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Computation of Lyapunov Quantities of Homogeneous Quartic Polynomial System

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Abstract The analysis of system behaviour near boundary of the stability domain requires the computation of Lyapunov quantities which determine a system behaviour in the neighbourhood of boundary This paper presents the computation of Lyapunov quantities and focal values of a homogeneous quartic polynomial system by using the Lyapunov-Poincare method where the polynomials are of degree four. The proposed two main theorems were proved to accomplish this goal.

Keywords Lyapunov quantities; focal values; Lyapunov-Poincare method; quartic polynomial system.

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1 Introduction

The computation of Lyapunov quantities is related to its importance in engineering and mechanics of the question on the behaviour of a dynamical system near to the boundary of a stability domain. From Bautin [1], one varies “dangerous” or “safe” limits, i.e. a small alteration of which implies a small (invertible) or noninvertible alterations of the system status correspondingly. Such alterations parallel, for example, to condition of “hard” or “soft” excitations of fluctuations of the system, as shown by Andronov [2].

The development of methods of computation and analysis of Lyapunov quantities (or focus values, Lyapunov coefficients, Poincare-Lyapunov constants) was greatly encouraged by firstly as a pure mathematic problems (such as investigation of stability in critical case of two purely imaginary roots of the first approximation system, Hilbert’s 16^{th} problem, cyclicity problem, and distinguishing between center and focus) and then as to the applied problems (such as the investigation of boundaries of domain of stability and excitation of oscillations). Poincare [3] and Lyapunov [4] in their classical works for the analysis of system, conducted the linking of neighbouring boundary of the stability domain and advanced the technique of calculation of the so-called Lyapunov coefficients, (or Lyapunov quantities, focus values, Poincare-Lyapunov constants), which determine the system behaviour in the region of the boundary. This method likewise permits us successfully to study the bifurcation of the birth of small cycles [6–15], which is comparable in mechanics to small vibrations.

In this work the Lyapunov quantities of a homogeneous quartic polynomial system is calculated by using the Lyapunov–Poincare method. This work will be suitably used later in the study of the stability of a general dynamical system and in computing of the limit cycles for the respective system.

2 Computation of Lyapunov Quantities in Euclidean Space

In the classical method of Lyapunov-Poincare for the calculation of representative terms of Lyapunov quantities, in the neighbourhood of zero equilibrium it is necessary to find Lyapunov function $V(x,\, y)$ for the following homogeneous polynomial system

$$\begin{aligned} \frac{dx}{dt} &= \dot{x} = y + P_4\,(x,y), \\ \frac{dy}{dt} &= \dot{y} = -x + Q_4, \end{aligned} \tag{1}$$

where P_4 and Q_4 are homogenous polynomial of degree four, i.e. the Lyapunov function $V(x,\, y)$ can be written in the form

$$V(x,y) \;=\; V_2\,(x,y) \;+\; V_3\,(x,y) \;+\; ... + V_{n+1}\,(x,y). \tag{2}$$

Here

$$V_2\,(x,y) \;=\; \frac{x^2+y^2}{2}$$

and

$$V_k\,(x,y), \quad k \;=\; 3, ..., n+1$$

are homogeneous polynomials

$$V_k\,(x,y) = \sum_{(i+j=k)} V_{i,j}\, x^i\, y^j$$

with unknown coefficients $\{V_{i,j}\}_{i+j=k}$, $i,\; j \geq 0$.

