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On Pi-prenormal Spaces

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Abstract The main aim of this paper is to obtain some characterizations of pi-prenormal spaces by using the notion of pi-generalized closed sets. Also, by using these characterizations we establish various preservation theorems of pi-prenormality under continuous and some generalized sense of continuous mappings. We give some characterizations of almost preregular spaces and present some relationships between prenormality and almost preregularity.

Keywords closed domain; p-closed; almost p-regular; p_3-paracompact.

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1 Introduction

Throughout this paper, a space X always means a topological space on which no separation axioms are assumed, unless explicitly stated. For a subset A of a space X, $X \setminus A$, $\overline{A}$ and $\text{int}(A)$ denote to the complement, the closure and the interior of A in X, respectively. A subset A of a space X is said to be a *regularly-open* or an *open domain* if it is the interior of its own closure, or equivalently if it is the interior of some closed set, [1]. A set A is said to be a *regularly-closed* or a *closed domain* if its complement is an open domain. A subset A of a space X is called π*-closed* if it is a finite intersection of closed domain sets, [2]. A subset A is called π*-open* if its complement is π-closed. Two sets A and B of a space X are said to be *separated* if there exist two disjoint open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$, [3–5]. A subset A of a space X is said to be *pre-open* (briefly; *p-open*), [6], if $A \subseteq \text{int}(\overline{A})$. A subset A of a space X is said to be *semi-open* if $A \subseteq \overline{\text{int}(A)}$, [7]. A space X is called *pre-normal* (briefly; *p-normal*), [8], if any two disjoint closed subsets A and B of X can be separated by two disjoint pre-open subsets. A space X is called an *almost p-normal*, [9], if any two disjoint closed subsets A and B of X, one of which is closed domain, can be separated by two disjoint pre-open subsets. A space X is called a *mildly p-normal*, [9], if any pair of disjoint closed domain subsets A and B of X, can be separated by two disjoint pre-open subsets. A space X is said to be a π*-prenormal* (briefly; πp*-normal*), [10], if any pair of disjoint closed subsets A and B of X, one of which is π-closed, can be separated by two disjoint pre-open subsets. A space X is said to be a π*-normal*, [11], if any pair of disjoint closed subsets A and B of X, one of which is π-closed, can be separated by two disjoint open subsets. The complement of pre-open (resp. semi-open) set is called pre-closed (resp. semi-closed). The intersection of all pre-closed sets containing A is called *pre-closure* of A, [12], and denoted by $p\,\text{cl}(A)$. Dually, the *pre-interior* of A denoted by $p\,\text{int}(A)$, is defined to be the union of all pre-open sets contained in A. Let A be a subset of a space X, then a subset V of a space X is said to be a *pre-neighborhood* (briefly; *p-neighborhood*) of A if there is a pre-open set U of X such that $A \subset U \subset V$, [13].

Clearly, every normal space is π-normal as well as p-normal, every π-normal space is πp-normal and we have:

$$p\text{-normal} \Longrightarrow \pi p\text{-normal} \Longrightarrow \text{almost } p\text{-normal} \Longrightarrow \text{mildly } p\text{-normal}$$

Observe that none of the above implications is reversible as shown by the examples in [10]. In this paper, we give various characterizations and preservation theorems of πp-normal spaces. Also, some characterizations of almost p-regularity as well as its relations with πp-normality are presented.

2 Characterizations of πp-normality

Some characterizations of πp-normality have been given in [10]. In this paper, we present various characterizations of it by using the notion of π-generalized closed sets. First, we need to recall the following definitions.

Definition 1 A subset A of a space X is called:

(i) *generalized closed* (briefly; *g-closed*), [14], if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is open.

(ii) *strongly generalized closed* (briefly; *g^*-closed*), [15], if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is g-open.

(iii) *π-generalized closed* (briefly; *πg-closed*), [16], if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is π-open.

(iv) *generalized pre-closed*, [17], (briefly; *gp-closed*) if $p\,\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open.

(v) *strongly generalized pre-closed*, [18], (briefly; *g^*p-closed*), if $p\,\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open.

(vi) *π-generalized pre-closed*, [19], (briefly; *πgp-closed*) if $p\,\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is π-open.

The complement of g-closed (resp. g^*-closed, πg-closed, gp-closed, g^*p-closed, πgp-closed) is called g-open (resp. g^*-open, πg-open, gp-open, g^*p-open, πgp-open). From the above definitions we have:

$$\text{closed} \Longrightarrow g^*\text{-closed} \Longrightarrow g\text{-closed} \Longrightarrow \pi g\text{-closed}$$
$$\text{closed} \Longrightarrow p\text{-closed} \Longrightarrow g^*p\text{-closed} \Longrightarrow gp\text{-closed} \Longrightarrow \pi gp\text{-closed}$$

Now, we give the following theorem, which is useful for giving some characterizations of πp-normal spaces.

