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On the Structure of Nil Graph of a Commutative Ring

[1]**Kuntala Patra** and [2]**Shazida Begum**
[1,2]Department of Mathematics, Gauhati University
Guwahati,781014,Assam, India
e-mail: [1]kuntalapatra@gmail.com, [2]shazida.begum17@gmail.com

Abstract Let R be a commutative ring and $N(R)$ be the set of all nil elements of index two. The nil graph of R denoted by $\Gamma_N(R)$, is an undirected graph with the vertex set $\mathcal{Z}_N(R)^* = \{x \in R^* | xy \in N(R)\ for\ some\ y\ in\ R^* = R - \{0\}\}$, and any two vertices x and y of $\mathcal{Z}_N(R)^*$ are adjacent if and only if $xy \in N(R)$.In this paper we determine the chromatic number of the nil graph $\Gamma_N(\mathbb{Z}_{p^\alpha q})$, where $\mathbb{Z}_{p^\alpha q}$is the cyclic group of order $p^\alpha q$. Also we study the diameter and girth of $\Gamma_N(\mathbb{Z}_{p^\alpha q})$.

Keywords Nil graph; proper coloring; chromatic number; diameter; girth.

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1 Introduction

A kind of graph structure on a commutative ring R was introduced by Chen [1] by considering the nil elements of R. A graph was defined with vertex set equal to all elements of R where any two distinct vertices x and y are adjacent if and only if $xy \in N(R)$, $N(R)$ denotes the set of all nil elements of R. This concept was modified by Li and Li [2]. In this modified definition the graph defined is an undirected graph $\Gamma_N(R)$ whose vertex set is the set $\mathcal{Z}_N(R)^* = \{x \in R^* | xy \in N(R)\ for\ some\ y\ in\ R^* = R - \{0\}\}$. Two vertices x and y in $\Gamma_N(R)$ are adjacent if and only if $xy \in N(R)$ or $yx \in N(R)$. Taking this concept, Nikmehr and Khojasteh [3] determined some results on the diameter and girth of $\Gamma_N(R)$ of matrix algebras.

Next we state some definitions and notations used throughout the paper.

A ring R is called non-reduced if there exists at least one non zero nil element in the ring. Let R be a non-reduced commutative ring and $N(R)$ be the set of all nil elements of R of index two. The Nil Graph of R, denoted by $\Gamma_N(R)$, is an undirected graph with the vertex set $\mathcal{Z}_N(R)^* = \{x \in R^* | xy \in N(R)\ for\ some\ y\ in\ R^* = R - \{0\}\}$ and any two vertices of $\mathcal{Z}_N(R)^*$ are adjacent if and only if $xy \in N(R)$. We recall that a graph is connected if there exists a path connecting any two distinct vertices. The distance between any two distinct vertices x and y, denoted by $d(x, y)$, is the length of the shortest path connecting them. The diameter of a graph Γ denoted by diam(Γ) is equal to $sup\{(x, y)\ :\ x\ and\ y\ are\ distinct\ vertices\ \}$. The girth of a graph, denoted by gr(Γ), is the length of the shortest cycle in Γ. A clique of a graph is a maximal complete subgraph.

A graph Γ is said to be r-partite if V (Γ) can be partitioned into r disjoint sets $V_1, V_2, \ldots, V_r$such that no two vertices within any V_i are adjacent, but for any $v \in V_i$, $u \in V_j$, u and v are adjacent.

A proper coloring of a graph is an assignment of k-colors $\{1, 2, \ldots, k\}$ to the vertices of Γ such that no two adjacent vertices have assigned with the same color. The chromatic number $\chi(\Gamma)$ of a graph Γ is the minimum k for which Γ has k-coloring.

A dominating set in a graph Γ is a subset D of the vertex set of Γ with the property that every vertex not in D is adjacent to one or more vertices of Γ.The domination number of Γ, denoted by Domn(Γ), is defined as the cardinality of a minimum dominating set of Γ.

In our paper we consider a non-reduced commutative ring where $N(R) = \{x \in R | x^2 = 0\}$ and call the graph $\Gamma_N(R)$ as nil graph of R. Taking the modified concept of the nil graph defined by Li and Li [2], we determine the chromatic number of the nil graph $\Gamma_N(\mathbb{Z}_{p^\alpha q})$, where $\mathbb{Z}_{p^\alpha q}$ is the cyclic group of order $p^\alpha q$. Also we determine the diameter and girth of $\Gamma_N(\mathbb{Z}_{p^\alpha q})$.

