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Robust Designs of Step-Stress Accelerated Life Testing Experiments for Reliability Prediction

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Abstract In this article, we discuss the optimal and robust designs for accelerated life testing (ALT) when a step-stress plan is performed. It is assumed that the time to failure of a product has a Weibull distribution with a log-linear life-stress relationship. We adopt a generalized Khamis-Higgins model for the effect of changing stress levels. Taking into account that the assumed life-stress relationship is possibly misspecified, we have derived the optimal stress changing time of the simple step-stress plans in order to minimize the asymptotic mean squared error of the maximum likelihood estimator for the reliability of a product at the normal use stress level and at a pre-specified time. The optimal 3-step-stress plans with minimum asymptotic squared bias are also discussed.

Keywords Two-step-stress Plans; Three-step-stress Plans; Fisher Information; Asymptotic Bias; Extrapolation; Model Misspecification; Reliability Estimation.

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1 Introduction

For highly reliable products or components, a life testing experiment takes too long to observe any failures under normal operating conditions (also called the normal use stress levels). ALT is often used to shorten the life so that the observed failures can be quickly obtained in a reasonable time period. ALT experiments are conducted at stress levels higher than normal use stress levels. It consists of a number of test methods for acceleration. In constant stress ALT, each unit is subjected to an accelerated stress level and this level remains unchanged during the testing period, although different units may be under different stress levels. In step-stress ALT, the stress subjected to each test unit is not constant but is changing in a stepwise manner. Compared to constant stress ALTs, step-stress ALTs can obtain information much more quickly.

1.1 Previous Work on Optimal Step-Stress ALT Plans

There are a number of research papers that have investigated the methods of optimal ALT experiments with step-stress plans. Miller and Nelson [1] provide the optimal design for simple step-stress under the assumption of an exponential distribution with complete data. Such plans minimize the asymptotic variance of the maximum likelihood estimator (MLE) of the mean lifetime at the normal use stress. Bai et al. [2] extend the results of [1] for censored data. Bai and Kim [3] obtain an optimal simple step-stress for a Weibull lifetime distribution under Type-I censoring. Ma and Meeker [4] extend the optimal step-stress plan construction to the general log-location-scale distributions. Dharmadhikari and Rahman [5] provide the optimal stress change time for Weibull distribution assuming the shape parameter is a function of stress, however, the scale parameter stays fixed when the stress level changes. Most recently, Hunt and Xu [6] have obtained the optimal stress-change time under a Weibull distribution when both shape and scale parameters are considered to be functions of the stress levels. All of these aforementioned articles consider that the assumed models are exactly correct.

For ALT experiments, all the test units are tested under accelerated stress levels; however, the estimation required is the reliability at the normal use level, which is lower than the test stresses. Therefore, the estimation is extrapolated based on the assumed the life-stress relationship. This relationship cannot be tested for its validity since there is no observation under normal use levels. Hence, this is a one-point extrapolation problem. Pascual [7] provides a review on the construction of ALT designs under the consideration of robustness following different directions. He also presents the asymptotic distribution of MLEs and the resulting estimation bias for an ALT experiment with a constant stress plan and when an ALT model is misspecified.

In a complete general setting, robust designs for one-point extrapolation have been discussed in Wiens and Xu [8], for least squares estimation of a mean response. Please also see the references therein. As Fang and Wiens [9] pointed out: "Extrapolation to regions outside of that in which observations are taken is, of course, an inherently risky procedure and is made even more so by an over-reliance on stringent model assumptions." The classical optimal designs minimize the variance alone. However, when the fitted models are incorrect, the estimation is biased. A robust design should be obtained in an optimal way so that even when the fitted model was not exactly correct, the designs can still be relatively efficient with a small bias.

In this paper, with the awareness of possible imprecision in the assumed life-stress relationship, we investigate the optimal stress changing time for simple step-stress plans so that the asymptotic mean squared error (AMSE) of the underlying reliability estimator can be minimized. We also discuss the best choices of 3-step-stress plans so that the asymptotic squared bias (ASB) can be minimized. This paper extends the results of constructing the optimal step-stress plans of others in the following ways: (1) it goes beyond [7] due to consideration of model uncertainty for step-stress plans; (2) it goes beyond [8] due to constructing the robust design for maximum likelihood estimation (MLE) for reliability.