Theorem 1 *For a space X, the following are equivalent:*

(a) *X is πp-normal.*

(b) *For each π-closed set A and each closed set B with $A \bigcap B = \emptyset$, there exist two gp-open subsets U and V of X such that $A \subseteq U$, $B \subseteq V$ and $U \bigcap V = \emptyset$.*

(c) *For each π-closed set A and each closed set B with $A \bigcap B = \emptyset$, there exist a πgp-open set U and a gp-open set V such that $A \subseteq U$, $B \subseteq V$ and $U \bigcap V = \emptyset$.*

(d) *For each π-closed set A and each open set B with $A \subseteq B$, there exists a gp-open subset U of X such that $A \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq B$.*

(e) *For each π-closed set A and each open set B with $A \subseteq B$, there exists a πgp-open subset U of X such that $A \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq B$.*

(f) *For each π-closed set A and each πg-open set B such that $A \subseteq B$, there exists a pre-open set U such that $A \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq \mathrm{int}(B)$.*

(g) *For each π-closed set A and each πg-open set B such that $A \subseteq B$, there exists a g^*p-open set U such that $A \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq \mathrm{int}(B)$.*

(h) *For each π-closed set A and each πg-open set B such that $A \subseteq B$, there exists a gp-open set U such that $A \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq \mathrm{int}(B)$.*

(i) *For each π-closed set A and each πg-open set B such that $A \subseteq B$, there exists a πgp-open set U such that $A \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq \mathrm{int}(B)$.*

(j) *For each πg-closed set A and each π-open set B such that $A \subseteq B$, there exists a pre-open set U such that $\overline{A} \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq B$.*

(k) *For each πg-closed set A and each π-open set B such that $A \subseteq B$, there exists a g^*p-open set U such that $\overline{A} \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq B$.*

(l) *For each πg-closed set A and each π-open set B such that $A \subseteq B$, there exists a gp-open set U such that $\overline{A} \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq B$.*

(m) *For each g-closed set A and each π-open set B such that $A \subseteq B$, there exists a pre-open set U such that $\overline{A} \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq B$.*

(n) *For each g^*-closed set A and each π-open set B such that $A \subseteq B$, there exists a pre-open set U such that $\overline{A} \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq B$.*

Proof In fact, $(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (d) \Longrightarrow (e) \Longrightarrow (f) \Longrightarrow (g) \Longrightarrow (h) \Longrightarrow (i) \Longrightarrow (j) \Longrightarrow (k) \Longrightarrow (l) \Longrightarrow (m) \Longrightarrow (n) \Longrightarrow (a)$. Now, we prove some of these implications and the rest can be proved as the same arguments.
$(c) \Longrightarrow (d)$. Let A be a π-closed set and B be an open set such that $A \subseteq B$. Then, $A \bigcap (X \setminus B) = \emptyset$, where $X \setminus B$ is closed. By (c), there exist a πgp-open set U and a gp-open set V such that $A \subseteq U$, $X \setminus B \subseteq V$ and $U \bigcap V = \emptyset$. Thus, $A \subseteq p\,\mathrm{int}(U)$, $X \setminus B \subseteq p\,\mathrm{int}(V)$ and $p\,\mathrm{int}(U) \bigcap p\,\mathrm{int}(V) = \emptyset$. Let $G = p\,\mathrm{int}(U)$. Then, G is pre-open set in X (hence gp-open) such that $A \subseteq G \subseteq p\,\mathrm{cl}(G) \subseteq B$.
$(e) \Longrightarrow (f)$. Let A be a π-closed and B be a πg-open such that $A \subseteq B$. Then, $A \subseteq \mathrm{int}(B)$ and $\mathrm{int}(B)$ is an open subset of X. Then by (e), there exists a πgp-open subset U of X such that $A \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq \mathrm{int}(B)$. Since A is π-closed, then we have $A \subseteq p\,\mathrm{int}(U)$. Now, let $V = p\,\mathrm{int}(U)$. Then, we obtain a pre-open set V such that $A \subseteq V \subseteq p\,\mathrm{cl}(V) \subseteq \mathrm{int}(B)$.
$(i) \Longrightarrow (a)$. Let A be a π-closed and B be a closed such that $A \bigcap B = \emptyset$. Then, $A \subseteq X \setminus B$, where $X \setminus B$ is open and hence πg-open. By (i), there exists a πgp-open set U such that