2 The Nil Graph $\Gamma_N(\mathbb{Z}_{p^\alpha q})$

Theorem 1 *Let $\Gamma_N(\mathbb{Z}_{p^\alpha q})$ be the nil graph of the commutative ring $\mathbb{Z}_{p^\alpha q}$, where p and q are two distinct primes and α is an odd positive integer greater than one. Then the graph $\Gamma_N(\mathbb{Z}_{p^\alpha q})$ is*

(a) $p^{3n} + 1$*- partite, if* $\alpha = 4n + 1, n = 1, 2, 3, \ldots$
(b) p^{3n+2}*- partite if* $\alpha = 4n + 3, n = 0, 1, 2, 3, \ldots$

Proof

(a) If $\alpha = 4n + 1$, then the vertex set of the graph $\Gamma_N(\mathbb{Z}_{p^\alpha q})$ can be partitioned into

$$V_k = \{x = mp^k q : p \nmid m, q \nmid m, 1 \leq m \leq p^{4n+1-k} - 1, \}, 0 \leq k \leq 4n,$$
$$V_i = \{x = mp^i : p \nmid m, q \nmid m, 1 \leq m \leq p^{4n+1-i} q - 1\}, 0 \leq i \leq 4n + 1.$$

Any two elements $x \in V_{k_1}$ and $y \in V_{k_2}$ are adjacent if $k_1 + k_2 \geq 2n + 1$. No two vertices of V_i are adjacent but are adjacent to the vertices of V_k if $i + k \geq 2n + 1$.

Now we can consider the following cases:

(i) Let $x, y \in V_i$, for $0 \leq i \leq 4n + 1$, such that $x = m_1 p^{i_1}$ and $y = m_2 p^{i_2}$, then $xy \notin N(\mathbb{Z}_{p^\alpha q})$ as $q \nmid x$ and y. Hence x and y are not adjacent. So the elements of $\cup_{i=0}^{4n+1} V_i$ are not adjacent to each other but are adjacent to the elements of V_k for some k such that $i + k \geq 2n + 1$.

(ii) Let $x, y \in V_k$, for $0 \leq k \leq n$, such that $x = m_1 p^{k_1} q \in V_{k_1}$ and $y = m_2 p^{k_2} q \in V_{k_2}$. Then $(xy)^2 = x^2 y^2 = m_1^2 m_2^2 p^{2(k_1+k_2)} q^4 \not\equiv 0 (mod\ p^{4n+1} q)$ as $k_1 + k_2 < 2n + 1$. So $xy \notin N(\mathbb{Z}_{p^\alpha q})$ and x and y are not adjacent. Therefore all the elements of $\cup_{k=0}^{n} V_k$ are not adjacent to each other. But they are adjacent to the elements of $V_{k'}$, for some k' such that $k + k' \geq 2n + 1$.

(iii) Let $x, y \in V_k$, for $n + 1 \leq k \leq 4n$,such that $x = m_1 p^{k_1} \in V_{k_1}$ and $y = m_2 p^{k_2} \in V_{k_2}$, then $(xy)^2 = x^2 y^2 = m_1^2 m_2^2 p^{2(k_1+k_2)} q^4 \equiv 0 (mod\ p^{4n+1} q)$ as $k_1 + k_2 \geq 2n + 1$. Thus $xy \in N(\mathbb{Z}_{p^\alpha q})$ and x and y are adjacent. Thus all the elements of the set $\cup_{k=n+1}^{4n} V_k$ are adjacent to each other. Total number of elements in the set $\cup_{k=n+1}^{4n} V_k$ is

$$\sum_{k=n+1}^{4n} (p^{4n+1-k} - p^{4n-k}) = p^{3n} - 1.$$

Now considering all the cases discussed above, we can arrange the vertices of the graph into the following independent sets

$$I_0 = \cup_{i=0}^{4n+1} V_i,$$
$$I_1 = \cup_{k=0}^{n} V_k,$$
$$A_t = \{x : x = tp^{n+1}q\}, 1 \leq t \leq p^{3n} - 1.$$

Thus the independent sets $A_1, A_2, A_3, \ldots, A_{p^{3n}-1}$ together with I_0 and I_1 form a $p^{3n} - 1 + 1 + 1 = p^{3n} + 1$ - partite graph.