1.2 Test Plans

In most ALT models, there is an implied transformation of the stress and such as reciprocal of the absolute Kelvin temperature, log of voltage, etc. We use s to denote this transformed stress, s_0 to denote the normal use stress level. All the test units will be tested between the use stress level and the highest possible stress level. Under a simple step-stress ALT plan, all test units are tested at a lower accelerated stress level s_1; then, at a stress changing time τ_1 all of the surviving units are moved to a predetermined highest stress level s_h until all units fail, where $s_0 < s_1 < s_h$. Under a 3-step-stress plan, there are three stress levels involved. All test units are tested at a lower accelerated stress level s_1; at a stress changing time τ_1, all of the surviving units are moved to a higher stress level s_2; at a stress changing time τ_2, all of the surviving units are moved to s_h until all units fail, where $s_0 < s_1 < s_2 < s_h$. For a simple step-stress plan, h = 2 and for a 3-step-stress plan, h = 3.

The rest of this article is arranged as follows: some mathematical preliminaries, notations, and derivations are detailed in §2. The constructions of the designs and the resulting optimal designs using an example are provided in §3. We conclude with a few remarks in §4.

2 Materials and Methods

2.1 Model assumptions

We first consider a simple step-stress plan. Let n_1 be the number of failures observed by the first stress changing time τ_1. We assume that the lifetimes of the test units are independent and follow a Weibull distribution with scale parameter θ_i and shape parameter ω, where θ_i varies when stress level changes. Let Y be the natural log lifetime of a test. Then Y has an extreme value distribution with location parameter $\mu_i = \log(\theta_i)$ at stress level s_i, i = 0, 1, … , h, and scale parameter $\sigma = 1/\omega$.

We utilize a Khamis-Higgins model for the effect of changing stress levels. For a 2-step-stress plan, we let $C_{11} = \frac{\log(\tau_1) - \mu_1}{\sigma}$, and $C_{21} = \frac{\log(\tau_1) - \mu_2}{\sigma}$. Then, the cumulative distribution function and the probability density function of Y become

$$F(y) = \begin{cases} 1 - \exp\left[-\exp\left(\frac{y-\mu_1}{\sigma}\right)\right], y < \log(\tau_1), \\ 1 - \exp\left[-\exp\left(\frac{y-\mu_2}{\sigma}\right) + \exp(C_{21}) - \exp(C_{11})\right], y \geq \log(\tau_1), \end{cases}$$

and

$$f(y) = \begin{cases} \frac{1}{\sigma}\exp\left[\frac{y-\mu_1}{\sigma} - \exp\left(\frac{y-\mu_1}{\sigma}\right)\right], y < \log(\tau_1), \\ \frac{1}{\sigma}\exp\left[\frac{y-\mu_2}{\sigma} - \exp\left(\frac{y-\mu_2}{\sigma}\right) + \exp(C_{21}) - \exp(C_{11})\right], y \geq \log(\tau_1). \end{cases}$$

The optimal step-stress designs rely on the assumed stress-life relationship. Pascal [7] has not only pointed out that the robustness respect to the life-stress relationship is crucial to ALT but also presents various examples to indicate the plausible reasons why the ALT model may fail to hold throughout the stress range. Due to extrapolation, it is impossible to test any departure of the assumed stress-life relationship at the use stress level. Therefore, the validity of the stress-life relationship in ALT is always in question because of extrapolation. However, the reliability estimation of interest under different relationships, for instance the linear and curved stress-life relationships, may appear similar within the test stress range while it may tend to diverge beyond this range. This is certainly problematic for the practitioner.

In this paper, we aim to derive the optimal step-stress ALT plans for the MLE of reliability which can be robust against the misspecification in the fitted life-stress relationship.