$A \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq \mathrm{int}(X \setminus B) = X \setminus B$. Thus, we get $A \subseteq p\,\mathrm{int}(U) \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq X \setminus B$. Let $G = p\,\mathrm{int}(U)$ and $H = X \setminus p\,\mathrm{cl}(U)$. Therefore, G and H are disjoint pre-open subsets of X such that $A \subseteq G$ and $B \subseteq H$. Hence, X is πp-normal.
$(i) \Longrightarrow (j)$. Let A be a πg-closed and B be a π-open sets in X such that $A \subseteq B$. Then, $X \setminus B \subseteq X \setminus A$, where $X \setminus B$ is π-closed and $X \setminus A$ is πg-open. Then by (i), there exists a πgp-open subset U of X such that $X \setminus B \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq \mathrm{int}(X \setminus A) = X \setminus \overline{A}$. Thus, we have $X \setminus B \subseteq p\,\mathrm{int}(U)$. Now, let $G = X \setminus p\,\mathrm{cl}(U)$ and $H = p\,\mathrm{int}(U)$. Then, G and H are disjoint pre-open subsets of X such that $\overline{A} \subseteq G$ and $X \setminus B \subseteq H$. Hence, we have $\overline{A} \subseteq G \subseteq p\,\mathrm{cl}(G) \subseteq B$.
$(l) \Longrightarrow (m)$. Let A be a g-closed and B be a π-open subsets of X such that $A \subseteq B$. Then, A is πg-closed. Thus by (l), there exists a gp-open set U such that $\overline{A} \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq B$. Then, we have $\overline{A} \subseteq p\,\mathrm{int}(U)$. Now, let $V = p\,\mathrm{int}(U)$. Then, V is pre-open subset of X such that $\overline{A} \subseteq V \subseteq p\,\mathrm{cl}(V) \subseteq B$.
$(n) \Longrightarrow (a)$. Let A and B be any disjoint closed subsets of X such that B is π-closed. Since $A \bigcap B = \emptyset$, then $A \subseteq X \setminus B$. Since A is g^*-closed and $X \setminus B$ is π-open, then by (n) there exists a pre-open subset U of X such that $\overline{A} = A \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq X \setminus B$. Put $V = X \setminus p\,\mathrm{cl}(U)$. Then, V is pre-open subset of X. Thus, we have $A \subseteq U$, $B \subseteq V$ and $U \bigcap V = \emptyset$. Hence, X is πp-normal. $\square$

The following result is obvious and it can be proved easily.

Theorem 2 *A space X is πp-normal if it satisfies one of the following:*

(1) *For each πgp-closed set A and each gp-open set B with $A \subseteq B$, there exists a pre-open set U such that $p\,\mathrm{cl}(A) \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq p\,\mathrm{int}(B)$.*

(2) *For each πgp-closed set A and each g^*p-open set B with $A \subseteq B$, there exists a pre-open set U such that $p\,\mathrm{cl}(A) \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq p\,\mathrm{int}(B)$.*

(3) *For each gp-closed set A and each g^*p-open set B with $A \subseteq B$, there exists a pre-open set U such that $p\,\mathrm{cl}(A) \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq p\,\mathrm{int}(B)$.*

(4) *For each gp-closed set A and each gp-open set B with $A \subseteq B$, there exists a pre-open set U such that $p\,\mathrm{cl}(A) \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq p\,\mathrm{int}(B)$.*

(5) *For each gp-closed set A and each πgp-open set B with $A \subseteq B$, there exists a pre-open set U such that $p\,\mathrm{cl}(A) \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq p\,\mathrm{int}(B)$.*

(6) *For each g^*p-closed set A and each gp-open set B with $A \subseteq B$, there exists a pre-open set U such that $p\,\mathrm{cl}(A) \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq p\,\mathrm{int}(B)$.*

(7) *For each g^*p-closed set A and each g^*p-open set B with $A \subseteq B$, there exists a pre-open set U such that $p\,\mathrm{cl}(A) \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq p\,\mathrm{int}(B)$.*

(8) *For each g^*p-closed set A and each πgp-open set B with $A \subseteq B$, there exists a pre-open set U such that $p\,\mathrm{cl}(A) \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq p\,\mathrm{int}(B)$.*

3 Characterizations of Almost p-regularity

Let us recall the following definition.

Definition 2 A space X is called a *p-regular* (resp. an *almost p-regular*) if for each closed (resp. closed domain) set F and each $x \notin F$, there exist disjoint pre-open sets U and V in X such that $x \in U$ and $F \subseteq V$, [6,12].