(b) If $\alpha = 4n + 3$, then the vertex set of the graph $\Gamma_N(\mathbb{Z}_{p^\alpha q})$ can be partitioned into

$$V_k = \{x = mp^k q : p \nmid m, q \nmid m, 1 \leq m \leq p^{4n+3-k} - 1, \}, 0 \leq k \leq 4n + 2,$$
$$V_i = \{x = mp^i : p \nmid m, q \nmid m, 1 \leq m \leq p^{4n+3-i}q - 1\}, 0 \leq i \leq 4n + 3.$$

Any two elements $x \in V_{k_1}$ and $y \in V_{k_2}$ are adjacent if $k_1 + k_2 \geq 2n + 2$. No two vertices of V_i are adjacent but are adjacent to the vertices of V_k if $i + k \geq 2n + 2$.

Now we can consider the following cases:

(i) Let $x, y \in V_i$, for $0 \leq i \leq 4n + 3$, such that $x = m_1 p^{i_1}$ and $y = m_2 p^{i_2}$, then $xy \notin N(\mathbb{Z}_{p^\alpha q})$ as $q \nmid x$ and y. Therefore x and y are the non adjacent vertices in $\Gamma_N(\mathbb{Z}_{p^\alpha q})$. So the elements of $\cup_{i=0}^{4n+3} V_i$ are not adjacent to each other but are adjacent to the elements of V_k for some k such that $i + k \geq 2n + 2$.

(ii) Let $x, y \in V_k$, for $0 \leq k \leq n$, such that $x = m_1 p^{k_1} q \in V_{k_1}$ and $y = m_2 p^{k_2} q \in V_{k_2}$. Then $(xy)^2 = x^2 y^2 = m_1^2 m_2^2 p^{2(k_1+k_2)} q^4 \not\equiv 0 (mod\ p^{4n+3} q)$ as $k_1 + k_2 < 2n + 2$. So $xy \notin N(\mathbb{Z}_{p^\alpha q})$ and x and y are not adjacent. Therefore all the elements of $\cup_{k=0}^{n} V_k$ are not adjacent to each other. These elements are adjacent to the elements of $V_{k'}$, for some k' such that $k + k' \geq 2n + 2$.

(iii) Let $x, y \in V_k$, for $n + 1 \leq k \leq 4n + 2$,such that $x = m_1 p^{k_1} q \in V_{k_1}$ and $y = m_2 p^{k_2} q \in V_{k_2}$, then $(xy)^2 = x^2 y^2 = m_1^2 m_2^2 p^{2(k_1+k_2)} q^4 \equiv 0 (mod\ p^{4n+3} q)$ as $k_1 + k_2 \geq 2n + 2$. Thus $xy \in N(\mathbb{Z}_{p^\alpha q})$ and x and y are adjacent. Thus all the elements of the set $\cup_{k=n+1}^{4n+2} V_k$ are adjacent to each other. Total number of elements in the set $\cup_{k=n+1}^{4n+2} V_k$ is

$$\sum_{k=n+1}^{4n+2} (p^{4n+3-k} - p^{4n+2-k}) = p^{3n+2} - 1.$$

Now considering all the cases discussed above, we can arrange the vertices of the graph into the following independent sets

$$I_0 = \cup_{i=0}^{4n+3} V_i,$$
$$I_1 = \cup_{k=0}^{n} V_k \cup \{x = p^{n+1} q\},$$
$$A_t = \{x : x = tp^{n+1} q\}, 2 \leq t \leq p^{3n+2} - 1.$$

Thus the independent sets $A_2, A_3, \ldots, A_{p^{3n+2}-1}$ together with I_0 and I_1 form a $p^{3n+2} - 2 + 1 + 1 = p^{3n+2}$ - partite graph. □

Corollary 1 *Let $\Gamma_N(\mathbb{Z}_{p^\alpha q})$ be the nil graph of the commutative ring $\mathbb{Z}_{p^\alpha q}$, where p and q are two distinct primes and α is an odd positive integer greater than one. Then*

(a) $\chi(\Gamma_N(\mathbb{Z}_{p^\alpha q}) = p^{3n}+1, if \alpha = 4n+1, n = 1,2,3,\dots.$

(b) $\chi(\Gamma_N(\mathbb{Z}_{p^\alpha q}) = p^{3n+2}, if \alpha = 4n+3, n = 0,1,2,3,\dots.$

