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Gracefully Harmonious Graphs

M. Murugan
School of Science, Tamil Nadu Open University
577, Anna Salai, Chennai–600 015, India
e-mail: muruganganesan@yahoo.in

Abstract Labelling problems are very useful in modelling, solving problems in other areas and enhance the study of algebraic structures. Nowadays, a more practical, accurate and efficient way of labeling a graph is using some mathematical methods like integer programming or constraint programming. This will improve the scope and quality of research. Here, considering the applications of Graceful and Harmonious labelings, we introduce a few new types of labeling, namely, gracefully harmonious labeling, gracefully felicitous labeling, gracefully elegant labeling, gracefully strongly c-elegant labeling and gracefully semi-harmonious labeling. In this paper, we prove that some complete multi-partite graphs is gracefully harmonious, gracefully felicitous and gracefully semi-harmonious.

Keywords Labeling, multi-partite graph, gracefully harmonious, gracefully felicitous, gracefully semi-harmonious.

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1 Introduction

Labeling of a graph is an assignment of labels (numbers) to its vertices or / and edges or faces, which satisfy some conditions. Graph labelings have several applications in X-ray crystallography, radar, communication networks, design theory, astronomy, computer science and combinatorics. Labeled graphs are very useful models for a broad range of applications. α-labelings are useful in graph decomposition problems. Graceful labelings arose in the characterization of finite neofields and in the study of perfect systems of difference sets. We can see the connection between graceful labeling and a variety of algebraic structures including cyclic difference sets, sequenceable groups, generalized complete mappings and near-complete mappings.

The best way of labeling a graph is using mathematical methods. There are two good techniques for graceful labeling problems, one involves integer programming and the other involves constraint programming and such techniques are available to harmonious labeling problems. Also, the complexity of determining whether a graph is harmonious or not was shown to be NP - complete, where as the complexity of determining whether a graph is graceful or not is unknown.

Rosa *et al.* [1] introduced many labeling methods and Golomb [2] was instrumental in coining the phraseology "Graceful Graphs". The famous Graceful Tree Conjecture (also known as Ringel-Kotzig or Rosa's or even Ringel-Kotzig – Rosa Conjecture) which says that "All Trees are Graceful" was first mentioned in [3] and this remains unsettled for many years. Gallian in his survey [4] of graph labeling, has mentioned the conjecture that every tree is (k,d)-graceful for some $k > 1$ and $d > 1$. Hegde [5] has conjectured that all trees are (k,d)-balanced for some values of k and d. A caterpillar is a tree with the property that the removal of its endpoints leaves a path. A lobster is a tree with the property that the removal of the endpoints leaves a caterpillar. Bermond [6] conjectured that lobsters are graceful and this is still open.

Graham and Sloane [7] have introduced harmonious graphs in their study of modular versions of additive bases problems stemming from error-correcting codes. The conjecture, "All Trees are Harmonious" is still open and is unsettled for many years. Gallian [4] in his survey of graph labeling, has mentioned that no attention has been given to analyze the harmonious property of lobsters. Many variations and generalizations of labeling graphs have been studied by many authors and in many ways. Since graceful labeling and harmonious labeling have a lot of applications, here we introduce a few new labelings based on the above two labelings and its variations.

Gallian [8] in his survey, had mentioned that no one has looked at the labeling of complete tri-partite graphs. Not much work has been done even today. Much work has to be done in n-partite graphs, in particular, complete n-partite graphs.

2 Different Labelings

In this section, we list a few existing labelings which are useful for the development of this paper. Here, we consider a graph G with p vertices and q edges.

Definition 1: Graceful Labeling
An injective function f from the vertices of G to $\{0, 1, 2, \ldots, q\}$ is called graceful labeling if the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is $\{1, 2, \ldots, q\}$.

Definition 2: Graceful Graph
A graph which has a graceful labeling is called a graceful graph.