In view of the fact that if A is a π-closed and $x \notin A$, then there exists a closed domain set D such that $A \subseteq D$ and $x \notin D$, we present the following result that gives a useful characterization of almost p-regular spaces and it can be proved easily.

Theorem 3 *A space X is an almost p-regular if and only if for each π-closed set F and each $x \notin F$, there exist disjoint pre-open sets U and V in X such that $x \in U$ and $F \subseteq V$.*

Observe that if U and V are disjoint pre-open sets in X, then $p\mathrm{cl}(U) \bigcap V = \emptyset$ and $U \bigcap p\mathrm{cl}(V) = \emptyset$. Now, the following theorem is useful for giving some other characterizations of almost p-regular spaces.

Theorem 4 *For a space X, the following statements are equivalent:*

(a) *X is almost p-regular.*

(b) *For each $x \in X$ and for each π-open set V with $x \in V$, there exists a pre-open set U such that $x \in U \subseteq p\,\mathrm{cl}(U) \subseteq V$.*

(c) *For each $x \in X$ and for each π-open set V with $x \in V$, there exists a g^*p-open set U such that $x \in U \subseteq p\,\mathrm{cl}(U) \subseteq V$.*

(d) *For each $x \in X$ and for each π-open set V with $x \in V$, there exists a gp-open set U such that $x \in U \subseteq p\,\mathrm{cl}(U) \subseteq V$.*

(e) *For each $x \in X$ and for each π-open set V with $x \in V$, there exists a πgp-open set U such that $x \in U \subseteq p\,\mathrm{cl}(U) \subseteq V$.*

(f) *Every π-closed subset A of X is expressible as an intersection of some pre-closed pre-neighborhoods of A.*

(g) *Every π-closed set A is identical with the intersection of all pre-closed pre-neighborhoods of A.*

(h) *For every set A and every π-open set B such that $A \bigcap B \neq \emptyset$, there exists a pre-open set G such that $A \bigcap G \neq \emptyset$ and $p\,\mathrm{cl}(G) \subseteq B$.*

(i) *For every non-empty set A and every π-closed set B such that $A \bigcap B = \emptyset$, there exist disjoint pre-open sets G and H such that $A \bigcap G \neq \emptyset$ and $B \subseteq H$.*

Proof Observe that $(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (d) \Longrightarrow (e) \Longrightarrow (f) \Longrightarrow (g) \Longrightarrow (h) \Longrightarrow (i) \Longrightarrow (a)$. Now, we prove some of these implications and the others can be proved as the same arguments.
$(a) \Longrightarrow (b)$. Let V be a π-open subset of X such that $x \in V$. Then, $x \notin X \setminus V$, where $X \setminus V$ is π-closed. Since X is almost p-regular, then by the Theorem 3 there exist pre-open sets U_1 and U_2 in X such that $x \in U_1$, $X \setminus V \subseteq U_2$ and $U_1 \bigcap U_2 = \emptyset$. Thus, we have $p\,\mathrm{cl}(U_1) \bigcap U_2 = \emptyset$. Let $U = U_1$. Then, we have $x \in U \subseteq p\,\mathrm{cl}(U) \subseteq V$.
$(b) \Longrightarrow (f)$. Let A be a π-closed subset of X. For each $x \notin A$, we have $x \in X \setminus A$, where $X \setminus A$ is π-open. By (b), there exist a pre-open set U_x such that $x \in U_x \subseteq p\,\mathrm{cl}(U_x) \subseteq X \setminus A$. Let $H_x = X \setminus p\,\mathrm{cl}(U_x)$. Then, H_x is pre-open subset of X such that $A \subseteq H_x$ and $U_x \bigcap H_x = \emptyset$. Thus, $U_x \bigcap p\,\mathrm{cl}(H_x) = \emptyset$. Therefore, for each $x \notin A$ we have $A \subseteq H_x$ and $x \notin p\,\mathrm{cl}(H_x)$. Now, we shall show that $A = \bigcap_{x \notin A} p\,\mathrm{cl}(H_x)$. Since $A \subseteq p\,\mathrm{cl}(H_x)$ for each $x \notin A$, then

$$A \subseteq \bigcap_{x \notin A} p\,\mathrm{cl}(H_x) \tag{1}$$

Now, let $y \in \bigcap_{x \notin A} p\,\mathrm{cl}(H_x)$. Then, $y \in p\,\mathrm{cl}(H_x)$ for each $x \notin A$. Thus, $y \notin U_x$ for each $x \notin A$. Therefore, $y \notin \bigcup_{x \notin A} U_x$. Since $X \setminus A \subseteq \bigcup_{x \notin A} U_x$, then $y \notin X \setminus A$. Hence, $y \in A$. Therefore,

