The Beta-spline equation can also be expressed as in equation (2) where the matrix for Beta-spline denoted as $[M_\beta]$ is,

$$[M_\beta]=\frac{1}{\delta}\begin{bmatrix} -2\beta_1^2 & 2(\beta_2+\beta_1^3+\beta_1^2+\beta_1) & -2(\beta_2+\beta_1^2+\beta_1+1) & 2 \\ 6\beta_1^3 & -3(\beta_2+2\beta_1^3+2\beta_1) & 3(\beta_2+2\beta_1^2) & 0 \\ -6\beta_1^3 & 6(\beta_1^3-\beta_1) & 6\beta_1 & 0 \\ 2\beta_1^3 & \beta_2+4(\beta_1^2+\beta_1) & 2 & 0 \end{bmatrix}$$

where

$$\delta=\beta_2+2\beta_1^3+4\beta_1^2+4\beta_1+2$$

Constants β_1 and β_2 are derived from G^2 continuity properties. The derivation of Bezier and Beta-spline equations are detailed by Rovenski [10].

Consider two curves, $P_1(t)$ and $P_2(t)$ connected together, the G^2 continuity properties are,

$$P_2\left(0\right)=P_1\left(1\right) \tag{6}$$

$$P_2'\left(0\right)=\beta_1 P_1'\left(1\right) \tag{7}$$

$$P_2''\left(0\right)=\beta_1^2 P_1''\left(1\right)+\beta_2 P_1'\left(1\right) \tag{8}$$

Figure 1 and Figure 2 below show the cubic Bezier and cubic Beta-spline curves respectively, with the same control polygon. The position of both curves was compared in Figure 3. It is clear that the position of cubic Beta-spline curve is closer to the control polygon segment than Bezier, but it does not connect the endpoints of the control polygon.

To make the Beta-spline curves touch the endpoints, the following condition must be fulfilled:

When

$$\beta_1=1,\ \text{ then }\ \beta_2=0. \tag{9}$$

At $t=0$, the Beta-spline equation is

$$P\left(0\right)=\frac{1}{6}v_0+\frac{2}{3}v_1+\frac{1}{6}v_2=v_0 \tag{10}$$

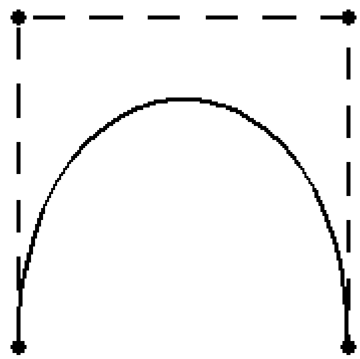

Figure 1: Cubic Bezier Curve with Control Polygon

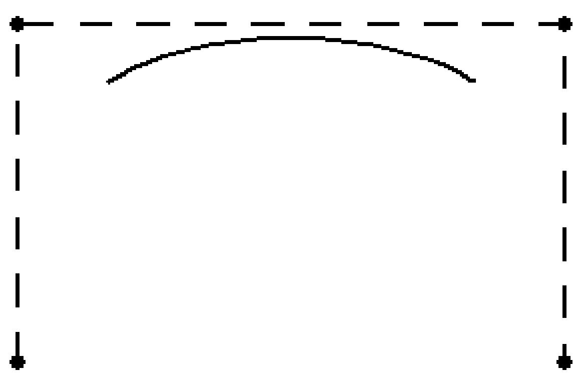

Figure 2: Cubic Beta-spline Curve with Control Polygon

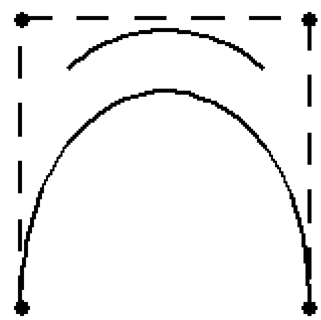

Figure 3: Cubic Bezier and Beta-Spline Curve with Control Polygon

This produces

$$4v_1 + v_2 = 5v_0 \tag{11}$$

Equation (11) is satisfied if

$$v_1 \text{ and } v_2 = v_0. \tag{12}$$

This condition makes the matrix $[V]$ be defined as the first Beta-spline curve as follows:

$$[V] = \begin{bmatrix} v_0 & v_0 & v_0 & v_1 \end{bmatrix} \tag{13}$$

The same process goes on defining the last Beta-spline curve at $t = 1$.

$$[V] = \begin{bmatrix} v_2 & v_3 & v_3 & v_3 \end{bmatrix} \tag{14}$$

Equations (13) and (14) need the user to fit five Beta-spline curves in the same control polygon to make it touches both endpoints which is v_0 and v_3.

Figure 4 shows a cubic Bezier curve and the combination of five cubic Beta-spline curves. Beta-spline curves approximate the control polygon better shape than Bezier curves.