Definition 3: Harmonious Labeling
An injective function f from the vertices of G to the group of integers modulo $q$is called harmonious if all edge labels are distinct when each edge xy is assigned the label $f(x)+f(y)$ (mod q); when G is a tree, exactly one label may be used on two vertices.

Definition 4: Harmonious Graph
A graph which has a harmonious labeling is called a harmonious graph.

This was introduced by Graham and Sloan [7].

Definition 5: Felicitous Labeling
An injective function f from the vertices of a graph G with q edges to the set $\{0,1,\ldots,q\}$is called felicitous if the edge labels induced by $f(x) + f(y)$ (mod q) for each edge xy are distinct.

Definition 6: Felicitous Graph
A graph which has a felicitous labeling is called a felicitous graph.

This was introduced by Lee *et al.* [9].

Definition 7: Elegant Labeling
An elegant labeling f of a graph G with q edges is an injective function from the vertices of G to the set $\{0,1,\ldots,q\}$such that when each edge xy is assigned the label $f(x) + f(y)$ (mod $(q + 1)$) the resulting edge labels are distinct and non-zero.

Definition 8: Elegant Graph
A graph which has a elegant labeling is called a elegant graph.

This notion was introduced by Chang et al.[10].

Definition 9: Strongly c-Elegant Labeling
A strongly c-elegant labeling f of a graph G with q edges is an injective function from the vertices of G to the set $\{0,1,\ldots,q\}$such that the edge labels induced by addition are $\{c,c+1,\ldots,c+q-1\}$.

Definition 10: Strongly c-Elegant Graph
A graph which has a strongly c-elegant labeling is called a strongly c-elegant graph.

This notion was introduced by Chang *et al.* [10]. We notice that every strongly c- elegant graph is felicitous.

3 Variations of Graceful and Harmonious Graphs

Considering the existing definitions in the literature, here we define a few new types of labeling. We expect the new labeling to satisfy the conditions of graceful and harmonious / felicitous / elegant labeling. We note that we use vertex labels $0,1,\ldots,q$ in case of graceful graphs and the vertex labels $0,1,\ldots,q-1$ in case of harmonious graphs; when G is a tree, exactly one label may be used on two vertices. The new labelings are interesting and also difficult to prove even for standard graphs. Now it will be so interesting and challenging to find some mathematical techniques and the complexity for the new family of graphs.

Definition 11: Gracefully Harmonious Labeling
A labeling which is both graceful and harmonious is called a gracefully harmonious labeling.

Trees may have such a labeling.

Definition 12: Gracefully Harmonious Graph
A graph which has a gracefully harmonious labeling is called a gracefully harmonious graph.

Definition 13: Gracefully Felicitous Labeling
A labeling f from the vertices of G to $\{0,1,2,\ldots, q\}$is called a gracefully felicitous labeling if

(i) the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is $\{1,2,..., q\}$.

(ii) the edge labels induced by $f(x) + f(y)$ (mod q) for each edge xy are distinct.

Definition 14: Gracefully Felicitous Graph
A graph which has a gracefully felicitous labeling is called a gracefully felicitous graph.

Definition 15: Gracefully Elegant Labeling
A labeling f from the vertices of G to {0,1,2,..., q}is called a gracefully elegant labeling if

(i) the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is {1,2, ..., q}.

(ii) the edge labels $f(x) + f(y)$ (mod $(q + 1)$) are distinct and non-zero.

Definition 16: Gracefully Elegant Graph
A graph which has a gracefully elegant labeling is called a gracefully elegant graph.

Definition 17: Gracefully Strongly c-Elegant Labeling
A labeling f from the vertices of G to {0,1,2,..., q}is called a gracefully strongly c-elegant labeling if

(i) the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is {1,2,...,q}.

(ii) the edge labels induced by addition are $\{c, c + 1, \ldots, c + q - 1\}$.

Definition 18: Gracefully Strongly c-Elegant Graph
A graph which has a gracefully strongly c-elegant labeling is called a gracefully strongly c-elegant graph.

