MATEMATIKA, 2013, Volume 29, Number 1, 65–71

(k,d)-Balanced of Uniform k-Distant Trees

M. Murugan
School of Science, Tamil Nadu Open University
577, Anna Salai, Saidapet – 600 015, Chennai, India
e-mail: muruganganesan @ yahoo.in

Abstract A k-distant tree consists of a main path called the "spine", such that each vertex on the spine is joined by an edge to at most one path on k-vertices. Those paths are called "tails" (i.e. each tail must be incident with a vertex on the spine). When every vertex on the spine has exactly one incident tail of length k, we call the tree a uniform k-distant tree. In this paper, we show that all uniform k-distant trees are graceful, skolem graceful, odd graceful, k-graceful, (k,d)-graceful and (k,d)-balanced.

Keywords Uniform k-distant tree, skolem-graceful, odd graceful, k-graceful, (k,d)-graceful, (k,d)-balanced.

2010 Mathematics Subject Classification 05C78.

1 Introduction

Labeled graphs are very useful models for a broad range of applications. α-labelings are useful in graph decomposition problems. Graceful labelings arose in the characterization of finite neofields and in the study of perfect systems of difference sets.

We can see connection between graceful labeling and a variety of algebraic structures including cyclic difference sets, sequenceable groups, generalized complete mappings and near-complete mappings. Graph labelings have several applications in X-ray crystallography, radar, communication networks, design theory, astronomy, computer science and combinatorics.

Rosa [1], Graham and Sloane [2] introduced many labeling methods and S.W. Golomb [3] was instrumental in coining the phraseology "Graceful Graphs". The famous Graceful Tree Conjecture (also known as Ringel − Kotzig, or Rosa's or even Ringel − Kotzig − Rosa conjecture) which says that all trees have a graceful labeling was first mentioned in [1]. Yao *et al.* [4] have conjectured that every tree is *(k,d)-graceful* for some $k > 1$ and $d > 1$. Hegde [5] has conjectured that all trees are *(k,d)-balanced* for some values of k and d. A caterpillar is a tree with the property that the removal of its endpoints leaves a path. A lobster is a tree with the property that the removal of the endpoints leaves a caterpillar. Bermond [6] conjectured that lobsters are graceful and this is still open. The harmoniousness of lobsters is also still an open problem. It is clear that uniform 2-distant trees are special lobsters. Atif Abueida and Dan Roberts [7] have proved that uniform *k-distant trees* admit a harmonious labeling, when they have even number of vertices.

2 k-Distant Tree

A *k-distant tree* consists of a main path called the "spine", such that each vertex on the spine is joined by an edge to at most one path on *k-vertices*. Those paths are called "tails" (i.e. each tail must be incident with a vertex on the spine). When every vertex on the spine has exactly one incident tail of length k, we call the tree a *uniform k-distant tree*.

In this paper, we show that every *uniform k-distant tree* admits a graceful, skolem graceful, odd graceful, *k-graceful, (k,d)-graceful* and *(k,d)-balanced labeling.* To prove this, we name the vertices of any *uniform k-distant tree* as in Figure 2 with the help of Figure 1. The arrows on the Figure 1 show the order of naming the vertices.

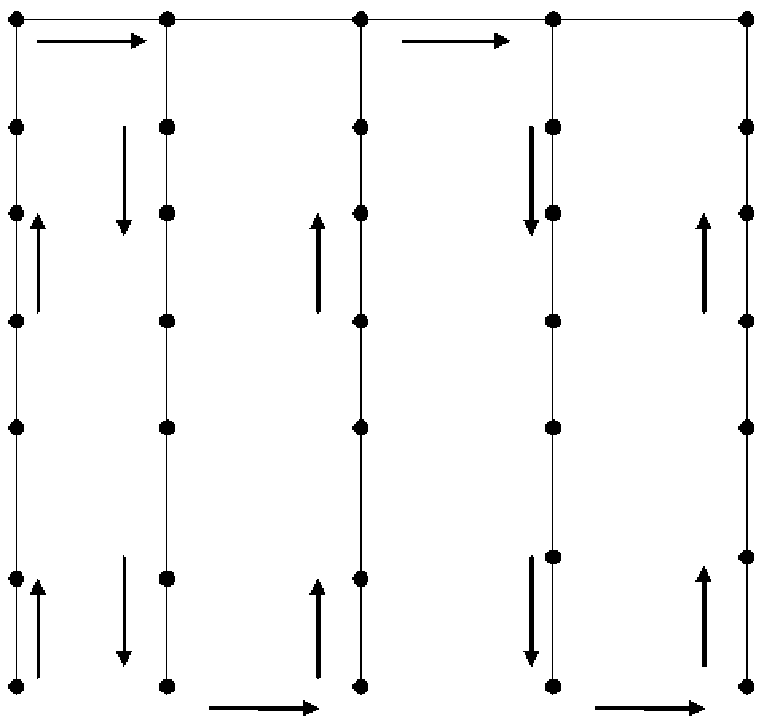

Figure 1: The Order of Naming the Vertices

3 Labelings

A graph G with q edges is graceful if there is a labeling f from the vertices of G to $\{0, 1, 2, \ldots, q\}$ such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is $\{1, 2, \ldots, q\}$.

