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Super-Open Sets and Super-Continuity of Maps in the Product Space

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Abstract In this short note we took another look at the concepts of super-open and super-closed sets and super-continuity and gave some properties of these concepts in the Cartesian product with the Tychonoff topology. Further, we characterized super-continuous functions from an arbitrary topological space into the product space. The result we obtained runs parallel to the one we have for continuous functions in the product space. Other results involving super-continuous functions in the product space are also given.

Keywords super-open; super-closed; super-closure; super-interior; super-continuous.

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1 Introduction

The first attempt to replace various concepts in topology with concepts possessing either of weaker or stronger properties was done by Levine [1] in 1963. In his work, Levine introduced the concept of semi-open set and used this to define other new concepts such as semi-closed set and semi-continuity of a function.

After this notable work of Levine on the concept of semi-open set, several mathematicians became interested in introducing other topological concepts which can replace the concept of open set. Over the years, a number of generalizations of the concept of open set have been coined and numerous results have been obtained. For instance, when open sets are replaced by semi-open sets, new results were generated some of which are generalizations of the existing ones.

In 1968, Volicko [2] introduced the concept of super-continuity between topological spaces. He also defined the concepts such as super-closure and super-interior of a subset of a topological space. Recently, Al-Hawary [3] characterized super-continuity and gave relationships between super-continuity and the other well-known variations of continuity such as strong continuity, semi-continuity, and closure continuity.

Let (X, τ) be a topological space and $A \subseteq X$. The *super-closure* and *super-interior* of A are, respectively, denoted and defined by

$$Cl_s(A) = \{x \in X : Cl(U) \cap A \neq \varnothing \text{ for every open set } U \text{ containing } x\}$$

and

$$Int_s(A) = \{x \in X : Cl(U) \subseteq A \text{ for some open set } U \text{ containing } x\},$$

where $Cl(U)$ is the closure of U in X. A subset A of X is *super-closed* if $Cl_s(A) = A$ and *super-open* if $Int_s(A) = A$. Equivalently, A is super-open if and only if $X \backslash A$ is super-closed. A function $f : X \to Y$ is *super-continuous* if $f^{-1}(G)$ is super-open in X for every open set

G in Y. Equivalently, $f : X \to Y$ is *super-continuous* if $f^{-1}(F)$ is super-closed in X for every closed set F in Y.

Now, let $\mathcal{A}$ be an indexing set and $\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a family of topological spaces. For each $\alpha \in \mathcal{A}$, let τ_α be the topology on Y_α. The *Tychonoff topology* on $\Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ is the topology generated by a subbase consisting of all sets $p_\alpha^{-1}(U_\alpha)$, where the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$, U_α ranges over all members of τ_α, and α ranges over all elements of $\mathcal{A}$. Corresponding to $U_\alpha \subseteq Y_\alpha$, denote $p_\alpha^{-1}(U_\alpha)$ by $\langle U_\alpha \rangle$. Similarly, for finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$, and sets $U_{\alpha_1} \subseteq Y_{\alpha_1}$, $U_{\alpha_2} \subseteq Y_{\alpha_2}, \ldots, U_{\alpha_n} \subseteq Y_{\alpha_n}$, the subset

$$\langle U_{\alpha_1} \rangle \cap \langle U_{\alpha_2} \rangle \cap \cdots \cap \langle U_{\alpha_n} \rangle = p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap p_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \cdots \cap p_{\alpha_n}^{-1}(U_{\alpha_n})$$

is denoted by $\langle U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n} \rangle$. We note that for each open set U_α subset of Y_α, $\langle U_\alpha \rangle = p_\alpha^{-1}(U_\alpha) = U_\alpha \times \Pi_{\beta \neq \alpha} Y_\beta$. Hence, a basis for the Tychonoff topology consists of sets of the form $\langle B_{\alpha_1}, B_{\alpha_2}, ..., B_{\alpha_k} \rangle$, where B_{α_i} is open in Y_{α_i} for every $i \in K = \{1, 2, ..., k\}$.

Now, the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$ for each $\alpha \in \mathcal{A}$. It is known that every projection map is a continuous open surjection. Also, it is well known that a function f from an arbitrary space X into the Cartesian product Y of the family of spaces $\{Y_\alpha : \alpha \in \mathcal{A}\}$ with the Tychonoff topology is continuous if and only if each coordinate function $p_\alpha \circ f$ is continuous, where p_α is the α-th coordinate projection map.

In this paper, we present some properties of the concepts introduced by Volicko in the Cartesian product with the Tychonoff topology. We also gave a necessary and sufficient condition for a function from an arbitrary topological space into the product space to be super-continuous.