Definition 19: Gracefully Semi-Harmonious Graph
A labeling f from the vertices of G to {0,1,2,...,q}is called a gracefully semi-harmonious labeling if

(i) the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is {1,2, ...,q}.

(ii) the edge labels induced by $f(x) + f(y)$ for each edge xy are distinct.

Definition 20: Gracefully Semi-Harmonious Graph
A graph which has a gracefully semi-harmonious labeling is called a gracefully semi-harmonious graph.

In all these cases, we denote the edge label as a, b where $a = |f(x) - f(y)|$ and $b = f(x) + f(y)$ or $f(x) + f(y) \pmod q$ or $f(x) + f(y) \pmod{q + 1}$.

Definition 21: Complete Bipartite Graph
A complete bipartite graph is a simple bipartite graph with bipartition (X,Y) in which each vertex of X is joined to each vertex of Y.

If $|X| = m$ and $|Y| = n$, then a complete bipartite graph with bipartition (X,Y) is denoted by $K_{m,n}$.

Definition 22: Complete n-Partite Graph
If G is an n-partite graph having partite sets V_1, V_2, ..., V_n such that every vertex of V_i is joined to every vertex of V_j, where $1 \leq i$, $j \leq n$ and $i \neq j$, then G is called a complete n-partite graph.

If $|V_i| = p_i$, for $i = 1,2,\ldots n$, then we denote G by $K_{p_1,p_2,..,p_n}$

These graphs are also called complete multipartite graphs.

Consider the complete bipartite graph $K_{1,6}$. The labeling which is shown in Figure 1 is gracefully harmonious and hence $K_{1,6}$ is a gracefully harmonious graph.

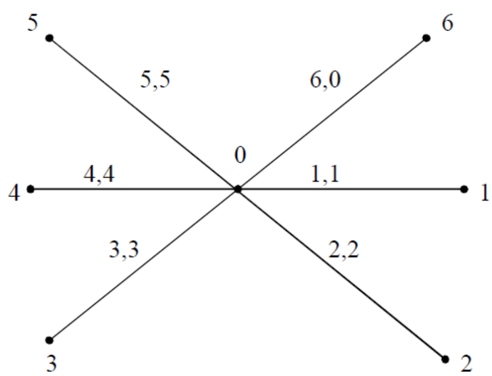

Figure 1: Gracefully Harmonious Labeling of $K_{1,6}$

Consider the binary tree on 7 vertices with the labeling shown in Figure 2. The labeling which is shown in Figure 2 is gracefully semi-harmonious and hence it is a gracefully semi-harmonious graph.

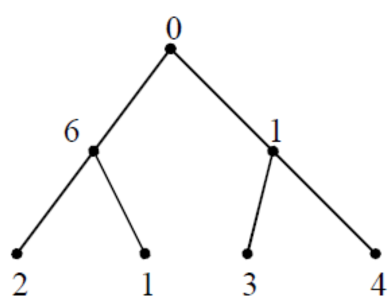

Figure 2: Gracefully Semi-Harmonious Labeling

Consider the cycle C_4 with the labeling shown in Figure 3. The labeling which is shown in Figure 3 is gracefully felicitous and hence C_4 is a gracefully felicitous graph.

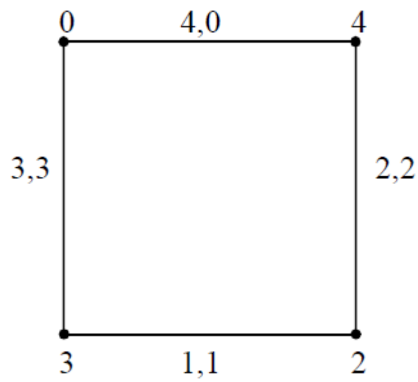

Figure 3: Gracefully Felicitous Labeling of C_4

4 Results

In this section, we prove that $K_{1,a}$ is gracefully harmonious, $K_{a,b}$ is gracefully felicitous and $K_{a,1,b}$, $K_{a,2,b}$ and $K_{a,1,1,b}$ are gracefully semi-harmonious.