A graph G with q edges is *k-graceful* if there is a labeling $f$from the vertices of G to $\{0, 1, 2, \ldots, q + k - 1\}$ such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is $\{k, k + 1, \ldots, q + k - 1\}$.

A graph G with $p$vertices and q edges is *Skolem-graceful* if there is a labeling f from the vertices of G to $\{1, 2, \ldots, p\}$ such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is $\{1, 2, \ldots, q\}$.

A graph G with q edges is odd graceful if there is a labeling f from the vertices of G to $\{0, 1, 2, \ldots, 2q - 1\}$ such that the set of edge labels induced by the absolute value of the

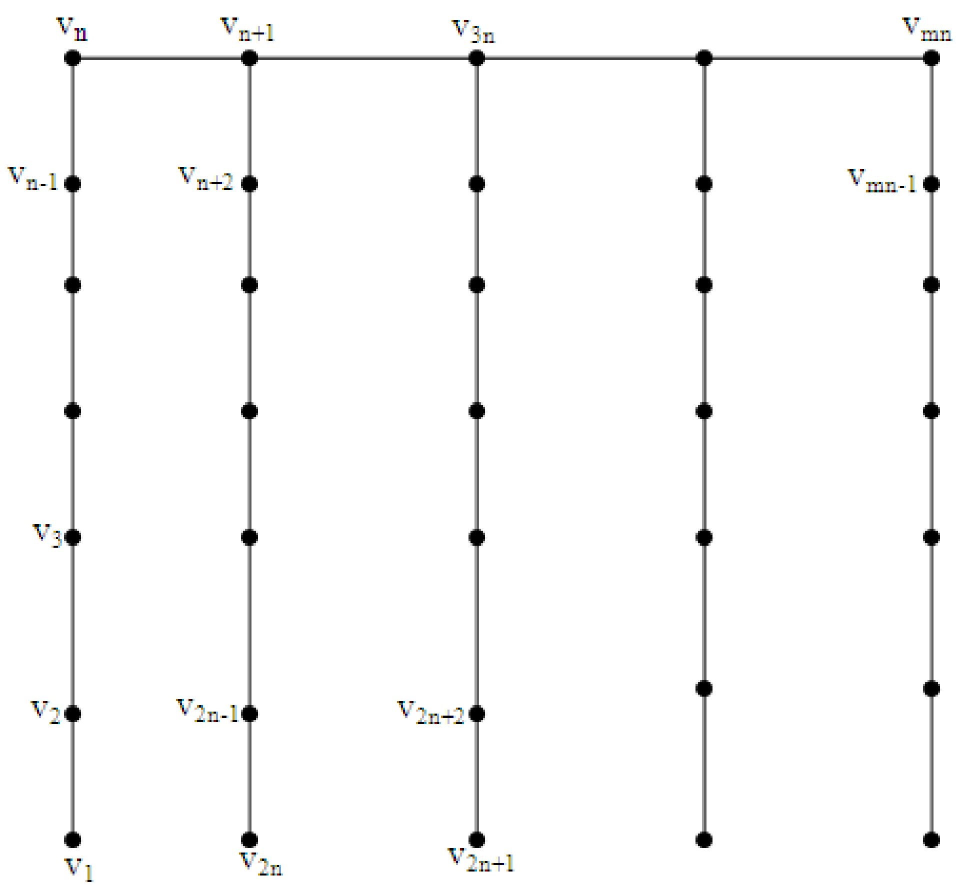

Figure 2: Uniform k-distant Tree

difference of the labels of adjacent vertices is $\{1, 3, 5, \ldots, 2q-1\}$.

A graph G with q edges is *(k,d) -graceful* if there is a labeling f from the vertices of G to $\{0, 1, 2, \ldots, k+(q-1)d\}$ such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is $\{k, k+d, k+2d, \ldots, k+(q-1)d\}$.

