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Some Properties of Nilpotent Lie Algebras

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Abstract In this paper we prove an analogue of Robinson Theorem for Lie algebras, which plays an important role to find the results connecting to the idea of nilpotency in Lie algebras. Also we find a criterion such that an extension of Lie algebras can be nilpotent.

Keywords Nilpotent Lie algebras, Non-abelian tensor product

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1 Introduction

Throughout this paper all Lie algebras are considered over a fixed field F and $[,]$ denotes the Lie bracket. A Lie algebra is called *nilpotent* if it has a *central series*, that is a series

$$\langle 0\rangle = L_0 \subseteq L_1 \subseteq ... \subseteq L_t = L$$

of ideals L such that L_{i+1}/L_i is contained in the center of L/L_i for all i (or equivalent by $[L_{i+1}, L] \subseteq L_i$, for all i).

Recall that the *lower central series* of a Lie algebra L is defined to be the series with terms

$$\gamma_1(L) = L \quad \text{and} \quad \gamma_{k+1}(L) = [\gamma_k(L), L] \quad \text{for any natural number } k.$$

Then

$$L = \gamma_1(L) \supseteq \gamma_2(L) \supseteq \dots.$$

One may see that a Lie algebra L is nilpotent if and only if the lower central series reaches the identity after a finite number steps. Robinson [1] showed that how the first lower central factor $G_{ab} = G/G'$ exerts a very strong influence on subsequent lower central factors of a group G. We prove an analogue of Robinson Theorem for Lie algebras. It is known that an extension of a nilpotent group by another nilpotent group may not be nilpotent in general. The following example shows that an extension of a nilpotent Lie algebra by another nilpotent Lie algebra may not be nilpotent in general.

Example 1 Let L be a 2-dimensional non-abelian Lie algebra. Then it have a basis $\{x, y\}$ with $[x, y] = x$. Hence, if put $M = \langle x\rangle$, then M is an ideal of L such that M and L/M are nilpotent but L is not.

Hall [1] obtained a criterion under which such an extension of groups can be nilpotent. We prove an analogue of the Hall Theorem for Lie algebras. Indeed we prove if M is an ideal of lie algebra L such that M and $L/[M, M]$ are nilpotent, then L is nilpotent.

2 Preliminaries

In this section we apply the notation and terminology of [2], which will be used in the proof of our results.

Definition 1 For any two arbitrary Lie algebras L and K, an *action* of L on K means a F-bilinear map $L \times K \to K$ sending (l, k) to ${}^{l}k$ satisfying

$$\begin{aligned} &(i) \quad {}^{[l,l']}k = {}^{l}({}^{l'}k) - {}^{l'}({}^{l}k), \\ &(ii) \quad {}^{l}[k, k'] = [{}^{l}k, k'] + [k, {}^{l}k'], \end{aligned}$$

for all $l, l' \in L$ and $k, k' \in K$.

If L is a subalgebra of some Lie algebra P and K is an ideal in P, then the Lie multiplication in P can induce an action of L on K via ${}^{l}k = [l, k]$, for all $l \in L$ and $k \in K$.

Definition 2 Let L and K be Lie algebras acting on each other and, on themselves by Lie multiplications. Then for each Lie algebra P a bilinear function $\alpha : L \times K \to P$ is called a *Lie pairing* if for all $l, l' \in L$ and $k, k' \in K$,

$$\begin{aligned} &(i) \quad \alpha([l, l'], k) = \alpha(l, {}^{l'}k) - \alpha(l', {}^{l}k), \\ &(ii) \quad \alpha(l, [k, k']) = \alpha({}^{k'}l, k) - \alpha({}^{k}l, k'), \\ &(iii) \quad \alpha({}^{k}l, {}^{l'}k') = -[\alpha(l, k), \alpha(l', k')]. \end{aligned}$$

Now we give the definition of the tensor product of Lie algebras due to Ellis[2].

Definition 3 The *non-abelian tensor product* $L \otimes K$ of the Lie algebras L and K is the Lie algebra generated by the symbols $l \otimes k (l \in L, k \in K)$ with the following defining relations:

$$\begin{aligned} &(i) && r(l \otimes k) = rl \otimes k = l \otimes rk, \\ &(ii) && (l + l') \otimes k = l \otimes k + l' \otimes k, \\ & && l \otimes (k + k') = l \otimes k + l \otimes k', \\ &(iii) && [l, l'] \otimes k = l \otimes ({}^{l'}k) - l' \otimes ({}^{l}k), \\ & && l \otimes [k, k'] = ({}^{k'}l) \otimes k - ({}^{k}l) \otimes k', \\ &(iv) && [(l \otimes k), (l' \otimes k')] = -({}^{k}l) \otimes ({}^{l'}k'), \end{aligned}$$

for all $r \in F, l, l' \in L$ and $k, k' \in K$.(see [2] and [3] for more information)

Lie pairings allow us to determine homomorphic images of $L \otimes K$ as follows. The proof of the following lemma is left to the reader.

