Super-Open Sets and Super-Continuity of Maps in the Product Space

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Abstract In this short note we took another look at the concepts of super-open and super-closed sets and super-continuity and gave some properties of these concepts in the Cartesian product with the Tychonoff topology. Further, we characterized super-continuous functions from an arbitrary topological space into the product space. The result we obtained runs parallel to the one we have for continuous functions in the product space. Other results involving super-continuous functions in the product space are also given.

Keywords super-open; super-closed; super-closure; super-interior; super-continuous.

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1 Introduction

The first attempt to replace various concepts in topology with concepts possessing either of weaker or stronger properties was done by Levine [1] in 1963. In his work, Levine introduced the concept of semi-open set and used this to define other new concepts such as semi-closed set and semi-continuity of a function.

After this notable work of Levine on the concept of semi-open set, several mathematicians became interested in introducing other topological concepts which can replace the concept of open set. Over the years, a number of generalizations of the concept of open set have been coined and numerous results have been obtained. For instance, when open sets are replaced by semi-open sets, new results were generated some of which are generalizations of the existing ones.

In 1968, Volicko [2] introduced the concept of super-continuity between topological spaces. He also defined the concepts such as super-closure and super-interior of a subset of a topological space. Recently, Al-Hawary [3] characterized super-continuity and gave relationships between super-continuity and the other well-known variations of continuity such as strong continuity, semi-continuity, and closure continuity.

Let \((X, \tau)\) be a topological space and \(A \subseteq X\). The super-closure and super-interior of \(A\) are, respectively, denoted and defined by

\[
Cl_s(A) = \{x \in X : Cl(U) \cap A \neq \emptyset \text{ for every open set } U \text{ containing } x\}
\]

and

\[
Int_s(A) = \{x \in X : Cl(U) \subseteq A \text{ for some open set } U \text{ containing } x\},
\]

where \(Cl(U)\) is the closure of \(U\) in \(X\). A subset \(A\) of \(X\) is super-closed if \(Cl_s(A) = A\) and super-open if \(Int_s(A) = A\). Equivalently, \(A\) is super-open if and only if \(X \setminus A\) is super-closed. A function \(f : X \to Y\) is super-continuous if \(f^{-1}(G)\) is super-open in \(X\) for every open set
Given $G$ in $Y$. Equivalently, $f : X \to Y$ is super-continuous if $f^{-1}(F)$ is super-closed in $X$ for every closed set $F$ in $Y$.

Now, let $\mathcal{A}$ be an indexing set and $\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a family of topological spaces. For each $\alpha \in \mathcal{A}$, let $\tau_\alpha$ be the topology on $Y_\alpha$. The Tychonoff topology on $\Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ is the topology generated by a subbase consisting of all sets $p^{-1}_\alpha(U_\alpha)$, where the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ is denoted by $p_\alpha(y_\alpha) = y_\alpha$, $U_\alpha$ ranges over all members of $\tau_\alpha$, and $\alpha$ ranges over all elements of $\mathcal{A}$. Corresponding to $U_\alpha \subseteq Y_\alpha$, denote $p^{-1}_\alpha(U_\alpha)$ by $\langle U_\alpha \rangle$. Similarly, for finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$, and sets $U_{\alpha_1} \subseteq Y_{\alpha_1}$, $U_{\alpha_2} \subseteq Y_{\alpha_2}, \ldots, U_{\alpha_n} \subseteq Y_{\alpha_n}$, the subset

$$\langle U_{\alpha_1} \rangle \cap \langle U_{\alpha_2} \rangle \cap \cdots \cap \langle U_{\alpha_n} \rangle = p^{-1}_{\alpha_1}(U_{\alpha_1}) \cap p^{-1}_{\alpha_2}(U_{\alpha_2}) \cap \cdots \cap p^{-1}_{\alpha_n}(U_{\alpha_n})$$

is denoted by $\langle U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n} \rangle$. We note that for each open set $U_\alpha$ subset of $Y_\alpha$, $\langle U_\alpha \rangle = p^{-1}_\alpha(U_\alpha) = U_\alpha \times \Pi_{\beta \neq \alpha} Y_\beta$. Hence, a basis for the Tychonoff topology consists of sets of the form $\langle B_{\alpha_1}, B_{\alpha_2}, \ldots, B_{\alpha_k} \rangle$, where $B_{\alpha_i}$ is open in $Y_{\alpha_i}$ for every $i \in K = \{1, 2, \ldots, k\}$.