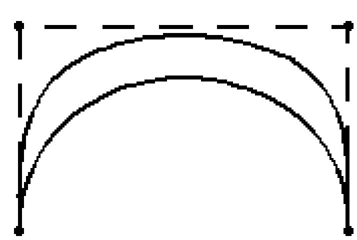

Figure 4: A Cubic Bezier and Five Beta-Spline Curves with Control Polygon

2.1 The Continuity of Bezier and Beta-spline Curves

Cubic degree curves can achieve as high as C^2 and G^2 continuity. Cubic Beta-spline always achieve G^2 continuity and when properties (9) are fulfilled, it will achieve C^2 continuity. Bezier curve continuity depends on the way the curves are connected.

The properties for C^n continuity is,

$$\frac{dP_{i+1}^{(n)}(t)}{dt^n} = \frac{dP_i^{(n)}(t)}{dt^n} \tag{15}$$

Figure 5a to Figure 7c show the comparison of Bezier and Beta-spline with the three types of continuity. Figure#b and #c are sharing the same control polygon #a. We can see that Bezier curves are acceptable for C^2 and C^1 continuity. But as the continuity decreases to C^0, the Bezier curves are not smooth while the Beta-spline curves maintain with C^2 continuity without any kink for all three types of continuity. This result explains the continuity of Bezier curve is influenced by the way of its control polygons are connected, while Beta-spline curve can preserve its continuity to degree two.

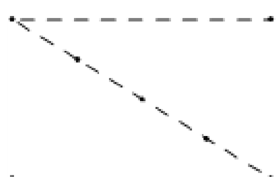

Figure 5a: C^2 Control Polygon

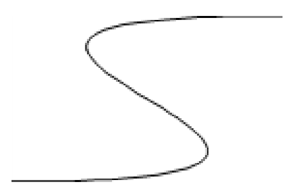

Figure 5b: Cubic Bezier Curves with C^2 Continuity

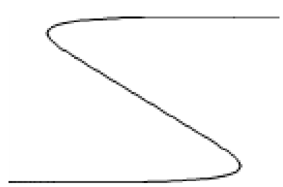

Figure 5c: Cubic Beta-spline Curves with C^2 Continuity

Figure 6a: C^1 Control Polygon

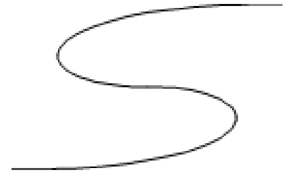

Figure 6b: Cubic Bezier Curves with C^1 Continuity

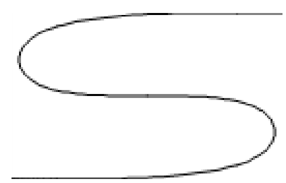

Figure 6c: Cubic Beta-spline Curves with C^2 Continuity

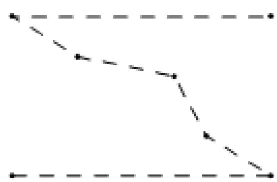

Figure 7a: C^0 Control Polygon

Figure 7b: Cubic Bezier Curves with C^0 Continuity

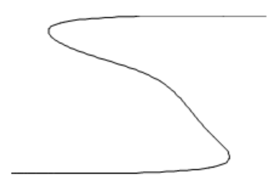

Figure 7c: Cubic Beta-spline Curves with C^2 Continuity

3 Bezier and Beta-spline Control Points Evaluation

This paper uses the common method in evaluating cubic Bezier control points. Some literatures those used this method are Yahya *et al.* [2], Sarfraz & Razzak [7] and Shao & Zhou [8].

The method is by minimizing the distance, S between the contour and the curve points. Let $P(t)$ as the fitted curve as in equation (1) and p_i as the contour points.

$$S = \sum_{i=1}^{m} \left[P\left(t_i\right) - p_i\right]^2 \tag{16}$$

For Beta-spline, the user can still use equation (16) but with two changes.

First, by replacing Bernstein's polynomial, $B_i\,(t)$ as:

$$\begin{aligned}
B_0\,(t) &= \frac{1}{\delta}(2\beta_1 - 6t\beta_1^3 + 6t^2\beta_1^3 - 2t^3\beta_1^3) \\
B_1\,(t) &= \frac{1}{\delta}\Big(4\left(\beta_1 + \beta_1^2\right) + 6t\left(-\beta_1 + \beta_1^3\right) + \beta_2 + 2t^3\left(\beta_1 + \beta_1^2 + \beta_1^3 + \beta_2\right) \\
&\qquad\qquad - 3t^2\left(2\beta_1^2 + 2\beta_1^3 + \beta_2\right)\Big) \qquad (17) \\
B_2\,(t) &= \frac{1}{\delta}\Big(2 + 6t\beta_1 - 2t^3\left(1 + \beta_1 + \beta_1^2 + \beta_2\right) + 3t^2(2\beta_1 + \beta_2)\Big) \\
B_3\,(t) &= \frac{1}{\delta}(2t^3)
\end{aligned}$$

Second, by substituting the following equation given by Liu *et al.* [11].