Lemma 1 For every Lie algebra P, L, K and each Lie pairing $\varphi : L \times K \to P$, there exists a unique homomorphism $\varphi^* : L \otimes K \to P$ such that $\varphi^*(l \otimes k) = \varphi(l, k)$ for all $l \in L, k \in K$.

In [2,3] the results on the non-abelian tensor product $L \otimes K$ are obtained by assuming the actions of L and K on each other are *compatible*, in the following sense.

Definition 4 The actions are *compatible* if

$$^{(^k l)}k' = [k', {}^l k] \text{ and } {}^{(^l k)}l' = [l', {}^k l]$$

for all $l, l' \in L$ and $k, k' \in K$.

The following definition is very useful in our further investigations.

Definition 5 Let L and K be Lie algebras such that L acts on K. Then we say L *acts nilpotently* on K if K having the series

$$\langle 0\rangle = K_0 \subseteq K_1 \subseteq ... \subseteq K_t = K$$

of ideals of K, where ${}^l k_{i+1} \in K_i, (i \geq 0)$ for all $l \in L$ and $k_{i+1} \in K_{i+1}$.

Remark 1 Let M and N be ideals of some Lie algebra L such that $N \subseteq M$. Then the Lie algebra L acts on $\frac{M}{N}$ by the following defined action:

$${}^l(m + N) = [l, m] + N \text{ for } l \in L \text{ and } m \in M.$$

Hence, L acts nilpotently on $\frac{M}{N}$, if $\frac{M}{N}$ having the series

$$\langle 0\rangle = \tfrac{M_0}{N} \subseteq \tfrac{M_1}{N} \subseteq ... \subseteq \tfrac{M_t}{N} = \tfrac{M}{N}$$

where $[L, M_{i+1}] \subseteq M_i$ for all $i \geq 0$.

3 The results

The following theorem is analogue to the work of Robinson for Lie algebras.

Theorem 1 *Let L be a Lie algebra and $F_i = \frac{\gamma_i(L)}{\gamma_{i+1}(L)}$ for $i \geq 1$. Then the map*

$$F_i \otimes \frac{L}{[L, L]} \to F_{i+1} \tag{1}$$

$$(x + \gamma_{i+1}(L)) \otimes (l + [L, L]) \mapsto [x, l] + \gamma_{i+2}(L) \tag{2}$$

is a well-defined epimorphism.

Note that in the above theorem it is obvious that the Lie algebras $\frac{\gamma_i(L)}{\gamma_{i+1}(L)}$ and $\frac{L}{[L,L]}$ act trivial on each other.

Proof We define a function α of the Lie algebra $F_i \times \frac{L}{[L,L]}$ to the Lie algebra F_{i+1}, given by

$$(x + \gamma_{i+1}(L), l + [L, L]) \mapsto [x, l] + \gamma_{i+2}(L)$$

for all $x \in \gamma_i(L)$ and $l \in L$. Since the Lie algebras $\frac{\gamma_i(L)}{\gamma_{i+1}(L)}$ and $\frac{L}{[L,L]}$ act trivial on each other and $\gamma_{i+1}(L) = [\gamma_i(L), L]$ for any natural number i, we have

$$\alpha([\overline{x}, \overline{y}], \overline{l}) = \alpha([\overline{x}, {}^{\overline{y}}\,\overline{l}]) - \alpha([\overline{y}, {}^{\overline{x}}\,\overline{l}])$$

such that $\overline{x} = x + \gamma_{i+1}(L)$, $\overline{y} = y + \gamma_{i+1}(L)$ and $\overline{l} = l + [L, L]$ for all $x, y \in \gamma_i(L)$ and $l \in L$. Similarly the function α satisfy in the other conditions of Definition 2. Hence, the function α is a Lie pairing. So, by the Lemma 1 there exists an induced homomorphism of the Lie algebra $F_i \otimes \frac{L}{[L,L]}$ to the Lie algebra F_{i+1}, given by

$$(x + \gamma_{i+1}(L)) \otimes (l + [L, L]) \mapsto [x, l] + \gamma_{i+2}(L)$$

for all $x \in \gamma_i(L)$ and $l \in L$. It can be easily seen that the induced homomorphism is onto, and the proof is complete. $\square$

In the following corollary, we intend to give sufficient conditions under which a Lie algebra can be finite-dimensional.

