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Some Ideal Convergent Sequence Spaces Defined by a Sequence of Modulus Functions Over n-Normed Spaces

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Abstract In the present paper we study some ideal convergent sequence spaces defined by a sequence of modulus functions in n-normed spaces and examine some topological properties and inclusion relations between these spaces.

Keywords Lacunary sequence, Difference sequence space, Modulus function, Ideal convergence, n-normed space.

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1 Introduction

The concept of 2-normed spaces was initially developed by Gähler [1] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [2]. Since then, many others have studied this concept and obtained various results, see Gunawan ([3], [4]) and Gunawan and Mashadi [5].

Definition 1 Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

(i) $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;

(ii) $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;

(iii) $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \; ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and

(iv) $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called a n-norm on X and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm

$$||x_1, x_2, \cdots, x_n||_E$$

to be equal to the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_n||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.

The standard n-norm on X, a real inner product space of dimension $d \geq n$ is as follows:

$$||x_1, x_2, \cdots, x_n||_S = \begin{vmatrix} \langle x_1, x_1 \rangle & . & . & . & \langle x_1, x_n \rangle \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ \langle x_n, x_1 \rangle & . & . & . & \langle x_n, x_n \rangle \end{vmatrix}^{\frac{1}{2}},$$

where $\langle ., . \rangle$ denotes the inner product on X. If $X = \mathbb{R}^n$, then this n-norm is exactly the same as the Euclidean n-norm $||x_1, x_2, \cdots, x_n||_E$ mentioned earlier. For $n = 1$, this n-norm is the usual norm $||x|| = \langle x_1, x_1 \rangle^{\frac{1}{2}}$.

Definition 2 A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k \to \infty} ||x_k - L, z_1, \cdots, z_n|| = 0 \text{ for every } z_1, \cdots, z_n \in X.$$

Definition 3 A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{\substack{k \to \infty \\ p \to \infty}} ||x_k - x_p, z_1, \cdots, z_n|| = 0 \text{ for every } z_1, \cdots, z_n \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

The notion of ideal convergence was introduced first by Kostyrko [6] as a generalization of statistical convergence which was further studied in topological spaces by Das *et al.* [7]. More applications of ideals can be seen in Das *et al.* [7] and Das *et al.* [8].

Let $(X, ||.||)$ be a normed space. Recall that a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of X is called statistically convergent to $x \in X$ if the set $A(\epsilon) = \Big\{n \in \mathbb{N} : ||x_n - x|| \geq \epsilon\Big\}$ has natural density zero for each $\epsilon > 0$.

Definition 4 A family $\mathcal{I} \subset 2^Y$ of subsets of a non empty set Y is said to be an ideal in Y if

(i) $\phi \in \mathcal{I}$

(ii) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$

(iii) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$ [9].

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a non trivial ideal in $\mathbb{N}$. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-convergent to $x \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \Big\{n \in \mathbb{N} : ||x_n - x|| \geq \epsilon\Big\}$ belongs to $\mathcal{I}$ [6].

A sequence of positive integers $\theta = (k_r)$ is called lacunary if $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r)$ and $q_r = \frac{k_r}{k_{r-1}}$. The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [10] as:

$$N_\theta = \Big\{x \in w : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - l| = 0, \text{ for some } \ l\Big\}.$$

Strongly almost convergent sequence was introduced and studied by Maddox [11] and Freedman et al. [10]. Parashar and Choudhary [12] have introduced and examined some properties of four sequence spaces defined by using an Orlicz function M, which generalized the well-known Orlicz sequence spaces $[C, 1, p]$, $[C, 1, p]_0$ and $[C, 1, p]_\infty$. It may be noted here that the space of strongly summable sequences were discussed by Maddox [13].

The notion of difference sequence spaces was introduced by Kızmaz [14], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [15] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all real or complex sequences $x = (x_k)$. Let m, n be non-negative integers, then for $Z = l_\infty$, c and c_0, we have sequence spaces,

$$Z(\Delta^n_m) = \{x = (x_k) \in w : (\Delta^n_m x_k) \in Z\}$$

where $\Delta^n_m x = (\Delta^n_m x_k) = (\Delta^{n-1}_m x_k - \Delta^{n-1}_m x_{k+m})$ and $\Delta^0_m x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^n_m x_k = \sum_{v=0}^{n} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} x_{k+mv}.$$

Taking $m = n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kızmaz [14].