A *(k,d)-graceful* graph is called a *(k,d)-balanced graph* if it has a *(k,d)-graceful labeling* f with the property that there is some integer m such that for every edge uv either *f(u)* $\leq$ *m and f(v) >m, or f(u) >m and f(v)* $\leq$ *m.*

In Figure -3 we have a (3,2) Graceful Graph.

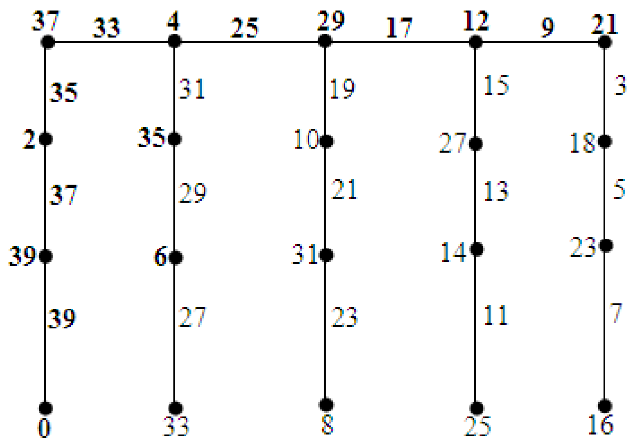

Figure 3: (3,2) Graceful Graph

4 Results

First, we prove that all *uniform k-distant tree* are *(k,d)-graceful.*

Theorem 1 *Every uniform k-distant tree is (k,d)-graceful.*

Proof Consider a *uniform k-distant tree* T with q edges. Since it is a tree, $q = p\text{-}1$, where $p$is the number of vertices of T.

Define a labeling f from $V(T)$ into $0, 1, 2, \ldots, k+(q-1)d$ such that

$$f(v_i) = \begin{cases} \left(\dfrac{i-1}{2}\right) d, & \text{if } i \text{ is odd,} \\ k + \left[p - \left(\frac{i}{2}+1\right)\right] d, & \text{if } i \text{ is even.} \end{cases}$$

We note that the absolute value of the difference of the labels of two consecutive vertices of the spine (that is, labels on the edges of the spine) is equal to the absolute value of the difference of the labels at the end vertices of the corresponding tail (for example, the absolute value of the difference of the labels at v_n and v_{n+1} is equal to the absolute value of the difference of the labels at v_1 and v_{2n}), by construction and labeling.

Case 1: p is even.

The odd vertices $v_{1,} v_3, \ldots, v_{p-1}$ receive the labels $0, d, 2d, \ldots, \left(\frac{p}{2}-1\right) d$ respectively.

The even vertices $v_{2,} v_{4,} \ldots, v_p$ receive the labels $k+(p-2)d, k+(p-3)d, \ldots,$ $k+\left(\frac{p}{2}-1\right) d$ respectively.

Therefore the edges receive the labels $k+(p-2)d, k+(p-3)d, \ldots, k$ respectively.

That is, the edges receive the labels $k, k+d, k+2d, \ldots, k+(q-1)d$.

Case 2: p is odd.

The odd vertices $v_{1,} v_{3,} \ldots, v_p$ receive the labels $0, d, 2d, \ldots, \left(\frac{p-1}{2}\right) d$ respectively.

The even vertices $v_{2,} v_{4,} \ldots, v_{p-1}$ receive the labels $k+(p-2)d, k+(p-3)d, \ldots, k+$ $\left(\frac{p-1}{2}\right) d$ respectively.

Therefore the edges receive the labels $k+(p-2)d, k+(p-3)d, \ldots, k$.

That is, the edges receive the labels $k, k+d, k+2d, \ldots, k+(p-3)d, k+(q-1)d$.

Hence every *uniform k-distant tree* is *(k,d)-graceful.* ■

Corollary 1 Every *uniform k-distant tree* is *k-graceful.*

Proof Since every *(k,1)-graceful* labeling is a *k-graceful* labeling the corollary is immediate. ■

Corollary 2 Every *uniform k-distant tree* is graceful.

Proof Since every (1,1)-graceful labeling is a graceful labeling the corollary is immediate.■

Lee and Wui [8] have shown that a connected graph is *Skolem-graceful* if and only if it is a graceful tree.

Corollary 3 Every *uniform k-distant tree* is *Skolem-graceful.*

Proof Since every *uniform k-distant tree* is a graceful tree, it is *Skolem-graceful.* ■

Theorem 2 *Every uniform k-distant tree is (k,d)-balanced.*

Proof Consider the *(k,d)-graceful labeling* f given in theorem 1. Take $m = \left(\frac{p}{2}-1\right) d$, if p is even and $m = \left(\frac{p-1}{2}\right) d$, if p is odd.