When $\beta_1 = 1, \beta_2 = 0$

$$P\,(0) = \frac{1}{6}\,(v_0 + 4v_1 + v_2) \tag{18}$$

$$P\,(1) = \frac{1}{6}\,(v_1 + 4v_2 + v_3) \tag{19}$$

Equations (18) and (19) are about the connection between the endpoints of Beta-spline curve and its first and last control points.

If the user has the Bezier control points, the points can be easily be converted to Beta-splines control points.

Referring to equation (2), the Beta-spline control points can be generated as follows:

$$[T]\,[M_\beta]\,[V_\beta] = [T]\,[M]\,[V] \tag{20}$$

The control points matrix of Beta-spline is

$$[V_\beta] = [M_\beta]^{-1}\,[M]\,[V] \tag{21}$$

Figure 8 shows a Beta-spline curve which is converted from Bezier curve in Figure 1. The comparison between these two control polygons is shown in Figure 9. To approximate the same curve, Beta-spline gives a bigger control polygon than Bezier.

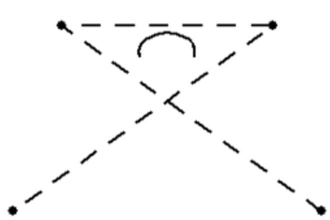

Figure 8: A Beta-spline Curve with Control Polygon

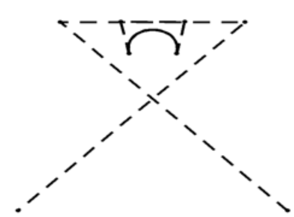

Figure 9: A Bezier and Beta-spline Curve with Control Polygons

4 Circle Approximation using Bezier and Beta-spline Curves

There are a few papers work on circle approximation (see Ahn, Y. J. *et al.* [20] and Piegl L. A. & Tiller, W [21]). This paper discuss the method done by Riskus, A. [22] which discusses a circle of four cubic Bezier curves.

$$v_0 = (r,0)(0,0)v_1 = (rk)v_2 = (kr)v_3 = (0,r)$$

Figure 10 shows a quarter circle for cubic Bezier curve with radius r and control points v_i $(i = 0,1,2,3)$. The value of k must satisfy the equation,

$$k = \frac{8r}{3\sqrt{2}} - \frac{4r}{3} \tag{22}$$

Four quarter circles are used to generate a full circle. Figure 11 shows a full circle consists four cubic Bezier curves.

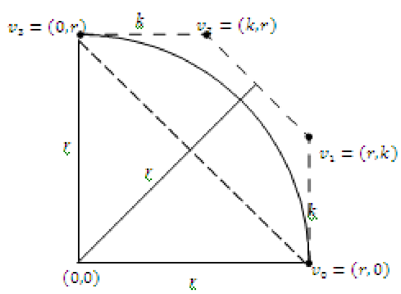

Figure 10: A Bezier Control Polygon for a Quarter Circle

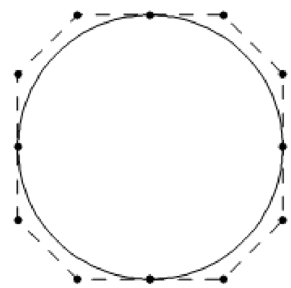

Figure 11: A Circle with Four Cubic Bezier Curves

For Beta-spline, the user cannot use the same control points as Bezier to generate a circle. The reason is clearly demonstrated in figure 4 before, which show the different position of Bezier and Beta-spline curves in the same control polygon. In this case, the user can simply convert the Bezier control points to Beta-spline control points.

Figure 12 shows a Beta-spline control polygon for a quarter circle. This control polygon generated from Bezier control polygon (bolded) as shown in Figure 10 using equation (21). The circle generated in Figure 13 is exactly the same as the circle in Figure 11. Compared to Figure 11 and Figure 13, it is clear that the control polygon of Beta-spline is bigger and messier than Bezier.

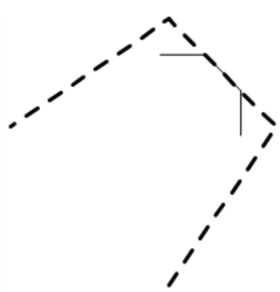

Figure 12: A Beta-spline Control Polygon for a Quarter Circle

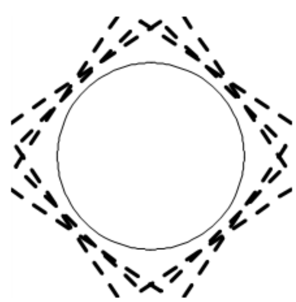

Figure 13: A Full Circle with Four Beta-spline Curves

5 Conclusion

In this paper, cubic Bezier and Beta-spline have been compared. Though Beta-spline seems quite expensive to generate, it provides appropriate smoothness which is very important in computer aided design. This paper also proposes a simple method which may help the user to generate the control points of Beta-spline. We believe that with further research, Beta-spline will become easier to handle.

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