Now, the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ is defined by $p_\alpha(y_\alpha) = y_\alpha$ for each $\alpha \in \mathcal{A}$. It is known that every projection map is a continuous open surjection. Also, it is well known that a function $f$ from an arbitrary space $X$ into the Cartesian product $Y$ of the family of spaces $\{Y_\alpha : \alpha \in \mathcal{A}\}$ with the Tychonoff topology is continuous if and only if each coordinate function $p_\alpha \circ f$ is continuous, where $p_\alpha$ is the $\alpha$-th coordinate projection map.

In this paper, we present some properties of the concepts introduced by Volicko in the Cartesian product with the Tychonoff topology. We also gave a necessary and sufficient condition for a function from an arbitrary topological space into the product space to be super-continuous.

## 2 Results

We begin by stating two simple lemmas.

**Lemma 1** Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then $x \in Cl_s(A)$ if and only if $Cl(B) \cap A \neq \emptyset$ for every basic open set $B$ in $X$ containing $x$.

**Proof** Let $x \in Cl_s(A)$. Then $Cl(U) \cap A \neq \emptyset$ for every open set $U$ in $X$ with $x \in U$. It follows that if $B$ is a basic open set in $X$ containing $x$, then $Cl(B) \cap A \neq \emptyset$.

Conversely, suppose that $Cl(B) \cap A \neq \emptyset$ for every basic open set $B$ in $X$ with $x \in B$. Let $U$ be an open set in $X$ with $x \in U$. Then there exists a basic open set $B_0$ contained in $U$ such that $x \in B_0$. By assumption, $Cl(B_0) \cap A \neq \emptyset$. Since $Cl(B_0) \subseteq Cl(U)$, it follows that $Cl(U) \cap A \neq \emptyset$. This shows that $x \in Cl_s(A)$. \qed

**Lemma 2** Let $(X, \tau)$ be a topological space and let $\tau^*$ be the family consisting of all the super-open subsets of $X$. Then $\tau^*$ is a topology on $X$.

**Proof** It is very clear that $\emptyset$ and $X$ are super-open sets. Now, let $\{G_\alpha : \alpha \in \mathcal{A}\}$ be a family of all super-open subsets of $X$ and let $O = \bigcup\{G_\alpha : \alpha \in \mathcal{A}\}$. Let $x \in O$. Then $x \in G_\alpha$ for some $\alpha \in \mathcal{A}$. Since $G_\alpha$ is super-open in $X$, there exists an open set $U$ in $X$ such that $x \in U$ and $Cl(U) \subseteq G_\alpha \subseteq O$. Therefore, $O$ is super-open in $X$. 

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Next, let $G_1$ and $G_2$ be super-open sets and let $x \in G_1 \cap G_2$. Then there exist open sets $V_1$ and $V_2$ with $x \in V_1 \cap V_2$ such that $\text{Cl}(V_1) \subseteq G_1$ and $\text{Cl}(V_2) \subseteq G_2$. Clearly, $V_1 \cap V_2$ is open, $x \in V_1 \cap V_2$ and $\text{Cl}(V_1 \cap V_2) \subseteq \text{Cl}(V_1) \cap \text{Cl}(V_2) \subseteq G_1 \cap G_2$. This proves that $\tau^*$ is a topology on $X$. \hfill \Box

We shall be needing the following result later.

**Theorem 1** A function $f : X \to Y$ is super-continuous on $X$ if and only if $f^{-1}(B)$ is super-open in $X$ for every basic open set $B$ in $Y$.

**Proof** Suppose that $f$ is super-continuous on $X$ and let $B$ be a basic open set in $Y$. Then, by definition, $f^{-1}(B)$ is super-open in $X$.

For the converse, suppose that $f^{-1}(B)$ is super-open in $X$ for all $B \in \Omega$, where $\Omega$ is a basis for the topology associated with $Y$. If $G$ is an open set, then $G = \bigcup \{B : B \in \Omega^*\}$, where $\Omega^* \subseteq \Omega$. It follows that $f^{-1}(G) = \bigcup \{f^{-1}(B) : B \in \Omega^*\}$. By Lemma 2, the result follows. \hfill \Box

The following remark follows from Lemma 1 and Theorem 1.

**Remark 1** A function $f : X \to Y$ is super-continuous on $X$ if and only if $f^{-1}(B)$ is super-open in $X$ for every subbasic open set $B$ in $Y$.

The following result can be found in Dugundji [4].