For the derivative of $V(x,\, y)$, in virtue of system (1) we have

$$\begin{aligned} \dot{V}\,(x,y) = & \frac{\partial V(x,y)}{\partial x}\sum_{j=0}^{n}(-y + a_j x^{k-j} y^j) \\ & + \frac{\partial V(x,y)}{\partial y}\sum_{j=0}^{n}(x + b_j x^{k-j} y^j) + O[(|x|+|y|)^{n+1}]. \end{aligned}$$

Denoting the homogeneous terms of order k by $V_k(x,y)$, we obtain

$$\dot{V}(x,y) = V_3(x,y) + \ldots + V_{n+1}(x,y) + O(|x|+|y|)^{n+1}.$$

Note that the terms of the second degree in $V(x,\, y)$ are cancelled out. Then the equations $V_k(x,y) \;=\; 0$ for $k \;=\; 2p \,+\, 1$, where $p\!=\!1,...,$ and $V_k(x,y) \;=\; \omega_k\;(x^2 \,+\, y^2)p$ for $k \;=\; 2p$, where $p\!=\!2,...$ (ω_{k} are unknown coefficients) are solved sequentially.

If for certain $k = 2p^* \leq n+1$ the relation $\omega_k \neq 0$ is satisfied, then the quantity $2\pi\omega_{2p^*}$ is equal to (p^*-1)-th Lyapunov quantity [17] of system (1).

Detailed justification of this method can be found, for example, in the works of Lynch[8] and Leonove & Kuznetsova [18].

Below is given the computational technique of Lyapunov-Poincare, based on the classical method of Lyapunov-Poincare in Euclidean space.

3 The Technique of Lyapunov-Poincare

We can write the system (1) as follows:

$$\dot{x} = \lambda x + y + P_2(x,y) + ... + P_n(x,y),$$
$$\dot{y} = -x + \lambda y + Q_2(x,y) + ... + Q_n(x,y),$$

where $P_n(x,y)$ and $Q_n(x,y)$ are two homogeneous polynomials of degree k in a neighbourhood of the origin. Here the function $V(x,\, y)$ is given as follows

$$V(x,y) = \frac{x^2+y^2}{2} + V_3(x,y) + ... + V_{n+1}(x,y)...,$$

where for $k \geq 3, V_k$ is a homogeneous polynomial of degree k, and we further write

$$V_k(x,y) = \sum_{i=0}^{k} V_{k-i,i}\, x^{k-i}y^i.$$

The rate of change of $V(x,\, y)$ along an orbit is given by

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V(x,y)}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial V(x,y)}{\partial y}\cdot\frac{dy}{dt} \\ &= [x + (V_3)_x + ...][\lambda x + y + P_2(x,y) + ... + P_n(x,y)] \\ &\quad + [y + (V_3)y + ...][-x + \lambda y + Q_2(x,y) + ... + Q_n(x,y)]. \end{aligned} \tag{3}$$

We denote by D_k the terms of degree $k\,(k \geq 3)\,\mathrm{in}\, V$ by

$$D_k = [\,y(V_k)_x - x(V_k)_y] + [(V_{k-1})_x\, P_2 + (V_{k-1})_y\, Q_2 + ... + xP_{k-1} + yQ_{k-1}]$$

and without loss of generality, D_k can be written as

$$D_k = [\,y(V_k)_x - x(V_k)_y] + R_k(x,y),$$

where

$$R_k\,(x,y) = [(V_{k-1})_x\, P_2 + (V_{k-1})_y Q_2 + ... + xP_{k-1} + yQ_{k-1}].$$

Note that R_k is denoted on V_j with $j < k$ but not on V_k and $V_{k-j,j}$=0 if $j < 0$ or $k-j < 0$.

The idea is to choose coefficients $V_{k-j,\,j}$ and quantities ω_{k} so that

$$D_k = \begin{cases} 0, & \text{if } k \text{ is odd}, \\ \omega_k(x^2+y^2)^{k/2}, & \text{if } k \text{ is even}. \end{cases} \tag{4}$$

For convenience, we say that $V_{k-j,\,j}$ is odd or even coefficient of V_k in accordance to whether i is odd or even.