Definition 5 A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$, $y \geq 0$,

(iii) f is increasing,

(iv) f is continuous from right at 0.
It follows from (i) and (iv) that f must be continuous everywhere on $[0, \infty)$. For a sequence of modulus function $F = (f_k)$, we give the following conditions:

(v) $\sup_k f_k(x) < \infty$ for all $x > 0$,

(vi) $\lim_{x\to 0} f_k(x) = 0$ uniformly in $k \geq 1$.

We remark that in case $f = (f_k)$ for all k, where f is a modulus, the conditions (v) and (vi) are automatically fulfilled. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus $f(x)$ is unbounded. Subsequently, modulus function has been discussed [16–21].

Let $\mathcal{I}$ be an admissible ideal, $F = (f_k)$ be a sequence of modulus functions, $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space, $p = (p_k)$ be a sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. By $w(n-X)$ we denote the space of all sequences defined over n- normed space $(X, ||\cdot, \cdots, \cdot||)$. In the present paper, we define the following classes of sequences:

$$[N_\theta, \Delta_n^m, F, u, p, ||\cdot, \cdots, \cdot||, X_S]^{\mathcal{I}} =$$

$$\Big\{x = (x_k) \in w(n-X) : \Big[r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \big[f_k\big(||\Delta_n^m x_k - L, z_1, z_2, \cdots, z_{n-1}||\big)\big]^{p_k} \geq \varepsilon,$$

$$\text{for some } L \text{ and for every } z_1, z_2, \cdots, z_{n-1} \in X\Big] \in \mathcal{I}\Big\},$$

$$[N_\theta, \Delta_n^m, F, u, p, ||\cdot, \cdots, \cdot||, X_S]_0^{\mathcal{I}} =$$

$$\Big\{x = (x_k) \in w(n-X) : \Big[r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \big[f_k\big(||\Delta_n^m x_k, z_1, z_2, \cdots, z_{n-1}||\big)\big]^{p_k} \geq \varepsilon,$$

$$\text{for every } z_1, z_2, \cdots, z_{n-1} \in X\Big] \in \mathcal{I}\Big\}.$$

If $F(x) = x$, we get

$$[N_\theta, \Delta_n^m, u, p, ||\cdot, \cdots, \cdot||, X_S]^{\mathcal{I}} =$$

$$\Big\{x = (x_k) \in w(n-X) : \Big[r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \big(||\Delta_n^m x_k - L, z_1, z_2, \cdots, z_{n-1}||\big)^{p_k} \geq \varepsilon,$$

$$\text{for some } L \text{ and for every } z_1, z_2, \cdots, z_{n-1} \in X\Big] \in \mathcal{I}\Big\},$$

$$[N_\theta, \Delta_n^m, u, p, ||\cdot, \cdots, \cdot||, X_S]_0^{\mathcal{I}} =$$

$$\Big\{x = (x_k) \in w(n-X) : \Big[r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \big(||\Delta_n^m x_k, z_1, z_2, \cdots, z_{n-1}||\big)^{p_k} \geq \varepsilon,$$

$$\text{for every } z_1, z_2, \cdots, z_{n-1} \in X\Big] \in \mathcal{I}\Big\}.$$

If $p = (p_k) = 1$, we get

$$[N_\theta, \Delta_n^m, F, u, ||\cdot, \cdots, \cdot||, X_S]^{\mathcal{I}} =$$

$$\Big\{x = (x_k) \in w(n-X) : \Big[r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \big[f_k\big(||\Delta_n^m x_k - L, z_1, z_2, \cdots, z_{n-1}||\big)\big] \geq \varepsilon,$$

$$\text{for some } L \text{ and for every } z_1, z_2, \cdots, z_{n-1} \in X\Big] \in \mathcal{I}\Big\},$$

$[N_\theta, \Delta_n^m, F, u, ||\cdot,\cdots,\cdot||, X_S]_0^{\mathcal{I}} =$

$$\Big\{x = (x_k) \in w(n-X) : \Big[r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} u_k\big[f_k\big(||\Delta_n^m x_k, z_1, z_2, \cdots, z_{n-1}||\big)\big] \geq \varepsilon,$$

for every $z_1, z_2, \cdots, z_{n-1} \in X\Big] \in \mathcal{I}\Big\}$.