Proof

(a) If $\alpha = 4n+1$, then by the proof of the previous Theorem 1(a) we see that the all the vertices of the set $\cup_{k=n+1}^{4n} V_k$ are adjacent to each other and total number of elements in this set is $p^{3n}-1$. The elements of the set $V_{k=n}$ are not adjacent among themselves but are adjacent to every member of the set $\cup_{k=n+1}^{4n} V_k$. Again the elements of the set $\cup_{i=n+1}^{4n+1} V_i$ are not adjacent among themselves but are adjacent to every element of the set $\cup_{k=n+1}^{4n} V_k$ and $V_{k=n}$. Therefore all the elements of $\cup_{k=n+1}^{4n} V_k$ along with any one element from $V_{k=n}$ and one from $\cup_{i=n+1}^{4n+1} V_i$ will form a clique of order $p^{3n}-1+1+1 = p^{3n}+1$. Therefore $\chi(\Gamma_N(\mathbb{Z}_{p^\alpha q})) \geq p^{3n}+1$.

By Theorem 1(a) the graph $\Gamma_N(\mathbb{Z}_{p^\alpha q})$ is a $p^{3n}+1$- partite graph which implies that $\chi(\Gamma_N(\mathbb{Z}_{p^\alpha q}) \leq p^{3n}+1$.. Therefore $\chi(\Gamma_N(\mathbb{Z}_{p^\alpha q}) = p^{3n}+1$.

(b) If $\alpha = 4n+3$, then by the proof of the previous Theorem 1(b), the elements of the set $\cup_{k=n+1}^{4n+2} V_k$ are adjacent with each other and total number of elements of this set is $p^{3n+2}-1$.Therefore the set $\cup_{k=n+1}^{4n+2} V_k \cup \{x\}$, where $x \in \cup_{i=n+1}^{4n+3} V_i$ will together form a complete subgraph of order $p^{3n+2}-1+1 = p^{3n+2}$ and hence $\chi(\Gamma_N(\mathbb{Z}_{p^\alpha q})) \geq p^{3n+2}$. Again by Theorem 2.1(b) we have the graph $\Gamma_N(\mathbb{Z}_{p^\alpha q})$ is a p^{3n+2} partite graph and $\chi(\Gamma_N(\mathbb{Z}_{p^\alpha q})) \leq p^{3n+2}$. Hence $\chi(\Gamma_N(\mathbb{Z}_{p^\alpha q})) = p^{3n+2}$. □

Example 1 Consider $\mathbb{Z}_{24} = \mathbb{Z}_{2^3 4}$. Then $\mathcal{Z}_N(\mathbb{Z}_{24})^* = \{1,2,3,\dots,23\}$. Then we can divide the vertex set into the following independent subsets

$V_1 = \{12\}$, $V_2 = \{18\}$, $V_3 = \{3,6,9,15,21\}$, $V_4 = \{1,2,4,5,7,8,10,11,13,14,16,17,19,20,22,23\}$. The nil graph $\Gamma_N(\mathbb{Z}_{24})$ is shown in Figure 1.

Theorem 2 *Let $\Gamma_N(\mathbb{Z}_{p^\alpha q})$ be the nil graph of the commutative ring $\mathbb{Z}_{p^\alpha q}$, where p and q are two distinct primes and α is an even positive integer. Then the graph $\Gamma_N(\mathbb{Z}_{p^\alpha q})$ is*

(a) p^{3n}- *partite, if* $\alpha = 4n, n = 1,2,3,\dots$

(b) $p^{3n+1}+1$- *partite if* $\alpha = 4n+2, n = 0,1,2,3,\dots$

Proof

(a) If $\alpha = 4n$, then the vertex set of the graph $\Gamma_N(\mathbb{Z}_{p^\alpha q})$ can be partitioned into

$$V_k = \{x = mp^k q : p \nmid m, q \nmid m, 1 \leq m \leq p^{4n-k}-1\}, 0 \leq k \leq 4n-1,$$
$$V_i = \{x = mp^i : p \nmid m, q \nmid m, 1 \leq m \leq p^{4n-i}q-1\}, 0 \leq i \leq 4n.$$

Any two elements $x \in V_{k_1}$ and $y \in V_{k_2}$ are adjacent if $k_1+k_2 \geq 2n$. No two vertices of V_i are adjacent but are adjacent to the vertices of V_k if $i+k \geq 2n$.