Theorem 1 *The star $K_{1,a}$ is gracefully harmonious.*

Proof Consider the star $K_{1,a}$. Let the vertex set of this graph be $V = V_1 \cup V_2$, where $|V_1| = 1, |V_2| = a$. Let the vertices of V_1 be $\{w\}$and V_2 be $\{v_1,\ v_2,\ldots,v_a\}$. Number of vertices in $K_{a,b} = a + 1$. Number of edges in $K_{a,b} = a$.

Define a labeling $f : \{w, v_1, v_2, \ldots, v_a\} \to \{0, 1, 2, \ldots, a\}$ such that $f(w) = 0$ and $f(v_i) = i, 1 \leq i \leq a$.

The edge labels induced by the absolute value of the difference of the labels of adjacent vertices are 1, 2,..., a. Hence this labeling is graceful.

Now we prove that this labeling is harmonious. Since $K_{1,a}$ is a tree, 0 is used on two vertices. The vertices of $K_{1,a}$ receive the labels $0, 1, \ldots, a - 1$ under addition modulo a.

Under addition modulo a, the edges receive the labels $0, 1, \ldots, a-1$. Hence this labeling is harmonious.

Hence the star $K_{1,a}$ is gracefully harmonious. □

Theorem 2 *Complete bipartite graph $K_{a,b}$ is gracefully felicitous.*

Proof Consider $K_{a,b}$. Let the vertex set of this graph be $V = V_1 \cup V_2$, where

$$|V_1| = a,\ |V_2| = b.$$

Let the vertices of V_1 be $\{v_1,\ v_2,\ldots,v_a\}$and V_2 be $\{w_1,\ w_2,\ldots,w_b\}$. Without loss of generality, we assume that $a \leq b$. Number of vertices in $K_{a,b} = a + b$. Number of edges in $K_{a,b} = a \cdot b$.

Define a labeling $f : \{v_1, v_2, \ldots, v_a, w_1, w_2, \ldots, w_b\} \to \{0, 1, 2, \ldots, ab\}$ such that

$$f(v_i) = (i-1)b,\ 1 \le i \le a$$
$$f(w_i) = ab - (i-1), 1 \le i \le b.$$

We prove that this labeling is both graceful and felicitous. Let

$$L = \{i \cdot b \,|\, 0 \le i \le a-1\} \cup \{ab - i \,|\, 0 \le i \le b-1\},$$

where

$$L_1 = \{i \cdot b \,|\, 0 \le i \le a-1\} \quad \text{and} \quad L_2 = \{ab - i \,|\, 0 \le i \le b-1\}.$$

Clearly $L_1 \cap L_2 = \varphi$ and different $v_i\, s$ are labeled by a distinct label of L_1 and different $w_i\, s$ are labeled by a distinct label of L_2. Hence all the vertex label are distinct and lie between 0 and ab.

Next we prove that the edge labels induced by the absolute value of the difference of the labels of adjacent vertices are distinct and lie between 1 and ab. Let

$$D = \bigcup_{\alpha=1}^{a} D_\alpha$$

where $D_\alpha = \{ab - (\alpha-1)b - i \ | \, 0 \le i \le b-1 \,\}$, $\alpha = 1,\, 2,\, \ldots, a$.

For $1 \le i \le a$, different $v_i w_j, j = 1, 2, \ldots, b$ are labeled by a distinct label of D_i. Since all D_is are mutually disjoint and

$$\bigcup_{i=1}^{a} D_i = \{1, 2, \ldots, ab\},$$

the edge labels are distinct and lie between 1 and ab. Hence f is a graceful labeling.