**Theorem 2** Let $\{Y_\alpha : \alpha \in A\}$ be a family of topological spaces and $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in A$. Then, in $\Pi\{Y_\alpha : \alpha \in A\}$ with the Tychonoff topology,

$$\text{Cl}(\Pi\{A_\alpha : \alpha \in A\}) = \Pi(\text{Cl}(A_\alpha) : \alpha \in A).$$

The next result says that Theorem 2 still holds even if closure is replaced by super-closure.

**Theorem 3** Let $\{Y_\alpha : \alpha \in A\}$ be a family of topological spaces and $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in A$. Then, in $Y = \Pi\{Y_\alpha : \alpha \in A\}$ with the Tychonoff topology,

$$\text{Cl}_s(\Pi\{A_\alpha : \alpha \in A\}) = \Pi(\text{Cl}_s(A_\alpha) : \alpha \in A).$$

**Proof** Let $A = \text{Cl}_s(\Pi\{A_\alpha : \alpha \in A\})$ and $B = \Pi(\text{Cl}_s(A_\alpha) : \alpha \in A)$. Let $x = \langle a_\alpha \rangle \in A$. Then, by Lemma 2 and Theorem 2,

$$\begin{align*}
\text{Cl}(\langle U_{\alpha_1}, \ldots, U_{\alpha_n} \rangle) & \cap \Pi\{A_\alpha : \alpha \in A\} \\
& = \langle \text{Cl}(U_{\alpha_1}), \text{Cl}(U_{\alpha_2}), \ldots, \text{Cl}(U_{\alpha_n}) \rangle \cap \Pi\{A_\alpha : \alpha \in A\} \\
& \neq \emptyset
\end{align*}$$

for every finite collection $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of indices from $A$. Suppose that there exists $\beta \in A$ such that $a_\beta \notin \text{Cl}_s(A_\beta)$. Then there exists an open set $U_\beta$ with $a_\beta \in U_\beta$ such that $\text{Cl}(U_\beta) \cap A_\beta = \emptyset$. It follows that $x = \langle a_\alpha \rangle \in \langle U_\beta \rangle$ and

$$\text{Cl}(\langle U_\beta \rangle) \cap \Pi\{A_\alpha : \alpha \in A\} = \Pi\{A_\alpha : \alpha \neq \beta\} \times (\text{Cl}(U_\beta) \cap A_\beta) = \emptyset.$$
This clearly contradicts our assumption. Therefore, \( x \in B \) and hence, \( A \subseteq B \).

To show the other inclusion, suppose that \( x = \langle a_\alpha \rangle \in B \). Then \( a_\alpha \in Cl_s(A_\alpha) \) for all \( \alpha \in A \). This means that for every \( \alpha \in A \) and for every open set \( U_\alpha \) in \( Y_\alpha \) with \( a_\alpha \in U_\alpha \), we have \( Cl(U_\alpha) \cap A_\alpha \neq \emptyset \). Choose \( b_\alpha \in Cl(U_\alpha) \cap A_\alpha \neq \emptyset \) for each \( \alpha \in A \).

Let \( V = \langle V_{a_1}, V_{a_2}, \ldots, V_{a_n} \rangle \) be a basic open set in \( Y \) with \( x \in V \). Since \( a_\alpha \in V_{a_\alpha} \) for each \( \alpha \in A \), it follows that \( Cl(V_{a_\alpha}) \cap A_\alpha \neq \emptyset \) for all \( \alpha \in A \). Choose \( c_\alpha \in Cl(V_{a_\alpha}) \cap A_\alpha \neq \emptyset \) for each \( \alpha \in A \). Define \( y = \langle d_\alpha \rangle \) by \( d_\alpha = b_\alpha \) for each \( \alpha \in A \setminus K \) and \( d_\alpha = c_\alpha \) for each \( \alpha \in K \), where \( K = \{ \alpha_1, \alpha_2, \ldots, \alpha_n \} \). Then

\[
y \in \Pi \{ A_\alpha : \alpha \notin K \} \times (Cl(V_{a_1}) \cap A_{a_1}) \times \cdots \times (Cl(V_{a_n}) \cap A_{a_n}).
\]

Thus,

\[
Cl(\langle V_{a_1}, \ldots, V_{a_n} \rangle) \cap \Pi \{ A_\alpha : \alpha \in A \}
= \Pi \{ A_\alpha : \alpha \notin K \} \times (Cl(V_{a_1}) \cap A_{a_1}) \times \cdots \times (Cl(V_{a_n}) \cap A_{a_n})
\neq \emptyset.
\]

Hence, by Lemma 1, \( x \in A \). This shows that \( B \subseteq A \). Accordingly, \( A = B \).