Now we can consider the following cases:

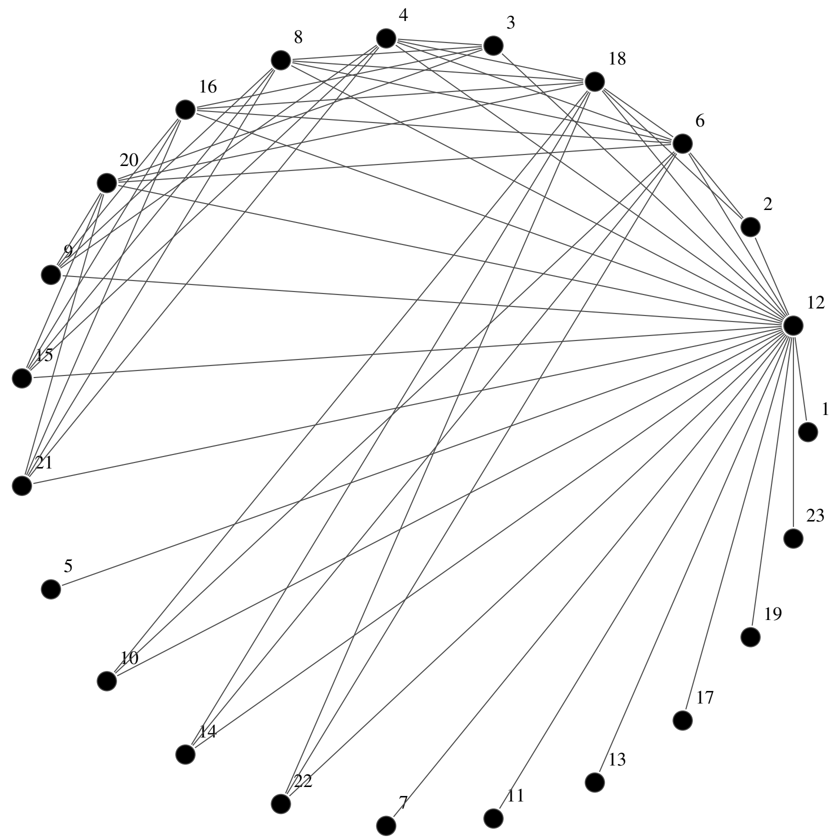

Figure 1: The Nil Graph $\Gamma_N(\mathbb{Z}_{24})$

(i) Let $x, y \in V_i$, for $0 \leq i \leq 4n$, such that $x = m_1 p^{i_1}$ and $y = m_2 p^{i_2}$, then $xy \notin N(\mathbb{Z}_{p^\alpha q})$ as $q \nmid x$ and y. Therefore x and y are not adjacent in $\Gamma_N(\mathbb{Z}_{p^\alpha q})$. So the elements of $\cup_{i=0}^{4n} V_i$ are not adjacent to each other but are adjacent to the elements of V_k for some k such that $i + k \geq 2n$.

(ii) Let $x, y \in V_k$, for $0 \leq k \leq n-1$, such that $x = m_1 p^{k_1} q \in V_{k_1}$ and $y = m_2 p^{k_2} q \in V_{k_2}$. Then $(xy)^2 = x^2 y^2 = m_1^2 m_2^2 p^{2(k_1+k_2)} q^4 \not\equiv 0 (mod\ p^{4n} q)$ as $k_1 + k_2 < 2n$. So $xy \notin N(\mathbb{Z}_{p^\alpha q})$ and x and y are not adjacent. Therefore all the elements of $\cup_{k=0}^{n-1} V_k$ are not adjacent to each other but are adjacent to the elements of $V_{k'}$, for some k' such that $k + k' \geq 2n$.