Next we prove that the edge labels induced by the sum of the vertex labels of the vertices incident with that edge are distinct. Let

$$S = \bigcup_{\alpha=1}^{a} S_\alpha$$

where $S_\alpha = \{ab + (\alpha-1)b - i \,|\, 0 \le i \le b-1\}$, $\alpha = 1, 2, \ldots, a$.

For $1 \le i \le a$, different $v_i w_j, j = 1, 2, \ldots, b$ are labeled by a distinct label of S_i . Since all $S_i s$ are mutually disjoint and

$$\bigcup_{i=1}^{a} S_i = \{ab - b + 1, \ldots, 2ab - b\},$$

the edge labels are distinct and lie between $ab - b + 1$ and $2ab - b$. Under modulo ab, the edges receive the labels $ab - b + 1, \ldots, 0, \ldots, ab - b$. Hence f is a felicitous labeling.

Hence complete bipartite graph $K_{a,b}$ is gracefully felicitous. □

Theorem 3 *$K_{a,1,b}$ is gracefully semi-harmonious.*

Proof Consider a complete tri-partite graph G with partitions A,B and C such that

$$|A| = a,\ |B| = 1,\ |C| = b.$$

Let the vertices of A be $\{v_1, v_2, v_3, \ldots, v_a\}$, the vertex of B be $\{w\}$and the vertices of C be $\{u_1, u_2, u_3, \ldots, u_b\}$.

Number of vertices in $G = a+b+1$. Number of edges in $G = a+b+ab$. Define a labeling

$$f : \{v_1, v_{2,\ldots,} v_a, w, u_1, u_2, \ldots, u_b\} \to \{0, 1, 2, 3, \ldots, a+b+ab\}$$

such that

$$f(v_i) = a+b+ab-(i-1), i = 1, 2, 3, \ldots a$$
$$f(w) = 0$$

and

$$f(u_i) = \begin{cases} (b+ab) - \dfrac{(i-1)(a+1)}{2}, & i \text{ is odd and } 1 \leq i \leq b \\ \dfrac{i(a+b)}{2}, & i \text{ is even and } 1 \leq i \leq b. \end{cases}$$

We prove that this labeling is gracefully semi-harmonious. Let us assume that b is odd. Let

$$L = L_1 \cup L_2 \cup L_3 \cup \{0\},$$

where

$$L_1 = \Big\{a+b+ab+1-i,\ 1 \leq i \leq a\Big\}$$
$$L_2 = \Big\{b+ab - \frac{(i-1)(a+1)}{2},\ i \text{ is odd and } 1 \leq i \leq b\Big\}$$
$$L_3 = \Big\{\frac{i(a+1)}{2},\ i \text{ is even and } 1 \leq i \leq b\Big\}.$$

Here L_is are mutually disjoint and different v_is are labeled with a distinct label of L_1, different u_is, i is odd and $i \leq b$, are labeled with a distinct label of L_2, different u_is, i is even and $i \leq b$, are labeled with a distinct label of L_3 and w is labeled with label 0. Hence all the vertex labels are distinct and lie between 0 and $a+b+ab$.

Next we prove that the edge labels induced by the absolute value of the difference of the labels of adjacent vertices are distinct and lie between 1 and $a+b+ab$. Let

$$D = D_\alpha \cup \left(\bigcup_{j=1}^{b} D_j\right) \cup D_\beta$$

where

$$
\begin{aligned}
D_\alpha &= \Big\{a+b+ab+1-i,\ 1\le\ i\le a\Big\},\\
D_j &= \left\{a+1-i+\frac{(j-1)(a+1)}{2},\ i=1,2,\ldots,a\right\},\ j=1,3,5,\ldots,b\\
&= \left\{a+b+ab+1-i-\frac{j(a+1)}{2},\ i=1,2,\ldots,a\right\},\ j=2,4,6,\ldots,b-1,\\
D_\beta &= \left\{b+ab-\frac{(j-1)(a+1)}{2},\ j \text{ is odd and } 1\le j\le b\right\}\\
&\quad\cup\left\{\frac{j(a+1)}{2},\ j\ \text{ is even and } 1\le j\le b\right\}.
\end{aligned}
$$