$$\bigcap_{x \notin A} p\,\mathrm{cl}(H_x) \subseteq A \tag{2}$$

From (1) and (2), we have $A = \bigcap_{x \notin A} p\,\mathrm{cl}(H_x)$, where each $p\,\mathrm{cl}(H_x)$ is pre-closed pre-neighborhood of A.
$(f) \Longrightarrow (g)$. Let A be a π-closed subset of X and $\{F_\alpha\}_{\alpha \in \Lambda}$ be a family of all pre-closed pre-neighborhoods of A. Then,

$$A \subseteq \bigcap\nolimits_{\alpha \in \Lambda} F_\alpha$$

But by (f), there is a subset $S \subseteq \Lambda$ such that

$$A = \bigcap\nolimits_{s \in S} F_s \supseteq \bigcap\nolimits_{\alpha \in \Lambda} F_\alpha$$

Thus, $A = \bigcap_{\alpha \in \Lambda} F_\alpha$. Hence, A is identical with intersection of all pre-closed pre-neighborhoods of it.
$(g) \Longrightarrow (h)$. Let A be any set and let B be a π-open subset of X such that $A \bigcap B \neq \emptyset$. Thus, there exists an element $x \in A \bigcap B$. Since $X \setminus B$ is π-closed, then by (g) we have $X \setminus B = \bigcap_{\alpha \in \Lambda} M_\alpha$, where $\{M_\alpha\}_{\alpha \in \Lambda}$ is a family of all pre-closed pre-neighborhoods of $X \setminus B$. Since $x \in B$, then $x \notin X \setminus B = \bigcap_{\alpha \in \Lambda} M_\alpha$. Then, $x \notin M_\alpha$ for some $\alpha \in \Lambda$. Since M_α is pre-neighborhood of $X \setminus B$, then there exists a pre-open set H such that $X \setminus B \subseteq H \subseteq M_\alpha$. Let $G = X \setminus M_\alpha$. Then, G is pre-open subset of X such that $x \in G$. Since $x \in A$, then $x \in G \bigcap A$. Hence, $G \bigcap A \neq \emptyset$. Also, $X \setminus H$ is pre-closed. Therefore, $G = X \setminus M_\alpha \subseteq X \setminus H \subseteq B$. Thus, $p\,\mathrm{cl}(G) \subseteq B$.
$(h) \Longrightarrow (i)$. Let A be any set and let B be a π-closed subset of X such that $A \bigcap B = \emptyset$. Then, $X \setminus B$ is π-open such that $A \subseteq X \setminus B$. So, we have $A \bigcap (X \setminus B) \neq \emptyset$. By (h), there exists a pre-open set G such that $A \bigcap G \neq \emptyset$ and $p\,\mathrm{cl}(G) \subseteq X \setminus B$. Let $H = X \setminus p\,\mathrm{cl}(G)$. Then, H is pre-open subset of X such that $G \bigcap H = \emptyset$. Therefore, there exist disjoint pre-open subsets G and H of X such that $A \bigcap G \neq \emptyset$ and $B \subseteq H$.

$(i) \Longrightarrow (a)$. Let A be a π-closed subset of X such that $x \notin A$. Then, $\{x\} \bigcap A = \emptyset$. By (i), there exist disjoint pre-open subsets G and H of X such that $\{x\} \bigcap G \neq \emptyset$ and $A \subseteq H$. Thus, G and H are disjoint pre-open subsets of X such that $x \in G$ and $A \subseteq H$. Hence, X is almost p-regular. □

Now, we have the following corollary.

Corollary 1 A space X is an almost p-regular if and only if for any π-closed set A and for each $x \notin A$, there exists a pre-open set U such that $x \in U$ and $p\,\mathrm{cl}(U) \bigcap A = \emptyset$.

4 Some Relationships Between πp-normality and Almost p-regularity

First, we recall the following definitions.

Definition 3 A space X is called *strongly-compact*, [6], if every pre-open cover of X has a finite subcover.

Definition 4 A space X is called *p_1-paracompact* (resp. *p_2-paracompact*), [6], if every pre-open cover of X has a locally finite open (resp. pre-open) refinement.

Definition 5 A collection $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ of subsets of X is called *pre-locally finite*, [20], if for each $x \in X$, there exists a pre-open set W_x in X such that $x \in W_x$ and W_x intersects at most finitely many members of $\mathcal{F}$.

Definition 6 A space X is called *p_3-paracompact*, [20], if every pre-open cover of X has a pre-locally finite pre-open refinement.