2 Results

We begin by stating two simple lemmas.

Lemma 1 Let (X, τ) be a topological space and $A \subseteq X$. Then $x \in Cl_s(A)$ if and only if $Cl(B) \cap A \neq \varnothing$ for every basic open set B in X containing x.

Proof Let $x \in Cl_s(A)$. Then $Cl(U) \cap A \neq \varnothing$ for every open set U in X with $x \in U$. It follows that if B is a basic open set in X contaning x, then $Cl(B) \cap A \neq \varnothing$.

Conversely, suppose that $Cl(B) \cap A \neq \varnothing$ for every basic open set B in X with $x \in B$. Let U be an open set in X with $x \in U$. Then there exists a basic open set B_0 contained in U such that $x \in B_0$. By assumption, $Cl(B_0) \cap A \neq \varnothing$. Since $Cl(B_0) \subseteq Cl(U)$, it follows that $Cl(U) \cap A \neq \varnothing$. This shows that $x \in Cl_s(A)$. □

Lemma 2 Let (X, τ) be a topological space and let τ^* be the family consisting of all the super-open subsets of X. Then τ^* is a topology on X.

Proof It is very clear that $\varnothing$ and X are super-open sets. Now, let $\{G_\alpha : \alpha \in \mathcal{A}\}$ be a family of all super-open subsets of X and let $O = \bigcup\{G_\alpha : \alpha \in \mathcal{A}\}$. Let $x \in O$. Then $x \in G_\alpha$ for some $\alpha \in \mathcal{A}$. Since G_α is super-open in X, there exists an open set U in X such that $x \in U$ and $Cl(U) \subseteq G_\alpha \subseteq O$. Therefore, O is super-open in X.

Next, let G_1 and G_2 be super-open sets and let $x \in G_1 \cap G_2$. Then there exist open sets V_1 and V_2 with $x \in V_1 \cap V_2$ such that $Cl(V_1) \subseteq G_1$ and $Cl(V_2) \subseteq G_2$. Clearly, $V_1 \cap V_2$ is open, $x \in V_1 \cap V_2$ and $Cl(V_1 \cap V_2) \subseteq Cl(V_1) \cap Cl(V_2) \subseteq G_1 \cap G_2$. This proves that τ^* is a topology on X. □

We shall be needing the following result later.

Theorem 1 *A function $f : X \to Y$ is super-continuous on X if and only if $f^{-1}(B)$ is super-open in X for every basic open set B in Y.*

Proof Suppose that f is super-continuous on X and let B be a basic open set in Y. Then, by definition, $f^{-1}(B)$ is super-open in X.

For the converse, suppose that $f^{-1}(B)$ is super-open in X for all $B \in \Omega$, where Ω is a basis for the topology associated with Y. If G is an open set, then $G = \bigcup\{B : B \in \Omega^*\}$, where $\Omega^* \subseteq \Omega$. It follows that $f^{-1}(G) = \bigcup\{f^{-1}(B) : B \in \Omega^*\}$. By Lemma 2, the result follows. □

The following remark follows from Lemma 1 and Theorem 1.

Remark 1 A function $f : X \to Y$ is super-continuous on X if and only if $f^{-1}(B)$ is super-open in X for every subbasic open set B in Y.

The following result can be found in Dugundji [4].

Theorem 2 *Let $\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a family of topological spaces and $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$. Then, in $\Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ with the Tychonoff topology,*

$$Cl(\Pi\{A_\alpha : \alpha \in \mathcal{A}\} = \Pi\{Cl(A_\alpha) : \alpha \in \mathcal{A}\}.$$

The next result says that Theorem 2 still holds even if closure is replaced by super-closure.

Theorem 3 *Let $\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a family of topological spaces and $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$. Then, in $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ with the Tychonoff topology,*

$$Cl_s(\Pi\{A_\alpha : \alpha \in \mathcal{A}\}) = \Pi\{Cl_s(A_\alpha) : \alpha \in \mathcal{A}\}.$$