(iii) Let $x, y \in V_k$, for $n \leq k \leq 4n-1$,such that $x = m_1 p^{k_1} \in V_{k_1}$ and $y = m_2 p^{k_2} \in V_{k_2}$, then $(xy)^2 = x^2 y^2 = m_1^2 m_2^2 p^{2(k_1+k_2)} q^4 \equiv 0 (mod\ p^{4n} q)$ as $k_1 + k_2 \geq 2n$. Thus $xy \in N(\mathbb{Z}_{p^\alpha q})$ and x and y are adjacent. Thus all the elements of the set $\cup_{k=n}^{4n-1} V_k$ are adjacent to each other. Total number of elements in the set $\cup_{k=n}^{4n-1} V_k$

is

$$\sum_{k=n}^{4n-1}(p^{4n-k}-p^{4n-1-k})=p^{3n}-1.$$

Now considering all the cases discussed above, we can arrange the vertices of the graph into the following independent sets

$$I_0=\cup_{i=0}^{4n}V_i,$$
$$I_1=\cup_{k=0}^{n-1}V_k\cup\{x=p^nq\},$$
$$A_t=\{x:x=tp^nq\},2\le t\le p^{3n}-1.$$

Thus the independent sets $A_2,A_3\ldots A_{p^{3n}}-1$ together with I_0 and I_1 form a p^{3n} -partite graph.

(b) If $\alpha=4n+2$, then the vertex set of the graph $\Gamma_N(\mathbb{Z}_{p^\alpha q})$ can be partitioned into
$V_k=\{x=mp^kq:p\nmid m,q\nmid m,1\le m\le p^{4n+2-k}-1,\},0\le k\le 4n+1.$
$V_i=\{x=mp^i:p\nmid m,q\nmid m,1\le m\le p^{4n+2-i}q-1\},0\le i\le 4n+2.$

Any two elements $x\in V_{k_1}$ and $y\in V_{k_2}$ are adjacent if $k_1+k_2\ge 2n+1$. No two vertices of V_i are adjacent but are adjacent to the vertices of V_k if $i+k\ge 2n+1$.

Now we can consider the following cases:

(i) Let $x,y\in V_i$, for $0\le i\le 4n+2$, such that $x=m_1p^{i_1}$ and $y=m_2p^{i_2}$, then $xy\notin N(\mathbb{Z}_{p^\alpha q})$ as $q\nmid x$ and y. Therefore x and y are the non adjacent vertices in $\Gamma_N(\mathbb{Z}_{p^\alpha q})$. Thus all the elements of the set $\cup_{i=0}^{4n+2}V_i$ are not adjacent to each other but are adjacent to the elements of the set V_k for some k such that $i+k\ge 2n+1$.

(ii) Let $x,y\in V_k$, for $0\le k\le n$, such that $x=m_1p^{k_1}q\in V_{k_1}$ and $y=m_2p^{k_2}q\in V_{k_2}$. Then $(xy)^2=x^2y^2=m_1^2m_2^2p^{2(k_1+k_2)}q^4\not\equiv 0(mod\ p^{4n+2}q)$ as $k_1+k_2<2n+1$. So $xy\notin N(\mathbb{Z}_{p^\alpha q})$ and x and y are not adjacent. Therefore all the elements of $\cup_{k=0}^{n}V_k$ are not adjacent to each other. But they are adjacent to the elements of $V_{k'}$, for some k' such that $k+k'\ge 2n+1$.

(iii) Let $x,y\in V_k$, for $n+1\le k\le 4n+1$,such that $x=m_1p^{k_1}q\in V_{k_1}$ and $y=m_2p^{k_2}q\in V_{k_2}$, then $(xy)^2=x^2y^2=m_1^2m_2^2p^{2(k_1+k_2)}q^4=m_1^2m_2^2p^{2(k_1+k_2)}q^4\equiv 0(mod\ p^{4n+2}q)$ as $k_1+k_2\ge 2n+1$. Thus $xy\in N(\mathbb{Z}_{p^\alpha q})$ and x and y are adjacent. Thus all the elements of the set $\cup_{k=n+1}^{4n+1}V_k$ are adjacent to each other. Total number of elements in the set $\cup_{k=n+1}^{4n+1}V_k$ is

$$\sum_{k=n+1}^{4n+1}(p^{4n+2-k}-p^{4n+1-k})=p^{3n+1}-1.$$

Now considering all the cases discussed above, we can arrange the vertices of the graph into the following independent sets

$$I_0=\cup_{i=0}^{4n+2}V_i,$$
$$I_1=\cup_{k=0}^{n}V_k,$$
$$A_t=\{x:x=tp^{n+1}q\},1\le t\le p^{3n+1}-1.$$

Thus the independent sets $A_1, A_2, A_3 \ldots, A_{p^{3n+1}} - 1$ together with I_0 and I_1 form a $p^{3n+1} - 1 + 1 + 1 = p^{3n+1} + 1$ - partite graph. □