Suppose first that k is odd. The requirement $D_k = 0$ is equivalent to a set of k+1 linear equations for k+1 unknowns

$$V_{k,0}, V_{k-1,1}, ..., V_{1,k-1}, V_{0,k}\,.$$

These can be uncoupled into two sets of (k+1)/2 equations, one set determines the odd coefficients of the V_k, and the other determines the even coefficients of V_k. But if k is even, the requirement $D_k = \omega_k(x^2 + y^2)^{k/2}$ gives $k+1$ linear equations, and then the equations also uncouple into two sets of $m + 1$ equations for $\omega_{2\mathrm{m}}$ and m odd coefficients of V_k, and m equations for $m + 1$even coefficients of V_k.

To obtain unique values for the even coefficients of V_k, we introduce the supplementary condition;

$$\begin{cases} V_{m,m} = 0, & \text{if } m \text{ is even,} \\ \text{and} & \\ V_{m+1,m-1} + V_{m-1,m+1} = 0, & \text{if } m \text{ is odd.} \end{cases} \tag{5}$$

Then the even coefficients of V_k are uniquely determined.

The calculation of the focal values is a recursive procedure. Each iteration of which consists of solving the two sets of linear equations for $V_{k-j,\,j}$ with $i + j = k$ (k is odd), and solving the two sets of linear equations for ω_k and $V_{k-j,\,j}$ with $i + j = k$ (k is even).

4 Homogeneous Polynomial System

To find the number of Lyapunov quantities we use the classical Lyapunov method as follows

$$V_k\,(x, y) = \sum_{i+j=k} V_{i,j},\ x^i\, y^j,$$

and for the homogeneous quartic polynomial system

$$\begin{aligned} \dot{x} &= y + P_4(x, y), \\ \dot{y} &= -x + Q_4(x, y), \end{aligned}$$

and

$$\dot{V} = V_x\dot{x} + V_y\dot{y}, \tag{6}$$

where

$$P_4 = a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4$$

and

$$Q_4 = b_1x^4 + b_2x^3y + b_3x^2y^2 + b_4xy^3 + b_5y^4.$$

The suffices x and y denote partial differentiation with respect to x and y, we write D_k for the collection of terms of degree k on the right hand side of (5), and clearly for $k \geq 3$ we have

$$\begin{aligned} D_k &= y(V_k)_x - x(V_k)_y + R_k(x, y), \\ R_k(x, y) &= \sum_{i=0}^{k} R_{k-i,i}\, x^{k-i}y^i. \end{aligned}$$

R_k depends on V_j with $j < k$ but not on V_k and the idea is to choose coefficients $V_{k-j,j}$ and quantities ω_{k} so that

$$D_k = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ \omega_k\,(x^2 + y^2)^{k/2}, & \text{if } k \text{ is even,} \end{cases}$$

and

$$D_k = y(V_k)_x - x(V_k)_y + (V_{k-3})_xP_4 - (V_{k-3})_yQ_4, \qquad k \geq 3.$$

Then the result is given as follows:

We have the following values (please see appendix)

$$\begin{aligned} &V_3 = V_6 = V_9 = V_{12} = V_{15} = 0, \\ &V_4 = V_7 = V_{10} = V_{13} = V_{16} = 0, \\ &\omega_4 = \omega_6 = \omega_{10} = \omega_{12} = \omega_{14} = 0, \end{aligned}$$

with $\omega_8 \neq 0, \omega_{14} \neq 0, V_8 \neq 0$ and $V_{14} \neq 0$.

Here we shall show that the Lyapunov quantities satisfy the following properties, i.e.

$$V_k \neq 0 \;\; \text{if}\, k = 2 \mod 3,$$

and

$$\omega_k \neq 0 \;\; \text{if}\, k = 2 \mod 6.$$

The proof is immediate from the following two theorems.

