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# Quadratic Bezier Homotopy Function for Solving System of Polynomial Equations

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**Abstract** We compare standard homotopy function with a proposed quadratic Bezier homotopy function to see which method has greater applicability and greater accuracy. We test the methods on system of polynomial equations by using Newton-Homotopy Continuation method. The results obtained indicate the superior accuracy of our proposed quadratic Bezier homotopy function.

**Keywords** Numerical method; Polynomial equations; Homotopy function.

**2010 Mathematics Subject Classification** 65H05; 65H10; 65H20.

## 1 Introduction

In this paper we consider the solution of the system of $n$ polynomial equations

$$F(x) = \begin{pmatrix} f_1(x_1, x_2, \cdots, x_n) \\ f_2(x_1, x_2, \cdots, x_n) \\ \vdots \\ f_n(x_1, x_2, \cdots, x_n) \end{pmatrix} = 0 \tag{1}$$

where $x = \{x_1, x_2, ..., x_n\}$ using homotopy continuation method (HCM). A popular method for solving (1) is the Newton method which is derived from the Taylor expansion series [1].

In recent years, the study of the solution of polynomial equations especially on system of polynomial equations using methods based on the concept of homotopy from Topology has attracted considerable interest. Rafiq and Awais [2] stated that the homotopy continuation method (HCM) has been known since the 1930s. However, more recent work in the development of homotopy concepts for nonlinear algebraic equation was developed in the 1970s [3]. This was followed by works of Garcia and Zangwill [4], Melhem and Rheinboldt [5], Morgan [6] and Watson [7,8].

Alexander and Yorke [3] stated that HCM involves numerically finding the solution of a problem by starting the solution from the solution of a known problem and continuing the solution as the known problem is homotoped to the given problem. Alexander and Yorke [3] also described the connection between algebraic topology and the continuation method.

Garcia and Zangwill [4] suggested a procedure to obtain all solutions to certain systems of $n$ equations in the complex domain. Let

$$z_i^{q_i} + P_i(z) = 0, \qquad i = 1, 2, ..., n. \tag{2}$$

where $z$ is the complex vector, $q_i$ is the large integer and $P(z) = F(z) - 1$. By rewriting (2), we obtain

$$(z_i^{q_i} - 1) + (P_i(z) + 1) = 0 \tag{3}$$

To start with, the authors defined homotopy function as

$$(z_i^{q_i} - 1) + t(P_i(z) + 1) = 0 \tag{4}$$

where $z$ is the complex vector, $q_i$ is the large integer and $t \in [0, 1]$ is a load parameter. Since $z_i^{q_i} - 1 \neq 0$, thus we will get a trivial solution when $t = 0$. Then, the authors rewrite (4) as

$$(1 - t)(z_i^{q_i} - 1) + t(P_i(z) + 1) = 0 \tag{5}$$

By assigning $z_i^{q_i} - 1 = G(z)$ and $P_i(z) + 1 = F(z)$, then we have

$$(1 - t)G(z) + tF(z) = 0 \tag{6}$$

From (6), we have

$$\frac{F(z)}{G(z)} = 1 - \frac{1}{t} \tag{7}$$

As $t \to 1$, we have

$$\lim_{t \to 1} \frac{F(z)}{G(z)} = 0 \tag{8}$$

Eq. (8) showed the relationship between the auxiliary homotopy function and the given function in the homotopy function (6). Garcia and Zangwill [4] also discussed the meaning of the terms - homotopy paths, monotonicity of the paths, simplicial pivoting and piecewise linear paths.

Melhem and Rheinboldt [5] compared several methods from local iterative and continuation path methods for determining turning points. For instance, Abbott's method [9], Moore and Spence's method [10], Seydel's method [11], Simpson's method [12], Rheninboldt's method [13] and so on. The results showed the Rheinboldt's method is most appropriate because it has the highest degree of reliability among all methods [5]. Since HCM is a method of solving divergence problem, therefore a body of knowledge about how to determine the turning points are become important.