2.2 MLEs and Fisher Information Matrix Under Possible Misspecification of the Stress-Life Relationship

Let M_l be the fitted model and M_q be true model for the stress-life relationship. In this paper, we assume the practitioner is fitting a commonly used model with the log-linear life-stress relationship, which is $\mu_{li} = \log(\theta_{li}) = \beta_0 + \beta_1 s_i$, where β_0 and β_1 are unknown parameters. However, we are aware of the possibility of the true relationship being a log-quadratic life-stress relationship, with $\mu_{qi} = \log(\theta_{qi}) = \alpha_0 + \alpha_1 s_i + \alpha_2 s_i^2$. Define $\boldsymbol{\alpha} = [\alpha_0, \alpha_1, \alpha_2]^T$ and $\boldsymbol{\beta} = [\beta_0, \beta_1]^T$. Let $\mathcal{L}(\boldsymbol{\alpha}, \boldsymbol{\xi})$ be the log-likelihood function under M_q, and $\mathcal{L}(\boldsymbol{\beta}, \boldsymbol{\xi})$ be the log-likelihood function under M_l, both with test plan $\boldsymbol{\xi}$. The expected log-likelihood ratio under the true model M_q is

$$I(\boldsymbol{\alpha}:\boldsymbol{\beta}) = E_{M_q}[\log \mathcal{L}(\boldsymbol{\alpha}, \boldsymbol{\xi}) - \log \mathcal{L}(\boldsymbol{\beta}, \boldsymbol{\xi})]. \quad (1)$$

We note that the MLE method is used for the parameter estimation.

We let y_{ik} be the log of the failure time of the *k*th unit under stress level s_i, and define $C_{ij}^{(l)}$ as C_{ij} in Section 2.1 but replacing μ_i with μ_{li}. For a 2-step-stress plan, the likelihood function and the negative log-likelihood function are

$$\mathcal{L}(\boldsymbol{\beta},\boldsymbol{\xi}) = \prod_{k=1}^{n_1}\left(\frac{1}{\sigma}\exp\left[\frac{y_{1k}-\mu_{l1}}{\sigma}-\exp\left(\frac{y_{1k}-\mu_{l1}}{\sigma}\right)\right]\right) \tag{2}$$

$$\times\prod_{k=1}^{n-n_1}\frac{1}{\sigma}\exp\left[\frac{y_{2k}-\mu_{li}}{\sigma}-\exp\left(\frac{y_{ik}-\mu_{li}}{\sigma}\right)+\exp(C_{21})-\exp(C_{11})\right], \tag{3}$$

$$l = -\log\ \mathcal{L}(\boldsymbol{\beta},\boldsymbol{\xi}) = n\ln(\sigma) - \frac{1}{\sigma}\left(\sum_{k=1}^{n_1} y_{ik} + \sum_{k=1}^{n-n_1} y_{ik} - n_1\mu_{l1} - (n-n_1)\mu_{l2}\right)$$

$$+\sum_{k=1}^{n_1}\exp\left(\frac{y_{1k}-\mu_{l1}}{\sigma}\right)+\sum_{k=1}^{n-n_1}\exp\left(\frac{y_{2k}-\mu_{l2}}{\sigma}\right)-(n-n_1)\left(\exp(C_{21}^{(l)})-\exp(C_{11}^{(l)})\right). \tag{4}$$

The expectation of $-\log\mathcal{L}(\boldsymbol{\beta},\boldsymbol{\xi})$ under the true model is

$$E_{M_q}(l) = n\ln(\sigma) - \frac{1}{\sigma}[\mu_{q1}E_{M_q}(n_1)+\mu_{q2}E_{M_q}(n-n_1)] + \frac{1}{\sigma}[\mu_{l1}E_{M_q}(n_1)+\mu_{l2}E_{M_q}(n-n_1)]$$
$$+\sum_{k=1}^{n_1}E_{M_q}\left[\exp\left(\frac{y_{1k}-\mu_{l1}}{\sigma}\right)\right]+\sum_{k=1}^{n-n_1}E_{M_q}\left[\exp\left(\frac{y_{2k}-\mu_{l2}}{\sigma}\right)\right]-[E_{M_q}(n-n_i)]\left(\exp(C_{21}^{(l)})-\exp(C_{11}^{(l)})\right), \tag{5}$$

where $E_{M_q}(n_1) = nF(\tau_1)$.