If $p = (p_k) = 1$ and $u = (u_k) = 1$, we get
$[N_\theta, \Delta_n^m, F, ||\cdot,\cdots,\cdot||, X_S]^{\mathcal{I}} =$

$$\Big\{x = (x_k) \in w(n-X) : \Big[r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} \big[f_k\big(||\Delta_n^m x_k - L, z_1, z_2, \cdots, z_{n-1}||\big)\big] \geq \varepsilon,$$

for some L and for every $z_1, z_2, \cdots, z_{n-1} \in X\Big] \in \mathcal{I}\Big\}$,
$[N_\theta, \Delta_n^m, p, ||\cdot,\cdots,\cdot||, X_S]_0^{\mathcal{I}} =$

$$\Big\{x = (x_k) \in w(n-X) : \Big[r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} \big[f_k\big(||\Delta_n^m x_k, z_1, z_2, \cdots, z_{n-1}||\big)\big] \geq \varepsilon,$$

for every $z_1, z_2, \cdots, z_{n-1} \in X\Big] \in \mathcal{I}\Big\}$.

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $D = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The aim of this paper is to study difference $\mathcal{I}$-convergent sequence spaces defined by a sequence of modulus functions in n-normed spaces and examine some topological properties and inclusion relations between the spaces $[N_\theta, \Delta_n^m, F, u, p, ||\cdot,\cdots,\cdot||, X_S]^{\mathcal{I}}$ and $[N_\theta, \Delta_n^m, F, u, p, ||\cdot,\cdots,\cdot||, X_S]_0^{I}$.

2 Main Results

Theorem 1 *Let $F = (f_k)$ be a sequence of modulus functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers, then $[N_\theta, \Delta_n^m, F, u, p, ||\cdot,\cdots,\cdot||, X_S]^{\mathcal{I}}$ and $[N_\theta, \Delta_n^m, F, u, p, ||\cdot,\cdots,\cdot||, X_S]_0^{\mathcal{I}}$ are linear spaces over the field of complex number $\mathbb{C}$.*

Proof Let $x = (x_k)$, $y = (y_k) \in [N_\theta, \Delta_n^m, F, u, p, ||\cdot,\cdots,\cdot||, X_S]_0^{\mathcal{I}}$ and $\alpha, \beta \in \mathbb{C}$, then there exist positive integers M_α and N_β such that $|\alpha| \leq M_\alpha$ and $|\beta| \leq N_\beta$. Since $||\cdot,\cdots,\cdot||$ is a n-norm and f_k is a modulus function for all k and also by using (1), the following inequality holds

$$\frac{1}{h_r}\sum_{k\in I_r} u_k\big[f_k\big(||\Delta_n^m(\alpha x_k+\beta y_k), z_1, z_2, \cdots, z_{n-1}||\big)\big]^{p_k}$$

$$\leq \quad D(M_\alpha)^H\frac{1}{h_r}\sum_{k\in I_r} u_k\big[f_k\big(||\Delta_n^m x_k, z_1, z_2, \cdots, z_{n-1}||\big)\big]^{p_k}$$

$$+ \quad D(N_\beta)^H\frac{1}{h_r}\sum_{k\in I_r} u_k\big[f_k\big(||\Delta_n^m y_k, z_1, z_2, \cdots, z_{n-1}||\big)\big]^{p_k}.$$

On the other hand from the above inequality, we get

$$\Big\{r\in\mathbb{N}: \tfrac{1}{h_r}\sum_{k\in I_r} u_k\big[f_k\big(||\Delta_n^m(\alpha x_k+\beta y_k), z_1, z_2, \cdots, z_{n-1}||\big)\big]^{p_k}\geq\varepsilon\Big\}$$

$$\subseteq \quad \Big\{r\in\mathbb{N}: D(M_\alpha)^H\frac{1}{h_r}\sum_{k\in I_r} u_k\big[f_k\big(||\Delta_n^m x_k, z_1, z_2, \cdots, z_{n-1}||\big)\big]^{p_k}\geq\varepsilon\Big\}$$

$$\cup \quad \Big\{r\in\mathbb{N}: D(N_\beta)^H\frac{1}{h_r}\sum_{k\in I_r} u_k\big[f_k\big(||\Delta_n^m y_k, z_1, z_2, \cdots, z_{n-1}||\big)\big]^{p_k}\geq\varepsilon\Big\}.$$

Two sets on the right side belongs to $\mathcal{I}$, so this completes the proof. □

Lemma 1 Let f be a modulus function and let $0<\delta<1$. Then for each $x>\delta$, we have $f(x)\leq 2f(1)\delta^{-1}x$. For detail see [13].