Observe that every p_1-paracompact is p_2-paracompact as well as paracompact, and every paracompact is p_3-paracompact, [20]. The following theorem can be proved easily.

Theorem 5 *Let $\{A_\alpha : \alpha \in \Lambda\}$ be a pre-locally finite collection of subsets of a space X. Then, $p\,\mathrm{cl}(\bigcup_{\alpha\in\Lambda} A_\alpha) = \bigcup_{\alpha\in\Lambda} p\,\mathrm{cl}(A_\alpha)$.*

Now, we prove the following result.

Theorem 6 *Every almost p-regular p_3-paracompact space is πp-normal.*

Proof Let X be an almost p-regular p_3-paracompact space. Let A be a π-closed and B be a closed set in X such that $A \bigcap B = \emptyset$. Then, for each $x \in B$ we have $x \notin A$. By almost p-regularity of X and by the Corollary 1, there exists a pre-open set U_x in X such that $x \in U_x$ and $p\,\mathrm{cl}(U_x) \bigcap A = \emptyset$. So, the family $\{U_x : x \in B\} \bigcup \{X \setminus B\}$ is pre-open cover for X. Since X is p_3-paracompact, then there exists a pre-locally finite pre-open refinement of it. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ denote to the members of the family which have a non-empty intersection with B. Let $V_1 = \bigcup_{\alpha\in\Lambda} U_\alpha$, which is pre-open set in X such that $B \subseteq V_1$. Let $V_2 = X \setminus \bigcup_{\alpha\in\Lambda} p\,\mathrm{cl}(U_\alpha)$, which is pre-open set in X, because $\{U_\alpha : \alpha \in \Lambda\}$ is pre-locally finite and $p\,\mathrm{cl}(\bigcup_{\alpha\in\Lambda} U_\alpha) = \bigcup_{\alpha\in\Lambda} p\,\mathrm{cl}(U_\alpha)$ (by the Theorem 5). Thus, $V_1 \bigcap V_2 = \emptyset$. Since $\mathcal{U}$

is a refinement and each member of it intersects B, then for each $U_\alpha \in \mathcal{U}$ there exists $x \in B$ such that $U_\alpha \subseteq U_x$. Now, $p\,\mathrm{cl}(U_\alpha) \subseteq p\,\mathrm{cl}(U_x) \subseteq X \setminus A$. Thus, $A \subseteq X \setminus p\,\mathrm{cl}(U_x) \subseteq X \setminus p\,\mathrm{cl}(U_\alpha)$ for each $U_\alpha \in \mathcal{U}$. So, $A \subseteq \bigcap_{\alpha\in\Lambda}(X \setminus p\,\mathrm{cl}(U_\alpha)) = X \setminus \bigcup_{\alpha\in\Lambda} p\,\mathrm{cl}(U_\alpha) = V_2$. Thus, $A \subseteq V_2$. So, we have two pre-open sets V_1 and V_2 in X such that $B \subseteq V_1$, $A \subseteq V_2$ and $V_1 \bigcap V_2 = \emptyset$. Therefore, X is πp-normal. □

Since every p_1-paracompact (resp. p_2-paracompact, paracompact) space is p_3-paracompact, then we have the following corollary.

Corollary 2 Every almost p-regular, p_2-paracompact (resp. p_1-paracompact, paracompact) space is πp-normal.

Now, we prove the following result.

Theorem 7 *Every almost p-regular strongly-compact space is πp-normal.*

Proof Let X be an almost p-regular strongly compact space. Let A and B be any disjoint closed subsets of X such that A is π-closed. Since $A \bigcap B = \emptyset$, then for each $x \in B$ we have $x \notin A$. By almost p-regularity of X and by the Corollary 1, we have for each $x \in B$ there exists a pre-open subset U_x of X such that $x \in U_x$ and $p\,\mathrm{cl}(U_x) \bigcap A = \emptyset$. Therefore, the family $\{U_x : x \in B\} \bigcup \{X \setminus B\}$ is a pre-open cover of X. Since X is strongly compact, then there exists a finite set $\{x_1, x_2, ..., x_n\} \subset B$ such that $X = (\bigcup_{i=1}^{n} U_{x_i}) \bigcup \{X \setminus B\}$. Now, let $G = \bigcup_{i=1}^{n} U_{x_i}$. Then, G is pre-open set in X such that $B \subseteq G$ and $p\,\mathrm{cl}(G) \bigcap A = \emptyset$. Thus, $A \subseteq X \setminus p\,\mathrm{cl}(G)$. Let $H = X \setminus p\,\mathrm{cl}(G)$. Then, H is pre-open set in X such that $A \subseteq H$. Therefore, G and H are pre-open sets in X such that $A \subseteq H$, $B \subseteq G$ and $H \bigcap G = \emptyset$. Hence, X is πp-normal. □

Since every regular (resp. almost regular) space is almost p-regular, then we get the following corollary.