Morgan [6] used SYMPOL (systems of polynomials) to conduct the implementation. SYMPOL was designed to find all solutions to a system of $n$ polynomial equations with complex coefficients in $n$ unknowns. The author also investigated three cases of choosing auxiliary homotopy function $G(x)$ having a theorem developed in [6]. The auxiliary functions selected to form the homotopy functions were

(i) $H(x, t) = (1 - t)(x - x_0) + t(-x^2 + 1)$ (9)

(ii) $H(x, t) = (1 - t)(x^2 - x_0^2) + t(-x^2 + 1)$ (10)

(iii) $H(x, t) = (1 - t)(x^3 - x_0^3) + t(-x^2 + 1)$ (11)

where the target function was $F(x) = -x^2 + 1 = 0$. Morgan [6] also described the basic concepts of HCM such as homotopy function and the auxiliary homotopy function. Homotopy function, denoted as $H(x, t)$, is a connection between the start and target functions, while the auxiliary homotopy function is the starting function.

According to Watson [7], homotopy methods are theoretically powerful, and if constructed and implemented properly, are robust, numerically stable, accurate, and practical.

In [8], the homotopy function was defined in the complex domain. The choice of auxiliary homotopy function is defined by

$$G(x) = b_i x_i^{d_i} - a_i \quad , \quad i = 1, 2, ..., n \tag{12}$$

where $a_i$ and $b_i$ are nonzero complex numbers. There are two types of total degree, denoted by $d_i$ and $d$. The total degree of polynomial $F(x)$, $d_i$ is

$$d_i = \max_k \sum_{j=1}^{n} d_{ijk} \tag{13}$$

and the total degree of the entire system (1), $d$ is

$$d = d_1 \times d_2 \times ... \times d_n. \tag{14}$$

According to Kotsireas [14], the total number of geometrically isolated solutions and solutions at infinity is no more than that given by equation (14). By using (12) and (13), the investigated homotopy functions by Morgan [6] can be simplified as

(i) $H(x,t) = (1-t)(x - x_0) + tF(x)$ (15)

(ii) $H(x,t) = (1-t)(x^{d_i} - x_0^{d_i}) + tF(x)$ (16)

(iii) $H(x,t) = (1-t)(x^{d_i+1} - x_0^{d_i+1}) + tF(x).$ (17)

Jalali and Seader [15] analyzed the stability of multiphase and reacting systems by using HCM. The accuracy of the initial guess was not important. Jalali and Seader focused on the use of the Newton, fixed-point, and affine homotopies which are as follows

(i) Newton homotopy

$$H(x,t) = tF(x) + (1-t)\left[F(x) - F(x_0)\right] \tag{18}$$

(i) Fixed-point homotopy

$$H(x,t) = tF(x) + (1-t)(x - x_0) \tag{19}$$

(i) Affine homotopy

$$H(x,t) = tF(x) + (1-t)F'(x_0)(x - x_0) \tag{20}$$

where $F(x)$ is the polynomial equations and $t \in [0, 1]$.

Gritton *et al.* [16] classified HCM as a global method which the user can use to find the solution from an arbitrary initial guess. For local methods such as the Newton method, the user should have sufficient knowledge regarding the location of a root to determine the initial guess. The closer the initial guess is to the solution, the more efficient is the local method. Otherwise, the iterative scheme of the local method will diverge away from the actual solution. Global methods overcome the problem of choosing the appropriate initial guess.

Gritton *et al.* [16] used HCM to study 16 chemical engineering problems involving isothermal flash, kinetics in a stirred reactor, azeotropic-point, flow in a smooth pipe, chemical equilibrium and others. One problem considered involved the equation,

$$F(x) = \frac{8(4-x)^2 x^2}{(6-3x)^2(2-x)} - 0.186 = 0 \tag{21}$$

where $x$ is the fractional conversion of nitrogen. The solutions were tracked by using HCM and adjusting auxiliary functions ( i.e. the initial guess) .

Allgower and Georg [17] defined the polynomial systems (1) as

$$F(z) = z^n + \sum_{j=0}^{n-1} a_j z^j \tag{22}$$

and equation (22) is a monic polynomial in which the coefficient of $z^n$ is equal to 1. Letting $G(z) = z^n + b_0$ where $b_0 \neq 0$, the authors defined homotopy function as

$$H(z,t) = (1-t)G(z) + tF(z). \tag{23}$$

Hence, we have

$$H(z,t) = z^n + t\sum_{j=0}^{n-1} a_j z^j + (1-t)b_0 + ta_0 \tag{24}$$

where $H(z,t)$ is measured in complex domain $H : C \times [0,1] \to C$.