Proof Let $A = Cl_s(\Pi\{A_\alpha : \alpha \in \mathcal{A}\})$ and $B = \Pi\{Cl_s(A_\alpha) : \alpha \in \mathcal{A}\}$. Let $x = \langle a_\alpha \rangle \in A$. Then, by Lemma 2 and Theorem 2,

$$\begin{aligned} & Cl(\langle U_{\alpha_1}, \ldots, U_{\alpha_n} \rangle) \cap \Pi\{A_\alpha : \alpha \in \mathcal{A}\} \\ = \ & \langle Cl(U_{\alpha_1}), Cl(U_{\alpha_2}), \ldots, Cl(U_{\alpha_n}) \rangle \cap \Pi\{A_\alpha : \alpha \in \mathcal{A}\} \\ \neq \ & \varnothing \end{aligned}$$

for every finite collection $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of indices from $\mathcal{A}$. Suppose that there exists $\beta \in \mathcal{A}$ such that $a_\beta \notin Cl_s(A_\beta)$. Then there exists an open set U_β with $a_\beta \in U_\beta$ such that $Cl(U_\beta) \cap A_\beta = \varnothing$. It follows that $x = \langle a_\alpha \rangle \in \langle U_\beta \rangle$ and

$$Cl(\langle U_\beta \rangle) \cap \Pi\{A_\alpha : \alpha \in \mathcal{A}\} = \Pi\{A_\alpha : \alpha \neq \beta\} \times (Cl(U_\beta) \cap A_\beta) = \varnothing.$$

This clearly contradicts our assumption. Therefore, $x \in B$ and hence, $A \subseteq B$.

To show the other inclusion, suppose that $x = \langle a_\alpha \rangle \in B$. Then $a_\alpha \in Cl_s(A_\alpha)$ for all $\alpha \in \mathcal{A}$. This means that for every $\alpha \in \mathcal{A}$ and for every open set U_α in Y_α with $a_\alpha \in U_\alpha$, we have $Cl(U_\alpha) \cap A_\alpha \neq \varnothing$. Choose $b_\alpha \in Cl(U_\alpha) \cap A_\alpha \neq \varnothing$ for each $\alpha \in \mathcal{A}$. Let $V = \langle V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_n} \rangle$ be a basic open set in Y with $x \in V$. Since $a_{\alpha_i} \in V_{\alpha_i}$ for each $i \in J = \{1, 2, \cdots, n\}$, it follows that $Cl(V_{\alpha_i}) \cap A_{\alpha_i} \neq \varnothing$ for all $i \in J$. Choose $c_{\alpha_i} \in Cl(V_{\alpha_i}) \cap A_{\alpha_i} \neq \varnothing$ for each $i \in J$. Define $y = \langle d_\alpha \rangle$ by $d_\alpha = b_\alpha$ for each $\alpha \in \mathcal{A} \backslash K$ and $d_{\alpha_i} = c_{\alpha_i}$ for each $i \in J$, where $K = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Then

$$y \in \Pi\{A_\alpha : \alpha \notin K\} \times (Cl(V_{\alpha_1}) \cap A_{\alpha_1}) \times \cdots \times (Cl(V_{\alpha_n}) \cap A_{\alpha_n}).$$

Thus,

$$\begin{aligned} & Cl(\langle V_{\alpha_1}, \cdots, V_{\alpha_n} \rangle) \cap \Pi\{A_\alpha : \alpha \in \mathcal{A}\} \\ = \; & \Pi\{A_\alpha : \alpha \notin K\} \times (Cl(V_{\alpha_1}) \cap A_{\alpha_1}) \times \cdots \times (Cl(V_{\alpha_n}) \cap A_{\alpha_n}) \\ \neq \; & \varnothing. \end{aligned}$$

Hence, by Lemma 1, $x \in A$. This shows that $B \subseteq A$. Accordingly, $A = B$. □

Theorem 4 *Let O be a non-empty super-open set in the product space $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$. Then $p_\alpha(O) = Y_\alpha$ for all but at most finitely many α and $p_\alpha(O)$ is super-open in Y_α for every $\alpha \in \mathcal{A}$.*

Proof Let $x \in O$. Then there exists a basic open set $\langle U_{\alpha_1}, \cdots, U_{\alpha_n} \rangle$ containing x and $\langle Cl(U_{\alpha_1}), Cl(U_{\alpha_2}), \cdots, Cl(U_{\alpha_n}) \rangle \subseteq O$ by Theorem 2. It follows that

$$p_\alpha(\langle Cl(U_{\alpha_1}), Cl(U_{\alpha_2}), \cdots, Cl(U_{\alpha_n}) \rangle) \subseteq p_\alpha(O)$$

for every $\alpha \in \mathcal{A}$. Now, since $p_\alpha(\langle Cl(U_{\alpha_1}), Cl(U_{\alpha_2}), \cdots, Cl(U_{\alpha_n}) \rangle) = Y_\alpha$ for each $\alpha \notin \{\alpha_1, \cdots, \alpha_n\}$, it follows that $p_\alpha(O) = Y_\alpha$ for all but at most a finite number of indices in $\mathcal{A}$.