Corollary 3 Every almost regular (resp. regular), strongly-compact space is πp-normal.

5 Preservation Theorems of πp-normal Spaces

In this section, we need to recall the definitions of some functions that help us to give various preservation theorems of πp-normality. The following definitions are in [21], [22], [12], [6], [9], [23], [24] and [25].

Definition 7 A function $f : X \longrightarrow Y$ is said to be:

(i) *almost continuous* (resp *almost π-continuous, almost p-continuous* or almost pre-continuous) if $f^{-1}(F)$ is closed (resp. π-closed, pre-closed) set in X for each closed domain subset F of Y.

(ii) *π-continuous* (resp. *p-continuous* or pre-continuous) if $f^{-1}(F)$ is π-closed (resp pre-closed) set in X for each closed subset F of Y.

(iii) *almost closed* (resp. *rc-preserving, almost π-closed*) function if $f(F)$ is closed (resp. closed domain, π-closed) set in Y for each closed domain subset F of X.

(iv) *weakly open* if for each open subset U of X, $f(U) \subseteq \operatorname{int}(f(\overline{U}))$.

(v) *pre gp-continuous* if $f^{-1}(F)$ is gp-closed in X for every pre-closed subset F of Y.

(vi) *R-map* (resp. *completely continuous*) if $f^{-1}(V)$ is open domain in X for every open domain (resp. open) subset V of Y.

(vii) *pre gp-closed* if $f(F)$ is gp-closed set in Y for every pre-closed subset F of X.

(viii) *almost pre-irresolute* if for each $x \in X$ and each pre-neighborhood V of $f(x)$ in Y, $p\operatorname{cl}(f^{-1}(V))$ is a pre-neighborhood of x in X.

(ix) *pre-closed* (resp *pre-open, semi-open*) if $f(F)$ is pre-closed (resp pre-open, semi-open) set in Y for each pre-closed (resp. pre-open, semi-open) subset F of X.

(x) *Mp-closed* or M-preclosed (resp. *Mp-open* or M-preopen) if $f(U)$ is pre-closed (resp. pre-open) set in Y for each pre-closed (resp. pre-open) set U in X.

Now, we give the following definition.

Definition 8 A function $f : X \longrightarrow Y$ is said to be *weakly p-open* (or *weakly pre-open*) if for each pre-open subset U of X, we have $f(U) \subseteq p\operatorname{int}(f(p\operatorname{cl}(U)))$.

The following lemmas, which are in [23], will be needed.

Lemma 1 If a function $f : X \longrightarrow Y$ is pre-open continuous function, then f is Mp-open.

Lemma 2 If a function $f : X \longrightarrow Y$ is weakly open continuous function, then f is Mp-open and R-map.

Lemma 3 If a function $f : X \longrightarrow Y$ is semi-open pre-continuous function, then f is pre-irresolute (or briefly; p-irresolute).

Lemma 4 A surjection $f : X \longrightarrow Y$ is pre gp-closed if and only if for each subset B of Y and each pre-open subset U of X containing $f^{-1}(B)$, there exists a gp-open subset V of Y such that $B \subset V$ and $f^{-1}(V) \subseteq U$.

Clearly, every pre-irresolute function is an almost pre-irresolute, every completely continuous function is R-map as well as π-continuous and also:

π-continuous $\Longrightarrow$ continuous $\Longrightarrow$ pre-continuous $\Longrightarrow$ gp-continuous

Now, we investigate various preserving theorems for πp-normal spaces.

Theorem 8 *If $f :\longrightarrow Y$ is an R-map pre-open continuous almost pre-irresolute surjection and X is πp-normal, then Y is πp-normal.*

Proof Let A be a closed and B be a π-open subset of Y such that $A \subseteq B$. Since f is R-map continuous function, then we have $f^{-1}(A)$ is closed and $f^{-1}(B)$ is π-open subsets of X such that $f^{-1}(A) \subseteq f^{-1}(B)$. Since X is πp-normal, then by the Theorem 1 there exists a pre-open subset U of X such that $f^{-1}(A) \subseteq U \subseteq p\,\mathrm{cl}(U) \subseteq f^{-1}(B)$. Then, $f(f^{-1}(A)) \subseteq f(U) \subseteq f(p\,\mathrm{cl}(U)) \subseteq f(f^{-1}(B))$. Since f is pre-open continuous almost pre-irresolute surjection, then by the Lemma 1, we have f is Mp-open. Therefore, $f(U)$ is pre-open subset of Y such that $A \subseteq f(U) \subseteq p\,\mathrm{cl}(f(U)) \subseteq B$. Hence by the Theorem 1, Y is πp-normal. $\square$