Wu [18-22] has conducted extensive research on HCM. Wu [18] introduced Ancient Chinese Homotopy method (ACHM) and Wu [19] studied the convergence of Newton-Homotopy continuation method (NHCM). Wu [20] compared the traditional Adomian Decomposition method with the Adomian-homotopy continuation method (AHCM). Further, Wu [21] investigated the criteria of adjustable auxiliary homotopy function and its guidelines and Wu [22] developed secant-homotopy continuation method (SHCM) from traditional secant method.

Palancz *et al.* [23] described HCM in a simpler way. HCM was defined as a method that deforms continuously from the known roots of the start system into the roots of the target system. Continuation graphs were used to illustrate the mathematical problems. A simple equation was used to demonstrate the homotopy continuation concepts. Then, Palancz *et al.* [23] extended the concepts by discussing nonlinear geodetic problems such as resection, GPS positioning, as well as affine transformation.

Rahimian *et al.* [24] concerned with the use of homotopy functions

$$H(x,t) = tF(x) + (1-t)\left[(x-x_0) + (F(x) - F(x_0)\right] \tag{25}$$

to track the approximate solutions. The authors chose $G(x)$ as a linear combination of fixed point and Newton functions.

In most of above research, the focus was on the auxiliary homotopy function as well as homotopy continuation method and not the homotopy function. The authors [1-2,4,6,14-24] were more comfortable to use standard homotopy function rather than others. This paper will introduce a new homotopy function which will be called the Quadratic Bezier Homotopy Function (QBHF).

## 2 Standard Homotopy Function

All homotopy function $H(x,t)$ mentioned on above used the standard homotopy function. Let us consider the standard homotopy function

$$H(x,t) = (1-t)G(x) + tF(x) \tag{26}$$

where

$$G(x) = \begin{pmatrix} g_1(x_1, x_2, \cdots, x_n) \\ g_2(x_1, x_2, \cdots, x_n) \\ \vdots \\ g_n(x_1, x_2, \cdots, x_n) \end{pmatrix} \tag{27}$$

$$F(x) = \begin{pmatrix} f_1(x_1, x_2, \cdots, x_n) \\ f_2(x_1, x_2, \cdots, x_n) \\ \vdots \\ f_n(x_1, x_2, \cdots, x_n) \end{pmatrix} \tag{28}$$

and $t$ is an arbitrary parameter which can vary from 0 to 1, i.e. $t \in [0,1]$. Thus, we will have

$$H(x,0) = G(x). \tag{29}$$

$$H(x,1) = F(x). \tag{30}$$

The homotopy function cannot stand alone; it must be followed by a method which is called homotopy continuation method. According to Burden and Faires [1], the formula of Newton HCM is as follows

$$x_{i+1} = x_i - [D_x H(x_i,t)]^{-1} H(x_i,t) \quad , \quad i = 1,2,...,k. \tag{31}$$

where

$$D_x H(x,t) = \begin{pmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x_2} & \cdots & \frac{\partial H_1}{\partial x_n} \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial x_2} & \cdots & \frac{\partial H_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial H_n}{\partial x_1} & \frac{\partial H_n}{\partial x_2} & \cdots & \frac{\partial H_n}{\partial x_n} \end{pmatrix}.$$

$D_x H(x,t)$ is called the Jacobian matrix.

To facilitate better understanding, let us consider the *2*-dimensional of homotopy function which is $n = 1$.

$$H(x,t) = (1-t)g(x) + tf(x) \tag{32}$$

where $g(x)$ is an auxiliary homotopy function and $f(x)$ is a given function which is scalar.

**Example 2.1.** Consider the following simple equation defined by Kotsireas [14]

$$f(x) = (x^2 - \frac{1}{4})(x^2 - 4) = 0 \tag{33}$$

and the auxiliary homotopy function is $g(x) = x^2 - 1$. So that, the homotopy function (30) will be

$$H(x,t) = (1-t)(x^2-1) + t(x^2 - \frac{1}{4})(x^2 - 4) \tag{34}$$

Since we need to solve $H(x,t) = 0$, there are a set equations of $H_i(x, t_{i/k}) = 0$ and there are a set solutions for $x_i$ where $i = 0, 1, 2, \cdots, k$ and $k$ = number of iterations. If there are $k$ iterations, the number of equation involved is *k+1*. Let $k$ =*10*, so that the equation varies from *0* until 1. Graphically, it can be represented as Figure 1.