Next, let $\alpha \in \mathcal{A}$. Then $p_\alpha(O) = Y_\alpha$ or $p_\alpha(O) \neq Y_\alpha$. If $p_\alpha(O) = Y_\alpha$, then it is super-open in Y_α. Suppose that $p_\alpha(O) \neq Y_\alpha$ and let $a \in p_\alpha(O)$. Then there exists $x = \langle a_\alpha \rangle \in O$ such that $p_\alpha(x) = a$. Since O is super-open there exists an open set $E = \langle V_{\alpha_1}, \cdots, V_{\alpha_k} \rangle$ containing x such that $\langle Cl(V_{\alpha_1}), Cl(V_{\alpha_2}), \cdots, Cl(V_{\alpha_k}) \rangle \subseteq O$. Then $a = p_\alpha(x) \in p_\alpha(E) = V_\alpha$, where $\alpha \in \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$, and $p_\alpha(Cl(E)) = Cl(V_\alpha) \subseteq p_\alpha(O)$. This shows that $p_\alpha(O)$ is super-open in Y_α. □

Remark 2 The converse of Theorem 4 is not true.

To see this, consider $Y_1 = \{1, 2, 3\}$ and $Y_2 = \{a, b, c\}$ with the respective topologies $\tau_1 = \{Y_1, \varnothing, \{2\}, \{1, 2\}, \{2, 3\}\}$ and $\tau_2 = \{Y_2, \varnothing, \{a\}, \{c\}, \{a, c\}\}$. Consider the set $O = \{(1, a), (2, c), (3, b)\}$. Then the family $\mathcal{B}$ consisting of the sets $Y_1 \times Y_2$, $\varnothing$, $Y_1 \times \{a\}$, $Y_1 \times \{c\}$, $Y_1 \times \{a, c\}$, $\{2\} \times Y_2$, $\{1, 2\} \times Y_2$, $\{2, 3\} \times Y_2$, $\{(1, a), (2, a)\}$, $\{(1, c), (2, c)\}$, $\{(2, a)\}$, $\{(2, c)\}$, $\{(2, a), (2, c)\}$, $\{(2, a), (3, a)\}$, $\{(2, c), (3, c)\}$, $\{(1, a), (1, c), (2, a), (2, c)\}$, and $\{(2, a), (2, c), (3, a), (3, c)\}$ is a basis for the Tychonoff topology on $Y_1 \times Y_2$. Since $p_1(O) = \{1, 2, 3\} = Y_1$ and $p_2(O) = \{a, b, c\} = Y_2$, it follows that $p_1(O)$ and $p_2(O)$ are super-open in Y_1 and Y_2, respectively. However, since there exists no basic open set V in Y containing $(1, a)$ with $Cl(V) \subseteq O$, it follows that O is not super-open in Y. □

Theorem 5 *Let $S = \{\alpha_1, \alpha_2, ..., \alpha_k\}$ be a finite subset of $\mathcal{A}$ and $\varnothing \neq O_{\alpha_i} \subseteq Y_{\alpha_i}$ for each $\alpha_i \in S$. Then $\langle O_{\alpha_1}, O_{\alpha_2}, ..., O_{\alpha_k} \rangle$ is super-open in $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ if and only if each O_{α_i} is super-open in Y_{α_i}.*

Proof Let $O = \langle O_{\alpha_1}, O_{\alpha_2}, ..., O_{\alpha_k} \rangle$ and suppose that each O_{α_i} is a non-empty super-open set in Y_{α_i}. Let $x = \langle a_\alpha \rangle \in O$. Then $a_{\alpha_i} \in O_{\alpha_i}$ for every $i \in J = \{1, 2, \cdots, k\}$. Hence, for each $i \in J$, there exists an open set V_{α_i} in Y_{α_i} with $a_{\alpha_i} \in V_{\alpha_i}$ such that $Cl(V_{\alpha_i}) \subseteq O_{\alpha_i}$. Let $V = \langle V_{\alpha_1}, \ldots, V_{\alpha_k} \rangle$. By Theorem 2, $Cl(V) = \langle Cl(V_{\alpha_1}), Cl(V_{\alpha_2}), \ldots, Cl(V_{\alpha_k}) \rangle$. Thus, V is an open set in Y with $x \in V$ such that $Cl(V) = \langle Cl(V_{\alpha_1}), Cl(V_{\alpha_2}), \ldots, Cl(V_{\alpha_k}) \rangle \subseteq O$. Thus, O is a super-open set in Y.