Theorem 1 (Lyapunov Quantities)

$$V_k = 0 \;\; \mathit{if}\;\; k = 3 \;\; \mathit{mod}\;\; 3$$

and

$$\omega_{2k} = 0 \;\; \mathit{if}\;\; k = 3 \;\; \mathit{mod}\;\; 3.$$

Proof We have already seen that

$$V_3 = V_6 = V_9 = V_{12} = V_{15} = 0.$$

Note that

$$D_k = y(V_k)_x - x(V_k)_y + (V_{k-3})_x P_4 - (V_{k-3})_y Q_4, \qquad k \geq 3.$$

The proof will be by induction

First when $k = 3i + 3$ or $k = 3i$,

$$D_{3i} = y(V_{3i})_x - x(V_{3i})_y + (V_{3i-3})_x P_4 - (V_{3i-3})_y Q_4.$$

For $i = 1$, we have $V_3 = 0$ the result is true for $i = l$, so that $V_{3l} = 0$, and we must show that it holds for $i = l + 1$. Consider the following two cases: if l is odd, and if l is even

Case I If l is odd:

$$\begin{aligned} D_{3l+3} &= -V_{3l+2,1} x^{3l+3} + (3l+3)\, V_{3l+3,0} - 2V_{3l+1,2}\, x^{3l+2} y \\ &\qquad + \ldots + 2V_{3l+1,2} - (3l+3) V_{0,3l+3}\; x y^{3l+2} + V_{1,3l+2} y^{3l+3} \\ &= 0 \end{aligned}$$

gives two sets of equations

$$\begin{aligned} &(3l+3)\, V_{3l+3,0} - 2V_{3l+1,2} = 0, \\ &(3l+1)\, V_{3l+1,2} - 4V_{3l-1,4} = 0, \\ &3V_{3,3l+3} - (3l+2) V_{1,3l+2} = 0, \\ &V_{1,3l+2} \qquad\qquad = 0, \end{aligned} \tag{7}$$

and

$$\begin{aligned} &-V_{3l+2,1} = 0,\\ &(3l+2)\,V_{3l+2,1} - 3V_{3l,3} = 0,\\ &3lV_{3l,3} - 5V_{3l-2,5} = 0,\\ &4V_{4,3l-1} - (3l+1)V_{2,3l+1} = 0,\\ &2V_{3l+1,2} - (3l+3)V_{0,3l+3} = 0. \end{aligned} \tag{8}$$

From equation (7), we get

$$V_{1,3l+2} = V_{3,3l} = ... = V_{3l+3,0} = 0,$$

and from equation (8), we get

$$V_{3l+2,1} = V_{3l,3} = ... = V_{0,3l+3} = 0,$$

$$V_{3l+3} = 0.$$

Case II If l is even:

Since

$$\begin{aligned} D_{3l+3} = &-V_{3l+2,1}x^{3l+3} + (3l+3)\,V_{3l+3,0} - 2V_{3l+1,2}\,x^{3l+2}\,y\\ &+ ... + 2V_{2,3l+1} - (3l+3)V_{0,3l+3}xy^{3l+2} + V_{1,3l+2}y^{3l+3}\\ = &\ \omega_{3l+3}\,(x^2+y^2\,)^{(3l+3)/2} \end{aligned}$$

and using binomial expansion we have the following two sets of equations

$$\begin{aligned} &(3l+3)V_{3l+3,0} - 2V_{3l+1,2} = 0,\\ &(3l+1)V_{3l+1,2} - 4V_{3l-1,4} = 0,\\ &4V_{4,3l-1} - (3l+1)V_{2,3l+1} = 0,\\ &2V_{2,3l+1} - (3l+3)\,V_{0,3l+3} = 0, \end{aligned}$$

and

$$\begin{aligned} &-\omega_{3l+3} - V\_(_{3l+2,1}) = 0,\\ &-\begin{pmatrix} \frac{3l+3}{2}\\ 1 \end{pmatrix}\omega_{3l+3} + (3l+2)\,V_{3l+2,1} - 3V_{3l,3} = 0,\\ &-\begin{pmatrix} \frac{3l+3}{2}\\ 2 \end{pmatrix}\omega_{3l+3} + (3l)\,V_{3l,3} - 5V_{3l-2,5} = 0,\\ &-\begin{pmatrix} \frac{3l+3}{2}\\ \frac{3l+1}{2} \end{pmatrix}\omega_{3l+3} + 3V_{3,3l} - (3l+2)V_{1,3l+2} = 0,\\ &-\omega_{3l+3} + V_{1,3l+2} = 0. \end{aligned}$$