Theorem 2 *Let $F=(f_k)$ be a sequence of modulus functions and $0<\inf_k p_k=h\leq p_k\leq \sup_k p_k=H<\infty$. Then*
(i)

$$[N_\theta, \Delta_n^m, u, p, ||\cdot, \cdots, \cdot||, X_S]^{\mathcal{I}}\subset[N_\theta, \Delta_n^m, F, u, p, ||\cdot, \cdots, \cdot||, X_S]^{\mathcal{I}}$$

and
(ii)

$$[N_\theta, \Delta_n^m, u, p, ||\cdot, \cdots, \cdot||, X_S]_0^{\mathcal{I}}\subset[N_\theta, \Delta_n^m, F, u, p, ||\cdot, \cdots, \cdot||, X_S]_0^{\mathcal{I}}.$$

Proof If $x=(x_k)\in[N_\theta, \Delta_n^m, u, p, ||\cdot, \cdots, \cdot||, X_S]^{\mathcal{I}}$, then for some $L>0$ and for every $z_1, z_2, \cdots, z_{n-1}\in X$. Thus

$$\Big\{r\in\mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} u_k\big(||\Delta_n^m x_k-L, z_1, z_2, \cdots, z_{n-1}||\big)^{p_k}\geq\varepsilon\Big\}.$$

Now let $\varepsilon>0$ be given. We can choose $0<\delta<1$ such that for every t with $0\leq t\leq\delta$ we have $f_k(t)<\varepsilon$ for all k. Now, using Lemma 1, we get

$$\Big\{r\in\mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} u_k\big[f_k\big(||\Delta_n^m x_k, z_1, z_2, \cdots, z_{n-1}||\big)\big]^{p_k}\geq\varepsilon\Big\}$$

$$= \quad \big\{r\in\mathbb{N}: \frac{1}{h_r}\big(h_r\max\{\varepsilon^h, \varepsilon^H\}\big)\geq\varepsilon\big\}$$

$$\cup \quad \Big\{r\in\mathbb{N}: \frac{1}{h_r}\max\{(2f_k(1)\delta^{-1})^h, (2f_k(1)\delta^{-1})^H\}\sum_{k\in I_r} u_k\big(||\Delta_n^m x_k-L, z_1, z_2, \cdots, z_{n-1}||\big)^{p_k}\Big\}.$$

This completes the proof of (i). Similarly we can prove (ii). □

Theorem 3 *Let $F = (f_k)$ be a sequence of modulus functions. If*

$$\limsup_t \frac{f_k(t)}{t} = A > 0 \quad \textit{for all } k, \quad \textit{then}$$

$$[N_\theta, \Delta_n^m, F, u, p, ||\cdot, \cdots, \cdot||, X_S]_0^{\mathcal{I}} = [N_\theta, \Delta_n^m, u, p, ||\cdot, \cdots, \cdot||, X_S]_0^{\mathcal{I}}$$

and

$$[N_\theta, \Delta_n^m, u, p, ||\cdot, \cdots, \cdot||, X_S]^{\mathcal{I}} = [N_\theta, \Delta_n^m, F, u, p, ||\cdot, \cdots, \cdot||, X_S]^{\mathcal{I}}.$$

Proof To prove $[N_\theta, \Delta_n^m, u, p, ||\cdot, \cdots, \cdot||, X_S]^{\mathcal{I}} = [N_\theta, \Delta_n^m, F, u, p, ||\cdot, \cdots, \cdot||, X_S]^{\mathcal{I}}$. It is sufficient to show that $[N_\theta, \Delta_n^m, F, u, p, ||\cdot, \cdots, \cdot||, X_S]^{\mathcal{I}} \subset [N_\theta, \Delta_n^m, u, p, ||\cdot, \cdots, \cdot||, X_S]^{\mathcal{I}}$. Let $x \in [N_\theta, \Delta_n^m, F, u, p, ||\cdot, \cdots, \cdot||, X_S]^{\mathcal{I}}$. Since $A > 0$, for every $t > 0$ we write $f_k(t) \geq At$ for all k. From this inequality