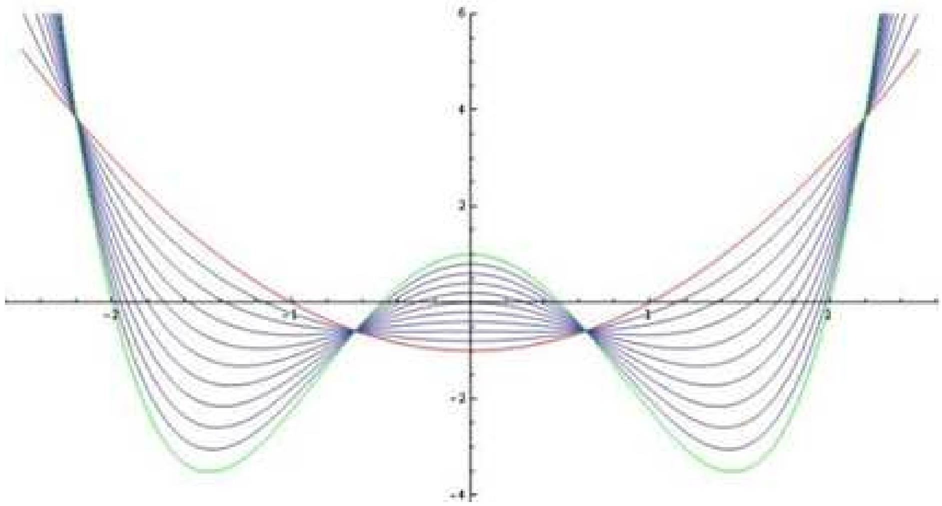


Figure 1: Homotopy Path of Equation (34)

Figure 1 shows the movement of $H(x_0, 0) = x^2 - 1 = g(x)$ to $H(x_{10}, 1) = (x^2 - \frac{1}{4})(x^2 - 4) = f(x)$ with $t$ being uniformly increased by 0.1 in equation (34). $x_0 = 1$ will move to the $x_{10} =$ 2.000618931 and $x_{10} =$ -2.000618931 when $x_0 = -1$. Both approximations will have the same value of $f(x_{10})$ i.e.

$$f(x_{10}) = 9.29 \times 10^{-3} \tag{35}$$

However, this approximation can be improved by using a new homotopy function which will be called the Quadratic Bezier Homotopy Function. This will be discussed in the next section.

## 3 Quadratic Bezier Homotopy Function

Suppose that we want to solve following a system of $n$ polynomial equations

$$F(x) = 0. \tag{36}$$

where $x = \{x_1, x_2, x_3, \cdots, x_{n-1}, x_n\}$. We introduce a new homotopy function viz

$$H_2(x,t) = (1-t)^2 G(x) + 2t(1-t)\left[(1-t)G(x) + tF(x)\right] + t^2 F(x). \tag{37}$$

This new homotopy function (37) can also be written as

$$H_2(x,t) = (1-t)^2 G(x) + 2t(1-t)H(x,t) + t^2 F(x) \tag{38}$$

where $H(x,t)$ is the standard homotopy function.

This new homotopy function fulfills the two boundary conditions i.e. (29) and (30) are still satisfied when we substitute $t=0$ and $t=1$ respectively into (38). It is interesting to note that there is a similar function of homotopy function between (26) and (38) when $t=\frac{1}{2}$ where

$$\begin{aligned} H(x,\tfrac{1}{2}) &= H_2(x,\tfrac{1}{2}) \\ &= \tfrac{1}{2}G(x)+\tfrac{1}{2}F(x). \end{aligned} \tag{39}$$

Since we want to solve $F(x)=0$, therefore $H_2(x,t)$ is set to zero by varying the parameter $t$ from 0 to 1. In other words, we start from auxiliary function, $G(x_0)=0$ and finish when $F(\tilde{x})=0$. As discussed before, the solution moves from $x_0$ until $\tilde{x}$ and the curves will move from $H_2(x_0,0)$ until $H_2(\tilde{x},1)$.