Since $\dfrac{y_{ik}-\mu_{qi}}{\sigma}$, i = 1, 2, has the standard extreme value distribution, we have

$$E_{M_q}\left[\exp\left(\frac{y_{ik}-\mu_{qi}}{\sigma}\right)\right] = 1.$$

Therefore,

$$E_{M_q}\left[\exp\left(\frac{y_{ik}-\mu_{li}}{\sigma}\right)\right] = \exp\left(\frac{\mu_{qi}-\mu_{li}}{\sigma}\right).$$

We make the use of the following quantities:

$$d_1 = F(\tau_1)\exp\left(\frac{\mu_{q1}-\mu_{l1}}{\sigma}\right),\qquad d_2 = [1-F(\tau_1)]\exp\left(\frac{\mu_{q2}-\mu_{l2}}{\sigma}\right),$$

$$c_1 = \exp(C_{21}^{(l)}) - \exp(C_{11}^{(l)}), \qquad c_2 = s_2 \exp(C_{21}^{(l)}) - s_1 \exp(C_{11}^{(l)}).$$

Then, (5) becomes

$$E_{M_q}(l) = n\left\{ \ln(\sigma) - \frac{1}{\sigma} F(\tau_1)(\mu_{q1} - \mu_{l1}) - \frac{1}{\sigma}[1 - F(\tau_1)](\mu_{q2} - \mu_{l2}) + d_1 + d_2 - [1 - F(\tau_1)]c_1 \right\}.$$

Let $\boldsymbol{\beta}^* = [\beta_0^*, \beta_1^*]^T$ be the value of $\boldsymbol{\beta}$ that minimizes (1), which is equivalent to minimizing (5). Setting $\frac{\partial E_{M_q}(l)}{\partial \beta_0} = 0$, $\frac{\partial E_{M_q}(l)}{\partial \beta_1} = 0$, and solving for β_0 and β_1 simultaneously will yield $\boldsymbol{\beta}^*$. Suppose that data is collected under a test plan $\boldsymbol{\xi}$ with sample size n and the practitioner fits Model M_l to the data using MLE methods. Let $\widehat{\boldsymbol{\beta}}$ denote the quasi-MLE of $\boldsymbol{\beta}$, since M_l is not the true model. By Theorem 3.2 in White [10], $\sqrt{n}\,(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*)$ is asymptotically normal with mean $\mathbf{0}$ and variance covariance matrix $C(\boldsymbol{\beta} = \boldsymbol{\beta}^*)$, which is defined below. Define the matrices

$$A(\boldsymbol{\beta}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \left[E_{M_q}\left(\frac{\partial^2 l}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}^T} \right) \right] = \begin{bmatrix} E_{M_q}\left(\frac{\partial^2 l}{\partial \beta_0^2} \right) & E_{M_q}\left(\frac{\partial^2 l}{\partial \beta_0 \partial \beta_1} \right) \\ E_{M_q}\left(\frac{\partial^2 l}{\partial \beta_1 \partial \beta_0} \right) & E_{M_q}\left(\frac{\partial^2 l}{\partial \beta_1^2} \right) \end{bmatrix}, \tag{6}$$

$$B(\boldsymbol{\beta}) = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \left[E_{M_q}\left(\frac{\partial l}{\partial \boldsymbol{\beta}} \cdot \frac{\partial l}{\partial \boldsymbol{\beta}^T} \right) \right] = \begin{bmatrix} E_{M_q}\left[\left(\frac{\partial l}{\partial \beta_0} \right)^2 \right] & E_{M_q}\left(\frac{\partial l}{\partial \beta_0} \cdot \frac{\partial l}{\partial \beta_1} \right) \\ E_{M_q}\left(\frac{\partial l}{\partial \beta_1} \cdot \frac{\partial l}{\partial \beta_0} \right) & E_{M_q}\left[\left(\frac{\partial l}{\partial \beta_0} \right)^2 \right] \end{bmatrix}, \tag{7}$$

and

$$C(\boldsymbol{\beta}) = [A(\boldsymbol{\beta})]^{-1} \times B(\boldsymbol{\beta}) \times [A(\boldsymbol{\beta})]^{-1}.$$