$$\frac{1}{h_r}\sum_{k\in I_r} u_k\big[f_k\big(\Delta_n^m x_k, z_1, z_2, \cdots, z_{n-1}||\big)\big]^{p_k} \geq A^H \frac{1}{h_r}\sum_{k\in I_r} u_k\big(\Delta_n^m x_k, z_1, z_2, \cdots, z_{n-1}||\big)^{p_k}$$

we get the result. Similarly we can prove the other part. □

Corollary 1 Let $F' = (f_k')$ and $F'' = (f_k'')$ be sequences of modulus functions. If

$$\limsup_t \frac{f_k'(t)}{f_k''(t)} < \infty$$

implies

$$[N_\theta, \Delta_n^m, F', u, p, ||\cdot, \cdots, \cdot||, X_S]_0^{\mathcal{I}} \subset [N_\theta, \Delta_n^m, F'', u, p, ||\cdot, \cdots, \cdot||, X_S]_0^{\mathcal{I}}$$

and

$$[N_\theta, \Delta_n^m, F', u, p, ||\cdot, \cdots, \cdot||, X_S]^{\mathcal{I}} \subset [N_\theta, \Delta_n^m, F'', u, p, ||\cdot, \cdots, \cdot||, X_S]^{\mathcal{I}}.$$

Proof It is trivial. □

Theorem 4 *Let $(X, ||\cdot, \cdots, \cdot||_{X_S})$ and $(X, ||\cdot, \cdots, \cdot||_{X_E})$ be standard and Euclid n-normed spaces, respectively. Then*

$$[N_\theta, \Delta_n^m, F, u, p, ||\cdot, \cdots, \cdot||, X_S]^{\mathcal{I}} \cap [N_\theta, \Delta_n^m, u, p, ||\cdot, \cdots, \cdot||, X_E]^{\mathcal{I}}$$

$$\subset [N_\theta, \Delta_n^m, F, u, p, (||\cdot, \cdots, \cdot||_{X_S} + ||\cdot, \cdots, \cdot||_{X_E})]^{\mathcal{I}}.$$

Proof We have the following inclusion

$$\Big\{r \in \mathbb{N} : \tfrac{1}{h_r} \sum_{k \in I_r} u_k \big[f_k\big((||\cdot, \cdots, \cdot||_{X_S} + ||\cdot, \cdots, \cdot||_{X_E})(||\Delta_n^m x_k - L, z_1, z_2, \cdots, z_{n-1}||)\big)\big]^{p_k} \geq \varepsilon\Big\}$$

$$\begin{aligned} &\subseteq \Big\{r \in \mathbb{N} : D\frac{1}{h_r} \sum_{k \in I_r} u_k \big[f_k\big(||\Delta_n^m x_k - L, z_1, z_2, \cdots, z_{n-1}||_{X_S}\big)\big]^{p_k} \geq \varepsilon\Big\} \\ &\cup \Big\{r \in \mathbb{N} : D\frac{1}{h_r} \sum_{k \in I_r} u_k \big[f_k\big(||\Delta_n^m x_k - L, z_1, z_2, \cdots, z_{n-1}||_{X_E}\big)\big]^{p_k} \geq \varepsilon\Big\} \end{aligned}$$

by using inequality (1). This completes the proof. □

Theorem 5 *Let $F' = (f'_k)$ and $F'' = (f''_k)$ be two sequences of modulus functions. Then*
(i)

$$[N_\theta, \Delta_n^m, F', u, p, ||\cdot, \cdots, \cdot||, X_S]_0^{\mathcal{I}} \subset [N_\theta, \Delta_n^m, F' \circ F'', u, p, ||\cdot, \cdots, \cdot||, X_S]_0^{\mathcal{I}}$$

and

$$[N_\theta, \Delta_n^m, F', u, p, ||\cdot, \cdots, \cdot||, X_S]^{\mathcal{I}} \subset [N_\theta, \Delta_n^m, F' \circ F'', u, p, ||\cdot, \cdots, \cdot||, X_S]^{\mathcal{I}}.$$