The first set with the additional condition (5) is given by

$$V_{(3l/2,3l/2)} = 0 \quad \text{when} \quad 3l/2 \quad \text{is even}$$

or

$$V_{3l+4/2,3l+2/2} + V_{3l+2/2,3l+4/2} = 0 \text{ when } 3l/2 \text{ is odd.}$$

Then we have

$$V_{0,3l+3} = V_{3l+1,2} = ... = V_{3l+3,0} = 0.$$

The second set gives us

$$\omega_{3l+3} = V_{3l+2,1} = V_{3l,3} = ... = V_{1,3l+2} = 0,$$

$$V_{3l+3} = 0.$$

So

$$V_k = 0 \quad \text{if } k = 3 \mod 3,$$

and

$$\omega_{2k} = 0 \quad \text{if } k = 3 \mod 3. \blacksquare$$

Theorem 2 *(Lyapunov quantities)*

$$V_k = 0 \quad \textit{if } k = 1 \, mod \; 3$$

and

$$\omega_{2k} = 0 \quad \textit{if } k = 1 \, mod \; 3$$

Proof We have already seen that

$$V_4 = V_7 = V_{10} = V_{13} = V_{16} = 0$$

note that

$$D_k = y(V_k)_x - x(V_k)_y + (V_{k-3})_x P_4 - (V_{k-3})_y Q_4 \qquad k \geq 3$$

The proof will be given by induction

First when $k = 3i + 1$,

$$D_{3i+1} = y(V_{3i+1})_x - x(V_{3i+1})_y + (V_{3i-2})_x P_4 - (V_{3i-2})_y Q_4$$

For $i = 1$, we have $V_4 = 0$ and the result is true for $i = l$, so that $V_{3l+1} = 0$, we must show that it holds $i = l + 1$, and we shall consider the following two cases: if l is odd, and if l is even

Case I If l is odd $D_{3i+4} = 0$ gives two sets of equations

$$\begin{aligned}
&(3l+4)\,V_{3l+4,0} - 2V_{3l+2,2} = 0\\
&(3l+2)\,V_{3l+2,2} - 4V_{3l,4} = 0\\
&3V_{3,3l+1} - (3l+3)V_{1,3l+3} = 0\\
&V_{1,3l+3} = 0
\end{aligned}$$

and

$$\begin{aligned}
&-V_{3l+3,1} = 0\\
&(3l+3)V_{3l+3,1} - 3V_{3l+1,3} = 0\\
&(3l+1)V_{3l+1,3} - 5V_{3l-1,5} = 0\\
&4V_{4,3l} - (3l+2)V_{2,3l+2} = 0\\
&2V_{2,3l+2} - (3l+3)V_{0,3l+4} = 0
\end{aligned}$$

From the above two sets of equation we obtain

$$V_{1,3l+3} = V_{3,3l+1} = ... = V_{3l+4,0} = 0$$

And

$$V_{3l+3,1} = V_{3l+1,3} = ... = V_{0,3l+4} = 0$$
$$V_{3l+4} = 0$$

Case II When l is even and since

$$D_{3i+4} = \omega_{3l+4}(x^2 + y^2)^{\frac{3l+4}{2}},$$

and by using binomial expansion we have the following two sets of equations

$$\begin{aligned}
&(3l+4)\,V_{3l+4,0} - 2V_{3l+2,2} = 0\\
&(3l+2)V_{3l+2,2} - 4V_{3l,4} \;\; = 0\\
&4V_{4,3l} - (3l+2)V_{2,3l+2} \;\; = 0\\
&2V_{2,3l+2} - (3l+4)V_{0,3l+4} = 0
\end{aligned}$$