The elements of (6) and (7) are as follows:

$$a_{11} = \frac{n}{\sigma^2}\{d_1 + d_2 - [1 - F(\tau_1)]c_1\},$$

$$a_{12} = a_{21} = \frac{n}{\sigma^2}\{s_1 d_1 + s_2 d_2 - [1 - F(\tau_1)]c_2\},$$

$$a_{22} = \frac{n}{\sigma^2}\left\{ s_1^2 d_1 + s_2^2 d_2 - [1 - F(\tau_1)]\left(s_2^2 \exp(C_{21}^{(l)}) - s_1^2 \exp(C_{11}^{(l)}) \right) \right\};$$

and

$$b_{11}=\frac{n}{\sigma^2}\left\{\begin{array}{c}\frac{n}{\sigma^2}+n(d_1+d_2)^2-2\{(n-1)[1-F(\tau_1)]d_1+[n-(n-1)F(\tau_1)]d_2\}c_1\\ +F(\tau_1)[1-F(\tau_1)]\left[\exp\left(\frac{2(\mu_{q1}-\mu_{l1})}{\sigma}\right)+\exp\left(\frac{2(\mu_{q2}-\mu_{l2})}{\sigma}\right)-\exp\left(\frac{(\mu_{q1}-\mu_{l1})+(\mu_{q2}-\mu_{l2})}{\sigma}\right)\right]\\ +[1-F(\tau_1)][n-(n-1)F(\tau_1)]c_1^2+\frac{2n}{\sigma}(d_1+d_2)-\frac{2n}{\sigma}[1-F(\tau_1)]c_1\end{array}\right\},$$

$$b_{12}=b_{21}=\frac{n}{\sigma^2}\left\{\begin{array}{l}\frac{n}{\sigma^2}\{s_1F(\tau_1)+s_2[1-F(\tau_1)]\}+\frac{n}{\sigma}(s_1d_1+s_2d_2)-\frac{n}{\sigma}[1-F(\tau_1)]c_2\\ +\frac{n}{\sigma}(d_1+d_2)\{s_1F(\tau_1)+s_2[1-F(\tau_1)]\}+n(d_1+d_2)(s_1d_1+s_2d_2)\\ -n(d_1+d_2)[1-F(\tau_1)]c_2-\frac{1}{\sigma}\{ns_2-(n-1)(s_2-s_1)F(\tau_1)\}[1-F(\tau_1)]c_1\\ -F(\tau_1)[1-F(\tau_1)][s_1d_1+s_2d_2]c_1+[n-(n-1)F(\tau_1)]c_1c_2\end{array}\right\},$$

$$b_{22}=\frac{n}{\sigma^2}\left\{\begin{array}{l}\frac{1}{\sigma^2}\left[n[s_1F(\tau_1)+s_2(1-F(\tau_1))]^2+(s_1^2+s_2^2-s_1s_2)F(\tau_1)[1-F(\tau_1)]\right]\\ +n(s_1d_1+s_2d_2)^2+\left[s_1F(\tau_1)\exp\left(\frac{2(\mu_{q1}-\mu_{l1})}{\sigma}\right)+s_2[1-F(\tau_1)]\exp\left(\frac{2(\mu_{q2}-\mu_{l2})}{\sigma}\right)\right]\\ +[1-F(\tau_1)][n-(n-1)F(\tau_1)]c_2^2-2[1-F(\tau_1)](s_1d_1+s_2d_2)c_2\\ +\frac{2n}{\sigma}\{s_1F(\tau_1)+s_2[1-F(\tau_1)]\}\left[s_1F(\tau_1)\exp\left(\frac{2(\mu_{q1}-\mu_{l1})}{\sigma}\right)+s_2[1-F(\tau_1)]\exp\left(\frac{2(\mu_{q2}-\mu_{l2})}{\sigma}\right)\right]\\ -\frac{2}{\sigma}\{ns_1F(\tau_1)+ns_2[1-F(\tau_1)]+(s_2-s_1)F(\tau_1)[1-F(\tau_1)]\}c_2\end{array}\right\},$$

The calculation of the values above for a 3-step-stress plan is similar to this for a 2-step-stress plan, so we omit it here due to page limit. It is also available upon request.