The idea of this new homotopy function comes from De Casteljau Algorithm [25]. It is known that, De Casteljau Algorithm describes the movement of point in a curve. Homotopy is a movement of a curve to another curve [14,23]. Therefore, we believe that there is relation between the De Casteljau algorithm and homotopy concepts.

In short, linear and quadratic curves for De Casteljau and homotopy can be formed as in Table 1.

Table 1: Linear and Quadratic for De Casteljau and Homotopy

| | De Casteljau, $P(t)$ | Homotopy, $H(x,t)$ |
|---|---|---|
| Linear | $(1-t)P_0+tP_1$ | $(1-t)G(x)+tF(x)$ |
| Quadratic | $(1-t)^2P_0+2t(1-t)P_1+t^2P_2$ | $(1-t)^2G(x)+$ $2t(1-t)\left[(1-t)G(x)+tF(x)\right]+t^2F(x).$ |

Let's observe how we obtain (38) with three reference curves. The recursive construction of quadratic Bezier homotopy function is illustrated in Figure 2.

We note that

$$A(x,t)=(1-t)G(x)+tH(x,t) \quad , \quad B(x,t)=(1-t)H(x,t)+tF(x)$$

Therefore

$$H_2(x,t)=(1-t)A+tB \tag{40}$$

Then, we have

$$\begin{aligned} H_2(x,t) &= (1-t)\left[(1-t)G(x)+tH(x,t)\right]+t\left[(1-t)H(x,t)+tF(x)\right] \\ &= (1-t)^2G(x)+2t(1-t)H(x,t)+t^2f(x) \\ &= H_0B_0^2(t)+H_1B_1^2(t)+H_2B_2^2(t) \\ &= \sum_{i=0}^{2} H_iB_i^2(t) \end{aligned} \tag{41}$$

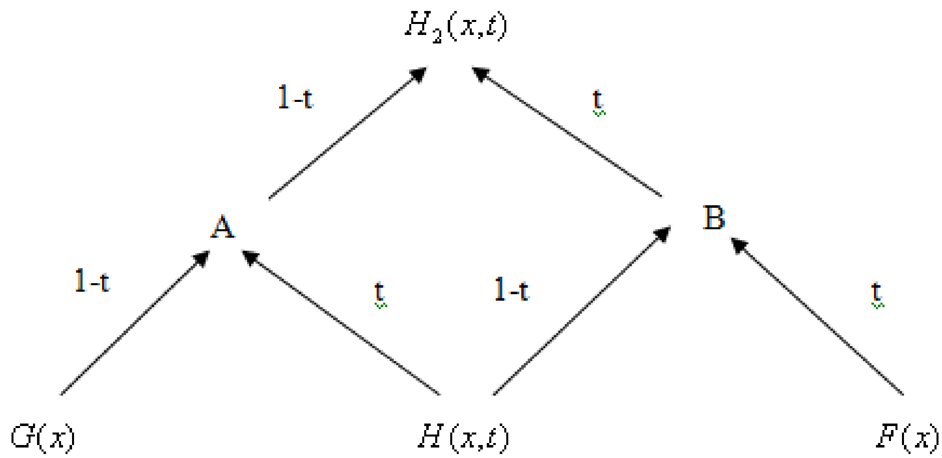


Figure 2: Recursive Construction of Quadratic Bezier Homotopy Function

where

$$\begin{aligned} H_0 &= G(x) && \text{for} \quad t = 0 \\ H_1 &= H(x,t) && \text{for} \quad t \in (0,1) \\ H_2 &= F(x) && \text{for} \quad t = 1. \end{aligned}$$

$B_i^2(t)$ is a Bernstein function which is defined as [25]

$$\begin{aligned} B_i^2(t) &= \begin{pmatrix} 2 \\ i \end{pmatrix} (1-i)^{2-i} t^i \\ &= \frac{2!}{i!\,(2-i)!} (1-i)^{2-i}\, t^i \end{aligned} \tag{42}$$

where $i : 0, 1, 2,\ t \in [0, 1]$.