When p is even, odd vertices $v_{1,} v_{3,} \ldots, v_{p-1}$ receive the labels. $0, d, 2d, \ldots, \left(\frac{p}{2}-1\right) d$ respectively and even vertices $v_{2,} v_{4,} \ldots, v_p$ receive the labels $k+(p-2)d, k+(p-3)d, \ldots,$ $k+\left(\frac{p}{2}-1\right) d$ respectively.

When p is odd, odd vertices $v_{1,} v_{3,} \ldots, v_p$ receive the labels $0, d, 2d, \ldots, \left(\frac{p-1}{2}\right) d$ respectively and even vertices $v_{2,} v_{4,} \ldots, v_{p-1}$ receive the labels $k+(p-2)d, k+(p-3)d, \ldots, k+$ $\left(\frac{p-1}{2}\right) d$ respectively. Hence this m satisfies the condition that for every edge uv either *f(u)* $\leq m$ and *f(v)* $>m$, *or f(u)* $>m$ *and f(v)* $\leq m$. Hence every *uniform k-distant tree* is *(k,d)-balanced.* ■

Theorem 3 *Every uniform k-distant tree is odd graceful.*

Proof Consider a *uniform k-distant tree* T with q edges. Since it is a tree, $q = p - 1$, where $p$is the number of vertices of T.

Define a labeling f from $V(T)$ into $\{0, 1, 2, \ldots, 2q-1\}$ such that

$$f(v_i) = \begin{cases} 2\left(\dfrac{i-1}{2}\right), & \text{if } i \text{ is odd} \\ 2q+1-i, & \text{if } i \text{ is even.} \end{cases}$$

We note that the absolute value of the difference of the labels of two consecutive vertices of the spine (that is, labels on the edges of the spine) is equal to the absolute value of the difference of the labels at the end vertices of the corresponding tail (for example, the absolute value of the difference of the labels at v_n and v_{n+1} is equal to the absolute value of the difference of the labels at v_1 and v_{2n}), by construction and labeling.

Case 1: p is even.

The odd vertices $v_1, v_3, \ldots, v_{p-1}$ receive the labels $0, 2, 4, \ldots, p-2$ respectively.

The even vertices $v_2, v_4, \ldots, v_p$ receive the labels $2q-1, 2q-3, 2q-5, \ldots, 2q+1-p$ respectively.

Therefore the edges receive the labels $2q-1, 2q-3, 2q-5, \ldots, 1$ respectively.

That is, the edges receive the labels $1, 3, \ldots, 2q-3, 2q-1$.

Case 2: p is odd.

The odd vertices $v_1, v_3, \ldots, v_p$ receive the labels $0, 2, 4, \ldots, p-1$ respectively.

The even vertices $v_2, v_4, \ldots, v_{p-1}$ receive the labels $2q-1, 2q-3, 2q-5, \ldots, 2q+2-p$ respectively.

Therefore the edges receive the labels $2q-1, 2q-3, 2q-5, \ldots, 1$ respectively.

That is, the edges receive the labels $1, 3, \ldots, 2q-3, 2q-1$.

Hence every *uniform k-distant tree* is odd graceful. ■

5 Conclusion

In this paper, we have shown that *uniform k-distant tree* possess many interesting properties. We believe that this paper will create an interest towards solving the open problems mentioned in this paper and other related problems.

Acknowledgement

The author would like to thank the referee for his valuable comments.

References

[1] Rosa, A. On certain valuations of the vertices of a graph, theory of graphs. In *Proceedings of the 1966 International Symposium, Rome.* New York: Gordon and Breach Science Publishers. 1967.

[2] Graham, R. L. and Sloane, N.J.A. On additive bases and harmonious graphs. *SIAM J. Alg. Discrete Meth.* 1980. 1:382–404.

[3] Golomb, S. W. (Ed.). *How to Number a Graph, Graph Theory and Computing.* New York: Academic Press. 1972.

[4] Gallian, J. A. A dynamic survey of graph labeling. *The Electronic Journal of Combinatorics.* 2007. 14: DS6.

[5] Hegde, S. M. On (k, d)-graceful graphs. *J. Combin., Inform. & Sys. Sci.* 2000.25: 255-265.

[6] Bermond, J. C. *Graph Theory and Combinatorics.* London: Pitman. 1979.

[7] Atif Abueida and Dan Roberts. Uniform k-distant even trees are harmonious. *Utilitas Mathematica.* 2009. 78: 279-285.

[8] Lee, S. M. and Wui, I. On Skolem- gracefulness of 2-stars and 3-stars. *Bull. Malaysian Math. Soc.* 1987. 10: 15-20.