2.3 Optimization Criteria

The robust and optimal test plan considered in this paper is to determine the stress changing times, τ_j, $j = 1, 2, \ldots, h-1$, so that the reliability estimation requested can be as precise as possible. Due to the model misspecification, the MLEs obtained using the fitted model are no longer asymptotically unbiased. Therefore, we consider the optimum criteria to be the minimizations of the AMSE, and the

ASB of the MLE of a reliability function at a predetermined time ς under the normal use stress level. In this paper, the optimization criteria are used under a transformation of the reliability estimate.

The MLE of the reliability estimate from the Weibull failure time distribution, at time ς under the normal use stress level s_0 is

$$R_{S_0}(\hat{\boldsymbol{\beta}},\varsigma)=\exp\left(-\exp\left(\frac{\log(\varsigma)-\log(\hat{\mu}_{l0})}{\sigma}\right)\right)=\exp\left(-\exp\left(\frac{\log(\varsigma)-[\hat{\beta}_0+\hat{\beta}_1 s_0]}{\sigma}\right)\right) \quad (8)$$

For simplicity, we transform the reliability estimate into a linear function of $\hat{\beta}_0$ and $\hat{\beta}_1$. We denote it as

$$N_{S_0}(\hat{\boldsymbol{\beta}},\varsigma)=\log(-\log[R_{S_0}(\hat{\boldsymbol{\beta}},\varsigma)]=\frac{1}{\sigma}[\log(\varsigma)-(\hat{\beta}_0+\hat{\beta}_1 s_0)]. \quad (9)$$

Then, the ASB of (9) is

$$\begin{aligned}ASB[N_{S_0}(\hat{\boldsymbol{\beta}},\varsigma)\,|\,M_q]&=\left[E_{M_q}[N_{S_0}(\hat{\boldsymbol{\beta}},\varphi)]-N_{S_0}(\boldsymbol{\alpha},\varphi)\right]^2 \quad (10)\\&=\left[\frac{1}{\sigma}E_{M_q}\left[\log(\varsigma)-\left(\hat{\beta}_0+\hat{\beta}_1 s_0\right)\right]-\frac{1}{\sigma}[\log(\varsigma)-\mu_{q0}\right]^2\\&=\frac{1}{\sigma^2}[\mu_{q0}-\beta_0^*-\beta_1^* s_0]^2,\end{aligned}$$

the asymptotic variance of (9) is

$$\begin{aligned}AVAR\left[N_{S_0}(\hat{\boldsymbol{\beta}},\varsigma)\,|\,M_q\right]&=\left[\frac{\partial N_{S_0}(\hat{\boldsymbol{\beta}},\varsigma)}{\partial\hat{\beta}_0},\frac{\partial N_{S_0}(\hat{\boldsymbol{\beta}},\varsigma)}{\partial\hat{\beta}_1}\right]C(\boldsymbol{\beta}=\boldsymbol{\beta}^*)\begin{bmatrix}\frac{\partial N_{S_0}(\hat{\boldsymbol{\beta}},\varsigma)}{\partial\hat{\beta}_0}\\\frac{\partial N_{S_0}(\hat{\boldsymbol{\beta}},\varsigma)}{\partial\hat{\beta}_1}\end{bmatrix}\\&=\frac{1}{\sigma^2}[1,\ s_0]C(\boldsymbol{\beta}=\boldsymbol{\beta}^*)\begin{bmatrix}1\\s_0\end{bmatrix} \quad (11)\end{aligned}$$

and the AMSE of (9) is

$$ASME\left[N_{S_0}(\hat{\boldsymbol{\beta}},\varsigma)\,|\,M_q\right]=ASB\left[N_{S_0}(\hat{\boldsymbol{\beta}},\varsigma)\,|\,M_q\right]+AVAR\left[N_{S_0}(\hat{\boldsymbol{\beta}},\varsigma)\,|\,M_q\right]. \quad (12)$$