An interesting property of Bernstein function for standard homotopy function and QBHF is that the summation is equal to one.

$$\text{Linear} \quad : \quad \sum_{i=0}^{1} B_i^1(t) \quad \begin{aligned} &= (1-t) + t \\ &= 1, \end{aligned} \tag{43}$$

$$\text{Quadratic} : \quad \sum_{i=0}^{1} B_i^2(t) \quad \begin{aligned} &= (1-t)^2 + 2t(1-t) + t^2 \\ &= 1. \end{aligned} \tag{44}$$

Both (43) and (44) show that the sum of binomial expansion always fulfill one of the convex hull property [25]. Another convex hull property is

$$B_i^2(t) \geq 0, \quad i : 0, 1, 2,\ t \in [0, 1]. \tag{45}$$

This continuation technique cannot stand alone in that it must be combined with other methods such as Newton, secant, Adomian method and so on. By these combinations, the name of the newly developed method will then change to Newton-Homotopy, Secant-Homotopy, Adomian-Homotopy and so on. Now, we will consider several examples that compares the standard and new homotopy functions. The method chosen is Newton-Homotopy continuation method (NHCM).

The formula of classical Newton method is well-known. The formula of NHCM is

$$x_{i+1} = x_i - \left[D_x H_2(x_i,t)\right]^{-1} H_2(x_i,t) \quad , \quad i = 1,2,...,k. \tag{46}$$

where $H_2(x,t)$ is Quadratic Bezier homotopy function as (38).

## 4 Numerical Experiments and Discussion

**Example 4.1.** Consider the following system of equations [2] :

$$\begin{aligned} f_1(x,y) &= x^2 - 2x - y + \tfrac{1}{2} = 0, \\ f_2(x,y) &= x^2 + 4y^2 - 4 = 0. \end{aligned} \tag{47}$$

The auxiliary homotopy function is $g_1(x) = x$, $g_2(y) = y$ and the initial value $(x_0, y_0) = (0,0)$are used for NHCM. The results are shown in Table 2 by varying the number of iterations.

Table 2: Comparison between Standard and New Homotopy Functions for Equation (47)

| Number of iterations | Standard Homotopy Function | Quadratic Bezier Homotopy Function | CPU Time, *second* |
|---|---|---|---|
| 10 | $f_1 = 4.90 \times 10^{-3}$<br>$f_2 = 2.93 \times 10^{-2}$ | $f_1 = 2.38 \times 10^{-4}$<br>$f_2 = 2.63 \times 10^{-3}$ | 0.0070004 |
| 100 | $f_1 = 3.83 \times 10^{-5}$<br>$f_2 = 2.96 \times 10^{-4}$ | $f_1 = 3.28 \times 10^{-8}$<br>$f_2 = 2.63 \times 10^{-7}$ | 0.0300017 |
| 1000 | $f_1 = 3.72 \times 10^{-7}$<br>$f_2 = 2.96 \times 10^{-6}$ | $f_1 = 3.33 \times 10^{-12}$<br>$f_2 = 2.66 \times 10^{-11}$ | 0.2180124 |

where $(\tilde{x}, \tilde{y}) = (1.90067672637127, 0.31121856542355)$

**Example 4.2.** Consider the following example [26]:

$$\begin{aligned} f_1(x,y,z) &= x^2 + y^2 + z^2 - 1 = 0 \\ f_2(x,y,z) &= 2x^2 + y^2 - 4z = 0 \\ f_3(x,y,z) &= 3x^2 - 4y^2 + z^2 = 0 \end{aligned} \tag{48}$$

The auxiliary homotopy function, $g_1(x) = x$, $g_2(y) = y$ and $g_3(z) = z$ and initial value $(x_0, y_0, z_0) = (0,0,0)$ are used for NHCM. The results are shown in Table 3.

Table 3: Comparison between Standard and New Homotopy Functions for Equation (48)

| Number of iterations | Standard Homotopy Function | Quadratic Bezier Homotopy Function | CPU Time, *second* |
|---|---|---|---|
| 10 | $f_1 = 2.12 \times 10^{-3}$<br>$f_2 = 1.21 \times 10^{-3}$<br>$f_3 = 1.75 \times 10^{-3}$ | $f_1 = 1.36 \times 10^{-4}$<br>$f_2 = 7.23 \times 10^{-5}$<br>$f_3 = 1.13 \times 10^{-4}$ | 0.0200011 |
| 100 | $f_1 = 1.82 \times 10^{-5}$<br>$f_2 = 1.04 \times 10^{-5}$<br>$f_3 = 1.56 \times 10^{-5}$ | $f_1 = 1.59 \times 10^{-8}$<br>$f_2 = 9.02 \times 10^{-9}$<br>$f_3 = 1.36 \times 10^{-8}$ | 0.1260072 |
| 1000 | $f_1 = 1.79 \times 10^{-7}$<br>$f_2 = 1.02 \times 10^{-7}$<br>$f_3 = 1.54 \times 10^{-7}$ | $f_1 = 1.61 \times 10^{-12}$<br>$f_2 = 9.14 \times 10^{-13}$<br>$f_3 = 1.38 \times 10^{-12}$ | 1.1730670 |