For $1\le i\le a$, different edge wv_i is labeled with a distinct label $a+b+ab+1-i$ of D_α. For j odd and $1\le j\le b$, different edge u_jv_i, $i=1,2,\ldots,a$ is labeled with a distinct label

$$a+1-i+\frac{(j-1)(a+1)}{2}\ \text{ of } D_j.$$

For j even and $1\le j\le b$, different edge $u_jv_i, i=1,2,\ldots,a$ is labeled with a distinct label

$$a+b+ab+1-i-\frac{j(a+1)}{2}\ \text{ of } D_j.$$

For j odd and $1\le j\le b$, different edge wu_j is labeled with a distinct label

$$b+ab-\frac{(j-1)(a+1)}{2}\ \text{ of } D_\beta$$

and for j even and $1\le j\le\ b$, different edge wu_j is labeled with a distinct label

$$\frac{j(a+1)}{2}\ \text{ of } D_\beta.$$

Since all D_is are mutually disjoint and $\cup D_i=\{1,2,\ldots,a+b+ab\}$, the edge labels are distinct and lie between 1 and $a+b+ab$. Hence f is a graceful labeling.

Next we prove that the edge labels induced by the sum of the vertex labels of the vertices incident with that edge are distinct. Let

$$S\quad=\quad S_\alpha\cup\left(\bigcup_{j=1}^{b}S_j\right)\cup S_\beta$$

where

$$S_\alpha = \left\{a+b+ab+1-i,\ 1\le i\le a\right\},$$
$$S_j = \left\{a+2b+2ab+1-i-\frac{(j-1)(a+1)}{2},\ i=1,2,...,a\right\}, j=1,3,5,\ldots,b$$
$$= \left\{a+b+ab+1-i+\frac{j(a+1)}{2},\ i=1,2,...,a\right\}, j=2,4,6,\ldots,b-1,$$
$$S_\beta = \left\{b+ab-\frac{(j-1)(a+1)}{2},\ i \text{ odd and } 1\le i\le b\right\}$$
$$\cup\left\{\frac{j(a+1)}{2}/i \text{ even and } 1\le i\le b\right\}.$$

For $1\le i\le a$, different edge wv_i is labeled with a distinct label $a+b+ab+1-i$ of S_α. For j odd and $1\le j\le b$, different edge u_jv_i, $i=1,2,\ldots,a$ is labeled with a distinct label

$$a+2b+2ab+1-i-\frac{(j-1)(a+1)}{2} \text{ of } S_j.$$

For j even and $1\le j\le b$, different edge $u_jv_i, i=1,2,\ldots,a$ is labeled with a distinct label

$$a+b+ab+1-i+\frac{j(a+1)}{2} \text{ of } S_j.$$

For j odd and $1\le j\le b$, different edge wu_j is labeled with a distinct label

$$b+ab+\frac{(j-1)(a+1)}{2} \text{ of } S_\beta$$

and for j even and $1\le j\le b$, different edge wu_j is labeled with a distinct label

$$\frac{j(a+1)}{2} \text{ of } S_\beta.$$

Since all S_is are mutually disjoint, the edge labels are distinct and lie between $a+1$ and $2b+2ab+a$.

When b is even, we can give a similar proof. Hence f is a gracefully semi-harmonious labeling. □