Theorem 9 *If $f : X \longrightarrow Y$ is a pre-open π-continuous almost pre-irresolute surjection and X is πp-normal, then Y is πp-normal.*

Proof The proof is entirely analogous to the proof of the Theorem 8. $\square$

Theorem 10 *If $f : X \longrightarrow Y$ is a weakly open π-continuous almost pre-irresolute surjection and X is πp-normal, then Y is πp-normal.*

Proof Let f be a weakly open π-continuous almost pre-irresolute surjection from a πp-normal space X to a space Y. Since f is weakly open continuous function, then by the Lemma 2 f is Mp-open and R-map. Therefore, by the Theorem 8 we have Y is πp-normal space. $\square$

Theorem 11 *If $f : X \longrightarrow Y$ is a pre-open π-continuous semi-open surjection and X is πp-normal, then Y is πp-normal.*

Proof The proof follows from Theorem 8 using the Lemma 2 and the Lemma 3. $\square$

Theorem 12 *If $f : X \longrightarrow Y$ is a pre gp-closed π-continuous surjection and X is πp-normal, then Y is πp-normal.*

Proof Let A and B be any disjoint closed subsets of Y such that A is π-closed. Then by π-continuity of f, we have $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint π-closed subsets of X. Since X is πp-normal, then there exist disjoint pre-open subsets U and V of X such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. By the Lemma 4, there exist gp-open (hence πgp-open) subsets G and H of Y such that $A \subseteq G$, $B \subseteq H$, $f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$. Since U and V are disjoint, then G and H are also disjoint. Thus, we have $A \subseteq p\,\mathrm{int}(G)$, $B \subseteq p\,\mathrm{int}(H)$ and $p\,\mathrm{int}(G) \bigcap p\,\mathrm{int}(H) = \emptyset$. Therefore, $p\,\mathrm{int}(G)$ and $p\,\mathrm{int}(H)$ are disjoint pre-open subsets of Y such that $A \subseteq p\,\mathrm{int}(G)$ and $B \subseteq p\,\mathrm{int}(H)$. Hence, Y is πp-normal. $\square$

Theorem 13 *If $f : X \longrightarrow Y$ is an R-map continuous pre gp-closed surjection and X is πp-normal, then Y is πp-normal.*

Proof Let A and B be any disjoint closed subsets of Y such that A is π-closed. Since f ia R-map continuous function, then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed subsets of X such that $f^{-1}(A)$ is π-closed. By πp-normality of X, there exist disjoint pre-open subsets U and V of X such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. By continuing as the same proof as that of the Theorem 12, we obtain two disjoint pre-open subsets G and H of Y such that $A \subseteq G$ and $B \subseteq H$. Hence, Y is πp-normal. □

The following statements can be proved easily by using arguments similar to those in the above theorems as well as by using the Definition 7.

Theorem 14 *Let $f : X \longrightarrow Y$ be a function. Then,*

(i) *If f is completely continuous pre gp-closed surjection and X is mildly p-normal, then Y is πp-normal.*

(ii) *If f is a continuous pre gp-closed surjection and X is pre-normal, then Y is πp-normal.*

(iii) *If f is a π-continuous, weakly open pre gp-closed surjection and X is πp-normal, then Y is πp-normal.*

(iv) *If f is a pre gp-continuous closed rc-preserving injection and Y is πp-normal, then X is πp-normal.*

(v) *If f is a pre gp-continuous closed injection and Y is pre-normal, then X is πp-normal.*

(vi) *If f is π-continuous, weakly p-open, pre-closed surjection and X is πp-normal, then Y is πp-normal.*

(vii) *If f is a pre-continuous, almost π-closed, open injection and Y is πp-normal, then X is πp-normal.*

(viii) *If f is an almost p-continuous closed rc-preserving injection function and Y is π-normal, then X is πp-normal.*

(ix) *If f is continuous, an almost π-continuous and pre gp-closed surjection and X is πp-normal, then Y is πp-normal.*

(x) *If f is a continuous, an almost continuous pre gp-closed surjection and X is pre-normal, then Y is πp-normal.*

6 Conclusion

We used the notion of π-generalized closed sets to obtain various characterizations of πp-normality and we established some various preservation theorems of it. Also, some characterizations of almost p-regularity were given and some relationships between πp-normality and almost p-regularity were presented.

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