where $(\tilde{x}, \tilde{y}, \tilde{z}) =$ (0.69828860997219,-0.62852429796055,0.34256418968992)

**Example 4.3.** Consider the following example [6]:

$$\begin{aligned} f_1(w,x,y,z) &= x + 10y = 0 \\ f_2(w,x,y,z) &= \sqrt{5}(z-w) = 0 \\ f_3(w,x,y,z) &= (y-2z)^2 = 0 \\ f_4(w,x,y,z) &= \sqrt{10}(x-w)^2 = 0 \end{aligned} \tag{49}$$

The auxiliary homotopy function, $g_1(w) = w-1$, $g_2(x) = x-4$, $g_3(y) = y-1$, $g_4(z) = z-2$ and initial value is $(w_0, x_0, y_0, z_0) = (1, 4, 1, 2)$ are used. The results are shown in Table 4.

The results in Table of Eq. (47), (48), and (49) show that Quadratic Bezier Homotopy Function has better performance than the standard homotopy function for solving the considered system of nonlinear algebraic equation. This has been ascertained by using the following stopping criteria

$$\left\| F(\tilde{X}^{k+1}) \right\|_\infty < \varepsilon \tag{50}$$

where $\varepsilon = 10^{-20}$. CPU time is assumed not to be an important consideration.

## 5 Conclusion

From the particular set of examples chosen, the use of Quadratic Bezier Homotopy Function (QBHF) is better than the use of standard homotopy function. It should be noted though that for the particular set of examples considered, the accuracy of approximate solutions increases when the numbers of iterations increase.

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Table 4: Comparison between Standard and New Homotopy Functions for Equation (49)

| Number of iterations | Standard Homotopy Function | Quadratic Bezier Homotopy Function | CPU Time, *second* |
|---|---|---|---|
| 10 | $f_1 = 2.78 \times 10^{-17}$<br>$f_2 = 7.14 \times 10^{-16}$<br>$f_3 = 2.93 \times 10^{-2}$<br>$f_4 = 7.21 \times 10^{-2}$ | $f_1 = -3.89 \times 10^{-16}$<br>$f_2 = 4.97 \times 10^{-16}$<br>$f_3 = 1.79 \times 10^{-2}$<br>$f_4 = 2.27 \times 10^{-2}$ | 0.1240071 |
| 100 | $f_1 = -6.37 \times 10^{-16}$<br>$f_2 = -2.72 \times 10^{-15}$<br>$f_3 = 2.88 \times 10^{-3}$<br>$f_4 = 5.61 \times 10^{-3}$ | $f_1 = 6.14 \times 10^{-16}$<br>$f_2 = -4.09 \times 10^{-16}$<br>$f_3 = 1.32 \times 10^{-4}$<br>$f_4 = 2.60 \times 10^{-4}$ | 0.4270244 |
| 1000 | $f_1 = -6.67 \times 10^{-16}$<br>$f_2 = -2.67 \times 10^{-15}$<br>$f_3 = 2.86 \times 10^{-4}$<br>$f_4 = 5.78 \times 10^{-4}$ | $f_1 = 4.20 \times 10^{-16}$<br>$f_2 = -4.00 \times 10^{-16}$<br>$f_3 = 1.33 \times 10^{-6}$<br>$f_4 = 2.65 \times 10^{-6}$ | 3.4601979 |

where $(\tilde{w}, \tilde{x}, \tilde{y}, \tilde{z})$ = (0.00050521152754,0.00142117446223,-0.00014211744622,0.00050521152754) and the actual solution is $(w, x, y, z) = (0, 0,0,0)$ as stated in Morgan [6]

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