(ii)

$$\begin{aligned} &[N_\theta, \Delta_n^m, F', u, p, ||\cdot, \cdots, \cdot||, X_S]_0^{\mathcal{I}} \cap [N_\theta, \Delta_n^m, F'', u, p, ||\cdot, \cdots, \cdot||, X_S]_0^{\mathcal{I}} \\ &\subset [N_\theta, \Delta_n^m, F' + F'', u, p, ||\cdot, \cdots, \cdot||, X_S]_0^{\mathcal{I}} \end{aligned}$$

and

$$\begin{aligned} &[N_\theta, \Delta_n^m, F', u, p, ||\cdot, \cdots, \cdot||, X_S]_0^{\mathcal{I}} \cap [N_\theta, \Delta_n^m, F'', u, p, ||\cdot, \cdots, \cdot||, X_S]_0^{\mathcal{I}} \\ &\subset [N_\theta, \Delta_n^m, F' + F'', u, p, ||\cdot, \cdots, \cdot||, X_S]_0^{\mathcal{I}}. \end{aligned}$$

Proof

(i) Let $x = (x_k) \in [N_\theta, \Delta_n^m, F', u, p, ||\cdot, \cdots, \cdot||, X_S]^{\mathcal{I}}$. Let $0 < \varepsilon < 1$ and δ with $0 < \delta < 1$ such that $f_k(t) < \varepsilon$ for $0 < t < \delta$. Let $y_k = f''_k(||\Delta_n^m x_k - L, z_1, z_2, \cdots, z_{n-1}||)$. Let

$$\frac{1}{h_r} \sum_{k \in I_r} \big[f'_k(y_k)\big]^{p_k} = \frac{1}{h_r} \sum_{1} \big[f'_k(y_k)\big]^{p_k} + \frac{1}{h_r} \sum_{2} \big[f'_k(y_k)\big]^{p_k},$$

where the first summation is over $y_k \leq \delta$ and the second summation is over $y_k > \delta$. Then $\frac{1}{h_r} \sum_{1} \big[f'_k(y_k)\big]^{p_k} \leq \varepsilon^H$ and for $y_k > \delta$, we use the fact that

$$y_k < \frac{y_k}{\delta} < 1 + \Big[|\frac{y_k}{\delta}|\Big],$$

where $[|z|]$ denotes the integer part of z. So that from the properties of modulus function, we have for $y_k > \delta$

$$f'_k(y_k) < \Big(1 + \Big[|\frac{y_k}{\delta}|\Big]\Big) f'_k(1) \leq 2 f'_k(1) \frac{y_k}{\delta}$$

for all k. Hence

$$\frac{1}{h_r} \sum_{2} \big[f'_k(y_k)\big]^{p_k} \leq \Big[2\frac{f'_k(1)}{\delta}\Big]^H \frac{1}{h_r} \sum_{2} [y_k]^{p_k}$$

which together with $\frac{1}{h_r}\sum\limits_{1}\left[f'_k(y_k)\right]^{p_k} \leq \varepsilon^H$ yields

$$\frac{1}{h_r}\sum_{k\in I_r}\left[f'_k(y_k)\right]^{p_k} \leq \varepsilon^H + \max(1,(2f'_k(1)\delta^{-1})^H)\sum_{k\in I_r}\left[f'_k(y_k)\right]^{p_k}$$

and this completes the proof of (i).

(ii) Let $x=(x_k)\in[N_\theta,\Delta_n^m,F',u,p,||\cdot,\cdots,\cdot||,X_S]_0^{\mathcal{I}}\cap[N_\theta,\Delta_n^m,F'',u,p,||\cdot,\cdots,\cdot||,X_S]_0^{\mathcal{I}}$. The fact that

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}u_k\Big[(f'_k+f''_k)(||\Delta_n^m x_k-L,z_1,z_2,\cdots,z_{n-1}||)\Big]^{p_k}\\
&\qquad\leq \frac{1}{h_r}\sum_{k\in I_r}u_k\Big[f'_k(||\Delta_n^m x_k-L,z_1,z_2,\cdots,z_{n-1}||)\Big]^{p_k}\\
&\qquad+ \frac{1}{h_r}\sum_{k\in I_r}u_k\Big[f''_k(||\Delta_n^m x_k-L,z_1,z_2,\cdots,z_{n-1}||)\Big]^{p_k}
\end{aligned}$$

gives the result. □

Conclusion

In this paper we have constructed $\mathcal{I}$-convergent sequence spaces defined by a sequence of modulus functions over n- normed spaces. We have studied some topological properties and interesting inclusion relation between these sequence spaces. The solutions obtained are potentially significant and important for the explanation of some practical physical problems. The method may also be applied to other sequence spaces.

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