and

$$\begin{aligned}
&-\omega_{3l+4} - V_{3l+3,1} = 0\\
&-\binom{\frac{3l+4}{2}}{1}\omega_{3l+4} + (3l+3)\,V_{3l+3,1} - 3V_{3l+1,3} = 0\\
&-\binom{\frac{3l+4}{2}}{2}\omega_{3l+4} + (3l+1)\,V_{3l+1,3)} - 5V_{3l-1,5} = 0\\
&-\binom{\frac{3l+4}{2}}{\frac{3l+3}{2}}\omega_{3l+4} + 3V_{3,3l+1} - (3l+3)\,V_{1,3l+3} = 0\\
&-\omega_{3l+4} + V_{1,3l+3} = 0
\end{aligned}$$

The first set with the additional condition and by (5), we have
$V_{(3l+4)/2,(3l+4)/2} = 0$ when $(3l+4)/2$ is even
or
$V_{(3l+6)/2,3l+2/2} + V_{(3l+2)/2,(3l+6)/2} = 0$ when $(3l+4)/2$ is odd
then we obtain

$$V_{0,3l+4} = V_{3l+2,2} = ... = V_{3l+4,0} = 0$$

From the second set, we have

$$\omega_{3l+4} = V_{3l+2,2} = V_{3l+1,3} = ... = V_{1,3l+3}0$$
$$V_{3l+4} = 0$$

So

$$V_k = 0 \quad \text{if} \;\; k = 1 \mod 3$$

and

$$\omega_{2k} = 0 \quad \text{if} \;\; k = 1 \mod 3. \;\blacksquare$$

Finally the Lyapunov quantities $L(k)$ for the system (1), are derivable from the focal values ω_{6k+4}, i.e. immediately from theorem 1 and theorem 2 in each of the above cases, we have

$$L(k) = \omega_{6k+4}, \;\; \text{where} \;\; \omega_{6k+4} \neq 0.$$

5 Conclusion

It has been shown in this paper that the general computation of Lyapunov quantities of homogenous quartic polynomial system of degree four can be achieved by using the classical method of Lyapunov-Poincare. In particular, two main theorems (Theorems 1 and Theorem 2) were proved to accomplish this goal. This work will suitably be used later in our study of the stability of a general dynamical system and computing of the limit cycles for the respective system.

At the present time there exist different methods for "construction" of limit cycles (the cycles, appearing from critical point, center, and homoclinic or heteroclinic orbits and from infinity). Historically, for more than a century in the framework of the solution of such a problem, numerous theoretical and numerical results were obtained. However the problem of visualization of limit cycles is still far from being resolved even for the simple classes of systems.

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Appendix (Calculation of V_3)

$$\begin{aligned} P_4 &= a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4 \\ Q_4 &= b_1x^4 + b_2x^3y + b_3x^2y^2 + b_4xy^3 + b_5y^4 \\ D_k &= y(V_k)_x - x(V_k)_y + (V_{k-3})_xP_4 + (V_{k-3})_yQ_4 \qquad k \geq 5. \end{aligned}$$

For $k = 3$ we have

$$\begin{aligned} D_3 &= y(V_3)_x - x(V_3)_y \\ &= 3V_{3,0}x^2y + 2V_{2,1}xy^2 + V_{1,2}\,y^3 - V_{2,1}x^3 - 2V_{1,2}x^2y - 3V_{0,3}xy^2 \\ &= -V_{2,1}\,x^3 + (3V_{3,0} - 2V_{1,2})\,x^2y + (2V_{2,1} - 3V_{0,3})xy^2 + V_{1,2}\,y^3 \\ &= 0 \end{aligned}$$

The above gives us the following two sets of equation:

$$\begin{aligned} -V_{2,1} \quad &= 0 \\ 2V_{2,1} - 3V_{0,3} &= 0 \end{aligned}$$

and

$$\begin{aligned} 3V_{3,0} - 2V_{1,2} &= 0 \\ V_{1,2} \quad &= 0. \end{aligned}$$

Hence we get

$$V_{2,1} = V_{0,3} = V_{3,0} = V_{1,2} = 0.$$

Therefore $V_3 = 0$

Similarly for V_i , $i = 4, 5, \ldots$.