Corollary 1 If L is a nilpotent Lie algebra such that $\frac{L}{[L,L]}$ is finite-dimensional, then L is finite-dimensional.

Proof Let $F_i = \frac{\gamma_i(L)}{\gamma_{i+1}(L)}$ be finite-dimensional. Then by Theorem 1, F_{i+1} is also finite-dimensional, since the finite-dimensional property is inherited by images of tensor products. Hence, by induction on i every lower central factor is finite-dimensional. Since L is nilpotent, then $\gamma_{c+1}(L) = \langle 0 \rangle$, for some non-negative integer c. As the finite-dimensional property is closed under forming extensions, thus L is finite-dimensional. $\square$

Lemma 2 Let A, B, C and D be ideals of a Lie algebra L such that $B \subseteq A$ and $D \subseteq C$. Also let $\frac{A}{B}$ and $\frac{C}{D}$ act compatibly on each other. If L acts nilpotently on $\frac{A}{B}$ and $\frac{C}{D}$, then L acts nilpotently on $\frac{A}{B} \otimes \frac{C}{D}$.

Proof Since L acts nilpotently on $\frac{A}{B}$ and $\frac{C}{D}$, then there are series

$$\langle 0 \rangle = \frac{A_0}{B} \subseteq \frac{A_1}{B} \subseteq ... \subseteq \frac{A_t}{B} = \frac{A}{B} \text{ and } \langle 0 \rangle = \frac{C_0}{D} \subseteq \frac{C_1}{D} \subseteq ... \subseteq \frac{C_s}{D} = \frac{C}{D}$$

of ideals $\frac{A}{B}$ and $\frac{C}{D}$ respectively, such that $[L, A_{i+1}] \subseteq A_i$ and $[L, C_{j+1}] \subseteq C_j$, for all $0 \leq i < t$ and $0 \leq j < s$. We claim L acts on $\frac{A}{B} \otimes \frac{C}{D}$ by the following defined action:

$${}^{l}(\overline{a} \otimes \overline{c}) = \overline{[l, a]} \otimes \overline{c} + \overline{a} \otimes \overline{[l, c]}$$

such that $l \in L$, $\overline{a} = a + B \in \frac{A}{B}$ and $\overline{c} = c + D \in \frac{C}{D}$. With using Remark 1 we have

$$\begin{aligned}
{}^{[l,l']}(\overline{a} \otimes \overline{c}) &= \overline{[[l, l'], a]} \otimes \overline{c} + \overline{a} \otimes \overline{[[l, l'], c]} \\
&= \overline{[l, [l', a]] - [l', [l, a]]} \otimes \overline{c} + \overline{a} \otimes \overline{[l, [l', c]] - [l', [l, c]]} \\
&= (\overline{[l, [l', a]]} \otimes \overline{c} - \overline{[l', [l, a]]} \otimes \overline{c}) + (\overline{a} \otimes \overline{[l, [l', c]]} - \overline{a} \otimes \overline{[l', [l, c]]}) \\
&= (\overline{[l, [l', a]]} \otimes \overline{c} + \overline{a} \otimes \overline{[l, [l', c]]}) - (\overline{[l', [l, a]]} \otimes \overline{c} + \overline{a} \otimes \overline{[l', [l, c]]}) \\
&= (\overline{[l, [l', a]]} \otimes \overline{c} + \overline{[l', a]} \otimes \overline{[l, c]} + \overline{[l, a]} \otimes \overline{[l', c]} + \overline{a} \otimes \overline{[l, [l', c]]}) - \\
&\quad (\overline{[l', [l, a]]} \otimes \overline{c} + \overline{[l, a]} \otimes \overline{[l', c]} + \overline{[l', a]} \otimes \overline{[l, c]} + \overline{a} \otimes \overline{[l', [l, c]]}) \\
&= {}^{l}(\overline{[l', a]} \otimes \overline{c} + \overline{a} \otimes \overline{[l', c]}) - {}^{l'}(\overline{[l, a]} \otimes \overline{c} + \overline{a} \otimes \overline{[l, c]}) \\
&= {}^{l}({}^{l'}(\overline{a} \otimes \overline{c})) - {}^{l'}({}^{l}(\overline{a} \otimes \overline{c})),
\end{aligned}$$