Conversely, suppose that O is a non-empty super-open set in Y. By Theorem 4, $p_{\alpha_i}(O) = O_{\alpha_i}$ is super-open in Y_{α_i} for every $i \in \{1, 2, ..., k\}$. □

We shall now characterize super-continuous functions from an arbitrary topological space X into the product space Y.

Theorem 6 *Let X be a topological space and $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ a product space. A function $f : X \to Y$ is super-continuous on X if and only if each coordinate function $p_\alpha \circ f$ is super-continuous on X.*

Proof Suppose that f is super-continuous on X. Let $\alpha \in \mathcal{A}$ and U_α be open in Y_α. Since p_α is continuous, $p_\alpha^{-1}(U_\alpha)$ is open in Y. Hence,

$$f^{-1}(p_\alpha^{-1}(U_\alpha)) = (p_\alpha \circ f)^{-1}(U_\alpha)$$

is a super-open set in X since f is super-continuous. Thus, $p_\alpha \circ f$ is super- continuous for every $\alpha \in \mathcal{A}$.

Conversely, suppose that each coordinate function $p_\alpha \circ f$ is super- continuous. Let G_α be open in Y_α. Then $\langle G_\alpha \rangle$ is a subbasic open set in Y and

$$(p_\alpha \circ f)^{-1}(G_\alpha) = f^{-1}(p_\alpha^{-1}(G_\alpha)) = f^{-1}(\langle G_\alpha \rangle)$$

is a super-open set in X. Therefore, f is super-continuous on X, by Remark 1. □

Corollary 1 Let X be a topological space, Y the product space and $f_\alpha : X \to Y_\alpha$ a function for each $\alpha \in \mathcal{A}$. Let $f : X \to Y$ be the function defined by $f(x) = \langle f_\alpha(x) \rangle$. Then f is super-continuous on X if and only if f_α is super-continuous for each $\alpha \in \mathcal{A}$.

Proof For each $\beta \in \mathcal{A}$ and each $x \in X$, we have

$$(p_\beta \circ f)(x) = p_\beta(f(x)) = p_\beta(\langle f_\alpha(x) \rangle) = f_\beta(x).$$

Thus, $p_\beta \circ f = f_\beta$ for every $\beta \in \mathcal{A}$. The result now follows from Theorem 6. □

Theorem 7 *Let X and Y be the product spaces of the families of spaces $\{X_\alpha : \alpha \in \mathcal{A}\}$ and $\{Y_\alpha : \alpha \in \mathcal{A}\}$, respectively, and for each $\alpha \in \mathcal{A}$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a function. If each f_α is super-continuous, then the function $f : X \to Y$, defined by $f(\langle x_\alpha \rangle) = \langle f_\alpha(x_\alpha) \rangle$, is super-continuous on X.*

Proof Let $\langle V_\alpha \rangle$ be a subbasic open set in Y. Then $f^{-1}(\langle V_\alpha \rangle) = \langle f_\alpha^{-1}(V_\alpha) \rangle$. Since f_α is super-continuous, $f_\alpha^{-1}(V_\alpha)$ is super-open in X_α. Now, we let an element $x = \langle x_\beta \rangle \in \langle f_\alpha^{-1}(V_\alpha) \rangle$. Then $x_\alpha \in f_\alpha^{-1}(V_\alpha)$. Hence, there exists an open set G_α in X_α with $x_\alpha \in G_\alpha$ such that $Cl(G_\alpha) \subseteq f_\alpha^{-1}(V_\alpha)$. Clearly, $\langle G_\alpha \rangle$ is open in X and $x \in \langle G_\alpha \rangle$. By Theorem 2, $Cl(\langle G_\alpha \rangle) = \langle Cl(G_\alpha) \rangle \subseteq \langle f_\alpha^{-1}(V_\alpha) \rangle$. This shows that $f^{-1}(\langle V_\alpha \rangle) = \langle f_\alpha^{-1}(V_\alpha) \rangle$ is super-open in X. Therefore, f is super-continuous on X. □

3 General Remarks

The paper has so far described super-open sets and established the formula of the super-closure of a set in a Cartesian product space. Moreover, the paper has formulated a necessary and sufficient condition for super-continuity of a function from an arbitrary space into the product space. This particular result is the counterpart of the known characterization of the ordinary continuity of a function into a product space. The results generated in this paper would certainly find importance as one explores further other possible concepts (e.g. separation axioms, axioms of countabilty, and compactness) involving super-open sets and studies them in the product space.

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