References

[1] Bautin, N. N. *Behaviour of Dynamic Systems Near the Boundaries of the Stability Region.* Moscow-Leningrad: Gostekhizdat. 1949.

[2] Andronov, A. A., Vitt, A. A. and Khaikin, S. E. *Theory of oscillators*, Oxford: Pergamon Press, 1966.

[3] Poincare, H. Memoiresur les courbes definies par lesequations diffeentielles, *J. de Mathemantiques Pures et Appliquees*. 1885. 1: 167–244.

[4] Lyapunov, A. M. *Stability of Motion*. New York and London: Academic Press. 1966

[5] Lloyd, N. G. *Limit cycles of polynomial systems – some recent developments.* In Bedford, T. and Swift, J. (Eds.) *In New Direction in Dynamical Systems.* Cambridge: Cambridge University Press. 1988. 182–234.

[6] Li, J. Hilbert's 16th problem and bifurcation of planar polynomial vector fields. *Int. J. Bifurcation Chaos*. 2003. 13: 47-106.

[7] Chavarriga, J. and Grau, M. Some open problems related to 16th Hilbert problem. *Sci. Ser. A Math. Sci. N.S.* 2003. 9: 1–26.

[8] Lynch, S. Symbolic computation of Lyapunov quantities and the second part of Hilbert's sixteenth problem. In Wang, D. and Zheng, Z. (Eds.). *Differential Equations with Symbolic Computations*. Basel: Birkhauser. 2005. 1–26.

[9] Gine, J. On some problems in planar differential systems and Hilbert's 16th problem. *Chaos, Solutions and Fractals*. 2007. 31: 1118–1134.

[10] Yu, P., Chen, G. Computation of focus values with applications. *Nonlinear Dynamics*. 2008. 51(3): 409-427.

[11] Leonov, G. A. *Criterion for cycles existence in quadratic systems.* Vestnik of St. Petersburg University: Mathematics. Volume 40, Issue 3. September 2007. 31-41.

[12] Kuznetsov, N. V. and Leonov, G. A., Computation of the first Lyapunov quantity for the Second-order dynamical system, In Leonov, G. and Fradkov, A. (Eds.) *Proceedings of Third IFAC Workshop Periodic Control System*, Vol. 3. Part I .2007. 87-89.

[13] Leonov, G. A. Hilbert's 16th problem for quadratic system. New method based on a transformation to the Lienard equation. *International Journal of Bifurcation and Chaos*. 2008. 18(3): 877–884.

[14] Kuznetsov, N. V. and Leonov, G. A. Computation of Lyapunov quantities. *Proceedings of the 6th EUROMECH Nonlinear Dynamics Conference*, IPACS Electronic Library (http://lib.physcon.ru/?item=1802). 2008. 1-10.

[15] Leonov, G. A., Kuznetsov, N. V. and Kudryashova, E. V. *Limit cycles of two dimensional Systems. Calculations, proofs, experiments*, Vestnik of St. Petersburg University: Mathematics. Issue 3. September 2008. 25–62.

[16] Dumortier, F., Llibre, J. and Artes, J. *Qualitative Theory of Planar Differential Systems*. Berlin: Springer-Verlag. 2006

[17] Frommer, M. Die Integralkurven einer gewonlichen Differential gleichungen erster Ordnung in der Umgebung rationaler Unbestimm the itsstellen *Math Annalen. Bd*. 1928. 99: 222–272.

[18] Leonov, G. A. and Kuznetsova, O. A. Lyapunov quantities and limit cycles of two-dimensional dynamical systems. *Analytical Methods and Symbolic Computation Regular and Chaotic Dynamics*. 2010. 15: 356–379.