3 Results and Discussion

3.1 Optimal Simple Step-Stress Plan for Minimizing AMSE

In order to demonstrate our construction of the robust design, we revisit the example presented by Fard and Li [11]. Suppose that a simple step-stress test of cable insulation is run to estimate reliability at a specified time ς = 2000 minutes. We obtain the optimal stress-changing time τ in order to minimize (12). Two accelerated stress levels of 24kV and 30kV are applied. The use level stress is s_0 = 20 kV. Suppose from a previous experiment or prior knowledge ω = 2.2, and the initial parameters are α_0 = 10.39264742, $\alpha_1 = -0.253190592$, α_2 = 0.004 which can be seen with the minimal curvature within the design space, [24kV , 30kV].

For a range of τ used in the minimization, we use an upper bound of 2500 minutes which is a slightly longer lifetime than the 99.9 percentile of the assumed lifetime distribution, 2458.5 minutes. A pattern search with a one minute time step interval is applied to find stress-changing times which minimize (12) over the range, $\tau \in$ (0, 2500). As a result, the AMSE is minimized when the stress-changing times is at 739 minutes. We note that the AMSE is an increasing function when τ exceeds 739 minutes. The result is displayed in Table 1 below. Also see Figure 1 for the graph of the function AMSE in terms of τ.

Table 1 The optimal stress-changing time

τ	The ASB	The minimum AMSE
739	0.123904	5.699109

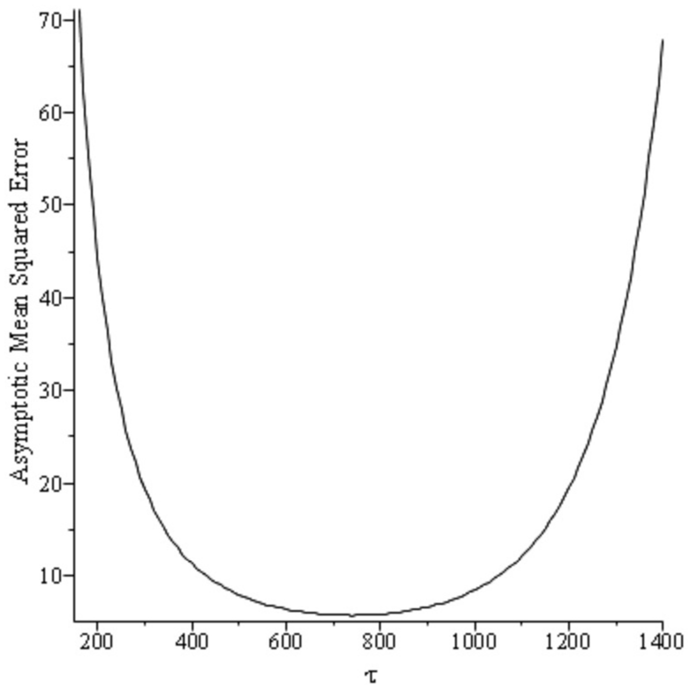

Figure 1 AMSE versus τ

In this example, we also observe that the ASB remains fairly constant as the stress-changing time varies, which indicates that the ASB of reliability estimation is not sensitive to the stress-changing time for an ALT with a simple step-stress plan. In order to reduce the asymptotic bias in estimation, we further investigate the effect of a multiple step-stress plan on the ASB in the next subsection.

3.2 Optimal Multiple Step-Stress Plan for Correcting the Asymptotic Bias

We continue to use the example presented in Section 3.1, however, instead we apply a 3-step-stress plan with three accelerated stress levels of 24kV, 27kV, and 30kV. In this example, all three stress levels are prespecified, and the middle stress level is set to be the average of the low and high stress levels. All other values of the parameters in the example remain the same as appeared in Section 3.1. We obtained the optimal stress-changing times τ_1 and τ_2 so that the ASB can be minimized. To be more practical, we search (τ_1, τ_2) within a certain range so that the stress changing times can have a gap between each other. Given such gaps, the failures are expected to be observed for each stress level.