\textbf{Theorem 4} Let \( O \) be a non-empty super-open set in the product space \( Y = \Pi \{ Y_\alpha : \alpha \in A \} \). Then \( p_\alpha(O) = Y_\alpha \) for all but at most finitely many \( \alpha \) and \( p_\alpha(O) \) is super-open in \( Y_\alpha \) for every \( \alpha \in A \).

\textbf{Proof} Let \( x \in O \). Then there exists a basic open set \( \langle U_{a_1}, \ldots, U_{a_n} \rangle \) containing \( x \) and \( (Cl(U_{a_1}), Cl(U_{a_2}), \ldots, Cl(U_{a_n})) \subseteq O \) by Theorem 2. It follows that

\[
p_\alpha(\langle Cl(U_{a_1}), Cl(U_{a_2}), \ldots, Cl(U_{a_n}) \rangle) \subseteq p_\alpha(O)
\]

for every \( \alpha \in A \). Now, since \( p_\alpha(\langle Cl(U_{a_1}), Cl(U_{a_2}), \ldots, Cl(U_{a_n}) \rangle) = Y_\alpha \) for each \( \alpha \notin \{ \alpha_1, \ldots, \alpha_n \} \), it follows that \( p_\alpha(O) = Y_\alpha \) for all but at most a finite number of indices in \( A \).

Next, let \( \alpha \in A \). Then \( p_\alpha(O) = Y_\alpha \) or \( p_\alpha(O) \neq Y_\alpha \). If \( p_\alpha(O) = Y_\alpha \), then it is super-open in \( Y_\alpha \). Suppose that \( p_\alpha(O) \neq Y_\alpha \) and let \( a \in p_\alpha(O) \). Then there exists \( x = \langle a_\alpha \rangle \in O \) such that \( p_\alpha(x) = a \). Since \( O \) is super-open there exists an open set \( E = \langle V_{a_1}, \ldots, V_{a_k} \rangle \) containing \( x \) such that \( Cl(V_{a_1}), Cl(V_{a_2}), \ldots, Cl(V_{a_k}) \subseteq O \). Then \( a = p_\alpha(x) \in p_\alpha(E) = V_\alpha \), where \( \alpha \in \{ \alpha_1, \alpha_2, \ldots, \alpha_k \} \), and \( p_\alpha(Cl(E)) = Cl(V_\alpha) \subseteq p_\alpha(O) \). This shows that \( p_\alpha(O) \) is super-open in \( Y_\alpha \).

\textbf{Remark 2} The converse of Theorem 4 is not true.

To see this, consider \( Y_1 = \{ 1, 2, 3 \} \) and \( Y_2 = \{ a, b, c \} \) with the respective topologies \( \tau_1 = \{ \{ 1 \}, \{ 2 \}, \{ 1, 2 \}, \{ 2, 3 \} \} \) and \( \tau_2 = \{ Y_2, \emptyset, \{ a \}, \{ c \}, \{ a, c \} \} \). Consider the set \( O = \{ \{ (1, a), (2, c), (3, b) \} \). Then the family \( \mathcal{B} \) consisting of the sets \( Y_1 \times Y_2, \emptyset, Y_1 \times \{ a \}, Y_1 \times \{ c \}, Y_1 \times \{ a, c \}, \emptyset, Y_1 \times Y_2, \{ 1, 2 \} \times Y_2, \{ 2, 3 \} \times Y_2, \{ (1, a), (2, a) \}, \{ (1, c), (2, c) \}, \{ (2, a) \}, \{ (2, c) \}, \{ (2, a), (2, c) \} \} \) is a basis for the Tychonoff topology on \( Y_1 \times Y_2 \). Since \( p_1(O) = \{ 1, 2, 3 \} = Y_1 \) and \( p_2(O) = \{ a, b, c \} = Y_2 \), it follows that \( p_1(O) \) and \( p_2(O) \) are super-open in \( Y_1 \) and \( Y_2 \), respectively. However, since there exists no basic open set \( V \) in \( Y \) containing \( (1, a) \) with \( Cl(V) \subseteq O \), it follows that \( O \) is not super-open in \( Y \).
Thus, \( O \) is an open set in \( V \).