Theorem 4 $K_{a,2,b}$ *is gracefully semi-harmonious.*

Proof Consider a complete tri-partite graph $G$with partitions A,B and C such that

$$|A|=a,\quad |B|=2,\quad |C|=b.$$

Let the vertices of A be $\{v_1, v_2, v_3, \ldots, v_a\}$, the vertex of B be $\{u,v\}$and the vertices of C be $\{u_1, u_2, u_3, \ldots, u_b\}$. Number of vertices in $G=a+b+2$. Number of edges in $G=ab+2a+2b$. Define a labeling

$$f:\{v_1, v_2, \ldots, v_a, u, v, u_1, u_2, \ldots, u_b\}\rightarrow\{0,1,2,\ldots,ab+2a+2b\}$$

such that

$$\begin{aligned} f(u) &= 0, \\ f(v) &= b+1, \\ f(u_i) &= ab+2a+2b-i,\ i=1,2,3,\ldots,b, \\ f(v_1) &= ab+2a+2b, \end{aligned}$$

$$f(v_i) = \begin{cases} 2(b+1) + \dfrac{(i-3)(b+2)}{2}, & i \text{ is odd and } 3 \le i \le a, \\ ab+2a-2-\left(\dfrac{i}{2}-1\right)(b+2), & i \text{ is even.} \end{cases}$$

Clearly the vertex labels are distinct and lie between 0 and $ab + 2a + 2b$. Now we have to prove that the edge labels induced by the absolute value of the difference of the labels of adjacent vertices are distinct and lie between 1 and $ab + 2a + 2b$ and also the edge labels induced by the sum of the vertex labels of the vertices incident with that edge are distinct, which is a routine verification. □

Theorem 5 *$K_{a,1,1,b}$ is gracefully semi-harmonious.*

Proof Consider a complete 4-partite graph G with partitions A,B,C and D such that

$$|A| = a, \quad |B| = 1, \quad |C| = 1 \text{ and } |D| = b.$$

Let the vertices of A be $\{v_1, v_2, v_3, \ldots, v_a\}$, the vertex of B be $\{v\}$and the vertices of C be $\{u\}$and the vertices of D be $\{u_1, u_2, u_3, \ldots, u_b\}$. Number of vertices in $G = a + b + 2$. Number of edges in $G = ab + 2a + 2b + 1$. Define a labeling

$$f : \{v_1, v_2, \ldots v_a, v, u, u_1, u_2, \ldots, u_b\} \to \{0, 1, 2, \ldots, ab + 2a + 2b + 1\}$$

such that

$$\begin{aligned} f(v) &= ab+2a+2b+1, \\ f(u) &= 0, \\ f(u_i) &= ab+2a+2b+1-i,\ i=1,2,\ldots,b \end{aligned}$$

$$f(v_i) = \begin{cases} b+1+\dfrac{(i-1)\ (b+2)}{2}, & i \text{ is odd and } 1 \le i \le a, \\ ab+a-1-\left(\dfrac{i}{2}-1\right)(b+2), & i \text{ is even and } 2 \le i \le a. \end{cases}$$

Clearly the vertex labels are distinct and lie between 0 and $ab + 2a + 2b + 1$. Now we have to prove that the edge labels induced by the absolute value of the difference of the labels of adjacent vertices are distinct and lie between 1 and $ab + 2a + 2b + 1$ and also the edge labels induced by the sum of the vertex labels of the vertices incident with that edge are distinct, which is a routine verification. □

We close this paper with the following conjectures.

Conjecture 1 $K_{a,b,c}$ is gracefully semi-harmonious.

Conjecture 2 $K_{a,b,c,d}$ is gracefully semi-harmonious.

5 Conclusion

In this paper, we introduced a few new types of labeling and proved that $K_{1,a}$ is gracefully harmonious, $K_{a,b}$ is gracefully felicitous and $K_{a,1,b}$,$K_{a,2,b}$ and $K_{a,1,1,b}$ are gracefully semi-harmonious. Graceful labeling and Harmonious labeling have been studied for over three decades and these topics continue to be a fascinating one in the field of graph theory. The scope of research is widening in new directions like labeling a graph using some mathematical methods, determining the complexity of these labeling problems, new types of labelings and so on. Though an abundance of papers exists, it provides many interesting problems and unproven conjectures and motivate researchers. We believe that this paper will sparkle interest in labelings, in particular, graceful labeling, harmonious labeling and their variations and motivate researchers in solving the open problems mentioned in this paper and other related problems.

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