Theorem 3 *Let $\Gamma_N(\mathbb{Z}_{p^\alpha q})$ be the nil graph of the commutative ring $\mathbb{Z}_{p^\alpha q}$, where p and q are two distinct primes and α is an even positive integer. Then*

(a) $\chi(\Gamma_N(\mathbb{Z}_{p^\alpha q}) = p^{3n}, if \alpha = 4n, n = 1, 2, 3, \ldots.$

(b) $\chi(\Gamma_N(\mathbb{Z}_{p^\alpha q}) = p^{3n+1} + 1, if \alpha = 4n + 2, n = 0, 1, 2, 3, \ldots.$

Proof

(a) If $\alpha = 4n$, the elements which are divisible by $p^n q$ are adjacent with each other. Therefore the set $\cup_{k=n}^{4n-1} V_k \cup \{a\}$, where $a \in \cup_{i=n}^{4n} V_i$ will together form a clique of order p^{3n} and hence $\chi(\Gamma_N(\mathbb{Z}_{p^\alpha q})) \geq p^{3n}$. Again by Theorem 2.2(a) we have the graph $\Gamma_N(\mathbb{Z}_{p^\alpha q})$ is a p^{3n}- partite which implies that $\chi(\Gamma_N(\mathbb{Z}_{p^\alpha q}) \leq p^{3n}$. Hence $\chi(\Gamma_N(\mathbb{Z}_{p^\alpha q}) = p^{3n}$.

(b) If $\alpha = 4n + 2$, then by the proof of the previous Theorem 2.2(b), the elements of the set $\cup_{k=n+1}^{4n+1} V_k$ are adjacent to each other and total number of elements in this set is $p^{3n+1} - 1$.Therefore the set $\cup_{k=n+1}^{4n+1} V_k \cup \{x = p^n q\} \cup \{x = p^n q\} \cup \{y\}$, where $y \in \cup_{i=n}^{4n+2} V_i$ will together form a clique of order $p^{3n+1} + 1$. Therefore $\chi(\Gamma_N(\mathbb{Z}_{p^\alpha q}) \geq p^{3n+1} + 1$. By Theorem 2.2(b) we have the graph $\Gamma_N(\mathbb{Z}_{p^\alpha q})$ is a $p^{3n+1} + 1$ partite graph which implies that $\chi(\Gamma_N(\mathbb{Z}_{p^\alpha q}) \leq p^{3n+1} + 1$. Hence $\chi(\Gamma_N(\mathbb{Z}_{p^\alpha q}) = p^{3n+1} + 1$. □

Example 2 Consider $\mathbb{Z}_{48} = \mathbb{Z}_{2^4 3}$. Then $\mathcal{Z}_N(\mathbb{Z}_{48})^* = \{1, 2, 3, \ldots, 23\}$. Then we can divide the vertex set into the following independent subsets

$V_1 = \{12\}$, $V_2 = \{24\}$, $V_3 = \{36\}$, $V_4 = \{18\}$,$V_5 = \{30\}$,$V_6 = \{42\}$,$V_7 = \{3, 6, 9, 15, 21, 27, 33, 39, 45\}$, $V_8 = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, 26, 28, 29, 31, 32, 33, 34, 35, 37, 38, 40, 41, 43, 44, 46, 47\}$. The nil graph $\Gamma_N(\mathbb{Z}_{48})$ is shown in Figure 2.

Theorem 4 *If p and q are distinct primes and α is any positive integer greater than one, then $diam(\Gamma_N(\mathbb{Z}_{p^\alpha q})) = 2$.*

Proof Since $\mathbb{Z}_{p^\alpha q}$ is non-reduced, there exists non-zero nil element in the ring. All the non-zero nil elements are adjacent among themselves and are also adjacent to every other vertices of the graph $\Gamma_N(\mathbb{Z}_{p^\alpha q})$. Therefore the non non-zero nil elements are connected through the non zero nil elements and hence the $diam(\Gamma_N(\mathbb{Z}_{p^\alpha q})) = 2$. □

Theorem 5 *If p and q are distinct primes and α is any positive integer greater than one, then $gr(\Gamma_N(\mathbb{Z}_{p^\alpha q})) = 3$.*