**Proof**  Let \( \alpha \) be a topological space. A function \( f : X \to Y \) is super-continuous on \( X \) if and only if each \( \alpha \) is super-continuous on \( X \) into the product space \( Y \).

Suppose that \( f \) is super-continuous on \( X \). Let \( \alpha \in A \) and \( U_\alpha \) be open in \( Y_\alpha \). Since \( p_\alpha \) is continuous, \( p_\alpha^{-1}(U_\alpha) \) is open in \( Y \). Hence,

\[
    f^{-1}(p_\alpha^{-1}(U_\alpha)) = (p_\alpha \circ f)^{-1}(U_\alpha)
\]

is a super-open set in \( X \) since \( f \) is super-continuous. Thus, \( p_\alpha \circ f \) is super-continuous for every \( \alpha \in A \).

Conversely, suppose that each \( \alpha \) is super-continuous. Let \( G_\alpha \) be open in \( Y_\alpha \). Then \( \langle G_\alpha \rangle \) is a subbasic open set in \( Y \) and

\[
    (p_\alpha \circ f)^{-1}(\langle G_\alpha \rangle) = f^{-1}(p_\alpha^{-1}(\langle G_\alpha \rangle)) = f^{-1}(\langle G_\alpha \rangle)
\]

is a super-open set in \( X \). Therefore, \( f \) is super-continuous on \( X \), by Remark 1.

**Corollary 1**  Let \( X \) be a topological space, \( Y \) the product space and \( f_\alpha : X \to Y_\alpha \) a function for each \( \alpha \in A \). Let \( f : X \to Y \) be the function defined by \( f(x) = \langle f_\alpha(x) \rangle \). Then \( f \) is super-continuous on \( X \) if and only if \( f_\alpha \) is super-continuous for each \( \alpha \in A \).

**Proof**  For each \( \beta \in A \) and each \( x \in X \), we have

\[
    (p_\beta \circ f)(x) = p_\beta(f(x)) = p_\beta(\langle f_\alpha(x) \rangle) = f_\beta(x).
\]

Thus, \( p_\beta \circ f = f_\beta \) for every \( \beta \in A \). The result now follows from Theorem 6.

**Theorem 7**  Let \( X \) and \( Y \) be the product spaces of the families of spaces \( \{X_\alpha : \alpha \in A\} \) and \( \{Y_\alpha : \alpha \in A\} \), respectively, and for each \( \alpha \in A \), let \( f_\alpha : X_\alpha \to Y_\alpha \) be a function. If each \( f_\alpha \) is super-continuous, then the function \( f : X \to Y \), defined by \( f(x_\alpha) = \langle f_\alpha(x_\alpha) \rangle \), is super-continuous on \( X \).
Proof Let $\langle V_\alpha \rangle$ be a subbasic open set in Y. Then $f^{-1}(\langle V_\alpha \rangle) = \langle f_\alpha^{-1}(V_\alpha) \rangle$. Since $f_\alpha$ is super-continuous, $f_\alpha^{-1}(V_\alpha)$ is super-open in $X_\alpha$. Now, let an element $x = \langle x_\beta \rangle \in (f_\alpha^{-1}(V_\alpha))$. Then $x_\alpha \in f_\alpha^{-1}(V_\alpha)$. Hence, there exists an open set $G_\alpha$ in $X_\alpha$ with $x_\alpha \in G_\alpha$ such that $Cl(G_\alpha) \subseteq f_\alpha^{-1}(V_\alpha)$. Clearly, $\langle G_\alpha \rangle$ is open in $X$ and $x \in \langle G_\alpha \rangle$. By Theorem 2, $Cl(\langle G_\alpha \rangle) = \langle Cl(G_\alpha) \rangle \subseteq (f_\alpha^{-1}(V_\alpha))$. This shows that $f^{-1}(\langle V_\alpha \rangle) = \langle f_\alpha^{-1}(V_\alpha) \rangle$ is super-open in $X$. Therefore, $f$ is super-continuous on $X$. \[\Box\]

3 General Remarks

The paper has so far described super-open sets and established the formula of the super-closure of a set in a Cartesian product space. Moreover, the paper has formulated a necessary and sufficient condition for super-continuity of a function from an arbitrary space into the product space. This particular result is the counterpart of the known characterization of the ordinary continuity of a function into a product space. The results generated in this paper would certainly find importance as one explores further other possible concepts (e.g. separation axioms, axioms of countability, and compactness) involving super-open sets and studies them in the product space.

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References