for all $l, l' \in L$, $\overline{a} = a + B \in \frac{A}{B}$ and $\overline{c} = c + D \in \frac{C}{D}$. Similarly

$$^{l}[\overline{a} \otimes \overline{c}, \overline{a'} \otimes \overline{c'}] = [^{l}(\overline{a} \otimes \overline{c}), \overline{a'} \otimes \overline{c'}] + [\overline{a} \otimes \overline{c}, {}^{l}(\overline{a'} \otimes \overline{c'})],$$

for all $l \in L$, $\overline{a} = a + B \in \frac{A}{B}$, $\overline{a'} = a' + B \in \frac{A}{B}$, $\overline{c} = c + D \in \frac{C}{D}$ and $\overline{c'} = c' + D \in \frac{C}{D}$.
Now let $\overline{a_i} = a_i + B \in \frac{A_i}{B}$ and $\overline{c_j} = c_j + D \in \frac{C_j}{D}$. Then we construct the following series of ideals $T = \frac{A}{B} \otimes \frac{C}{D}$:

$$\langle 0 \rangle = T_0 \subseteq T_1 \subseteq ... \subseteq T_{t+s} = T$$

where $T_r = \langle \overline{a_m} \otimes \overline{c_n} \mid m + n \leq r, 0 \leq m \leq t, 0 \leq n \leq s \rangle$. Hence, for any $l \in L$ and $\overline{a_m} \otimes \overline{c_n} \in T_{r+1}$ we have $^{l}(\overline{a_m} \otimes \overline{c_n}) = \overline{[l, a_m]} \otimes \overline{c_n} + \overline{a_m} \otimes \overline{[l, c_n]} \in T_r$. This completes the proof. □

Now we are able to prove the analogue of Hall Theorem for Lie algebras.

Theorem 2 *Let M be an ideal of Lie algebra L. If M and $\frac{L}{[M,M]}$ are nilpotent Lie algebras, then L is nilpotent.*

Proof Since $\frac{L}{[M,M]}$ is nilpotent, then it has a central series as follows:

$$\langle 0 \rangle = \frac{L_0}{[M,M]} \subseteq \frac{L_1}{[M,M]} \subseteq ... \subseteq \frac{L_t}{[M,M]} = \frac{L}{[M,M]} \qquad (1).$$

Now we construct the following series for $\frac{M}{[M,M]}$:

$$\langle 0 \rangle = \frac{M_0}{[M,M]} \subseteq \frac{M_1}{[M,M]} \subseteq ... \subseteq \frac{M_t}{[M,M]} = \frac{M}{[M,M]},$$

where $M_j = L_j \cap M$, $(0 \leq j \leq t)$. The Lie algebra L acts on $\frac{M}{[M,M]}$ by the following defined action:

$$^{l}(m + [M, M]) = [l, m] + [M, M] \text{ for } l \in L \text{ and } m \in M.$$

Hence, by the above central series, the action of L on $\frac{M}{[M,M]}$ is nilpotent. Put $F_i = \frac{\gamma_i(M)}{\gamma_{i+1}(M)}$ for $i \geq 1$. Then L acts nilpotently on F_1. Suppose that L acts nilpotently on F_i, then by Lemma 2, L acts nilpotently on $F_i \otimes \frac{M}{[M,M]}$. By Theorem 1, F_{i+1} is an image of $F_i \otimes \frac{M}{[M,M]}$ and hence, L acts nilpotently on F_{i+1}. Therefore by induction on i, L acts nilpotently on every lower central factor of M. Since M is nilpotent, then there exists a non-negative integer c such that $\gamma_{c+1}(M) = \langle 0 \rangle$. Now, combining the lower central series of M and (1) we obtain

$$\langle 0 \rangle = \gamma_{c+1}(M) \subseteq ... \subseteq \gamma_2(M) = [M, M] = L_0 \subseteq ... \subseteq L_t = L.$$

By the fact that L acts nilpotently on F_i, there is a series

$$\langle 0 \rangle = \frac{K_{i_1}}{\gamma_{i+1}(M)} \subseteq ... \subseteq \frac{K_{i_r}}{\gamma_{i+1}(M)} = F_i$$

such that $[L, K_{i_{j+1}}] \subseteq K_{i_j}$. Now we obtain a central series of L which provides the nilpotency of L, as required. □

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