Table 2 Optimal 3-step-stress testing plans for correcting ASB

Test plan	Stress range	τ_1	τ_2	The minimum ASB
ξ_1	$\tau_1 = 400$, and $\tau_2 \in$ [450, 2450]	400	2425	0.0607129
ξ_2	$\tau_1 \in$[50, 2400] and $\tau_1 \in [\tau_1 + 50$, 2450]	64	2445	0.0607121

In the test plans ξ_1 and ξ_2 shown in Table 2, we introduce a bounded design for the stress-changing times. These bounded designs are used to create practical ALT experiments. We create a time gap of at least 50 minutes in between the start of the test and τ_1, in between τ_1 and τ_2, and in between τ_2 and the upper bound. This type of design is implemented to allow for failures of the test units at each of the stress levels. We minimize the ASB of the transformed reliability in (10) to find the optimal stress-changing times τ_1 and τ_2. A pattern search is applied with a one minute time step interval for the stress-changing times. We use the same upper bound of 2500 minutes as it was in Section 3.1. The resulting optimal stress-changing times τ_1 and τ_2 are displayed in Table 2 for each of the two specified stress ranges.

Although the optimal test plan ξ_2, as a result of searching in a much wider stress range, provides a smaller ASB of the transformed reliability estimate, it suggests that the low stress-changing time should occur at 64 minutes and the high stress-changing time at 2445 minutes. However, its competitor, ξ_1, not only gives the minimum ASB almost as small as that for ξ_2, but also enjoys the longer testing time under the lowest test stress level, which may be required to observe sufficient failures.

3.3 Discussion and Future Research

Under the consideration of possibly underfitting an assumed stress-life model, the robust and optimal stress changing time is obtained in order to minimize the AMSE. For a simple step-stress plan involving only one stress changing time, the optimal stress changing time appears to be longer than the one obtained without such consideration. For the example discussed in Section 3.1, the optimal τ_1 obtained by minimizing the asymptotic variance alone was 686 minutes; see [11]. However, the robust and optimal τ_1 derived by minimizing AMSE is lengthened to 739 minutes.

According to the theory of robust design of experiments for linear statistical models, the robust designs against model misspecification tend to be more uniform, see [8]. Since the simple step stress plans only consist of two distinct stress levels, which is far from being uniform, the asymptotic bias appeared in the estimation due to the misspecification in the assumed stress-life relationship cannot be corrected by only optimizing the stress changing time. By introducing multiple step-stress plans, we are able to reduce the estimation bias. As discussed in the example above, implementing a 3-step-stress ALT with two stress-changing times and with three predetermined stress levels has already reduced the ASB by more than half, from 0.123904 for a simple step-stress plan to 0.060712 for a 3-step-stress plan in this example.

A future study can be done to construct promising robust designs for ALT against misspecification in the life-stress relationship by determining the stress-changing time(s) and the test stress level(s) simultaneously so that the AMSE or ASB can be minimized.

Furthermore, homoscedasticity is another commonly used assumption for ALT models. The method we developed in this paper is under assumption of a Weibull lifetime distribution with a constant shape parameter. For some products, the shape parameter of (possibly transformed) lifetime distribution varies when stress level changes. Such heteroscedastic manner would affect the resulting optimal deigns, so it should be taken into account in the design stage of the life-testing experiments. Therefore, another future study is to take robustness consideration of possible heteroscedasticity in the course of constructing the optimal designs for the ALT experiments.

4 Conclusion

We have provided the expected Fisher information and the asymptotic distribution of the MLEs for reliability prediction when the life-stress relationship is misspecified. Although the resulting components of these results are given for simple step-stress plans, the derivation for multiple step-stress is very similar to this and the authors make them available upon request due to page limit. A robust and optimal ALT design with simple step-stress plan is obtained for a practical example. In addition, a minimum bias design is also constructed for a 3-step-stress plan. In general, our method of deriving the robust and optimal plan can be applied to estimation or prediction of any quantity of interest as long as it is a given function of the unknown parameters, such as the mean life to failure or a percentile of the lifetime distribution.

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