Proof Let v_1,v_2,v_3 be the vertices of $\Gamma_N(\mathbb{Z}_{p^\alpha q})$ such that $v_1 = p^{\alpha-1} q$, $v_2 = q$, $v_3 = p^\alpha$. Then v_1 - v_2 - v_3 - v_1 is a 3-cycle. Hence $gr(\Gamma_N(\mathbb{Z}_{p^\alpha q})) = 3$. □

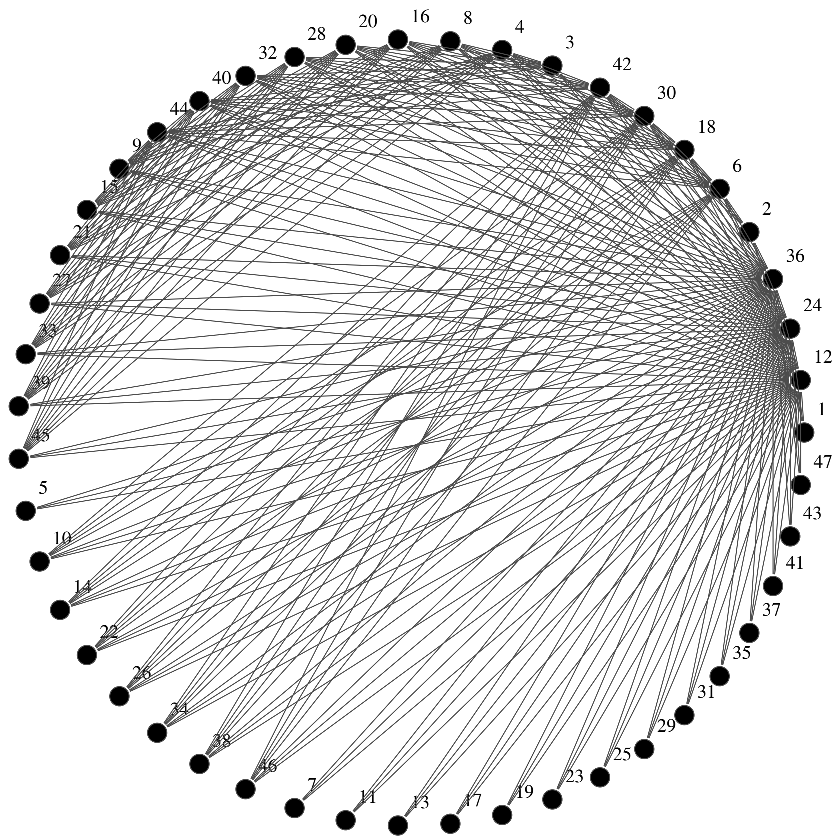

Figure 2: The Nil Graph $\Gamma_N(\mathbb{Z}_{48})$

3 Conclusion

The results and findings of our discussions can be summerized as follows:

(a) The nil graph $\Gamma_N(\mathbb{Z}_{p^\alpha q})$ is always a partite graph.

  (i) If $\alpha \equiv 0(mod4n+1)$, then the graph $\Gamma_N(\mathbb{Z}_{p^\alpha q})$ is $p^{3n}+1$- partite.

  (ii) If $\alpha \equiv 0(mod4n+3)$, then the graph $\Gamma_N(\mathbb{Z}_{p^\alpha q})$ is p^{3n+2}- partite.

  (iii) If $\alpha \equiv 0(mod4n)$, then the graph $\Gamma_N(\mathbb{Z}_{p^\alpha q}$ is p^{3n}- partite.

  (iv) If $\alpha \equiv 0(mod4n+2)$, then the graph $\Gamma_N(\mathbb{Z}_{p^\alpha q})$ is is $p^{3n+1}+1$- partite.

(b) The chromatic number of the graph depends on the p-partite structure of the graph and also the clique of the graph.

(c) Every nil-element of the graph individually constitutes an independent set.

(d) Since every nil-element of the graph is adjacent to all other vertices of the graph. Hence $Domn(\Gamma_N(\mathbb{Z}_{p^\alpha q})) = 1$.

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References

[1] Chen, P. W. A kind of graph structure of rings. *Algebra Colloq.* 2003. 10(2): 229–238.

[2] Li, A. H. and Li, Q. S. A kind of graph structure on Von-Neumann regular rings. *International Journal of Algebra*. 2004. 4(6): 291–302.

[3] Nikmehr, M. J. and Khojasteh, S. On the nilpotent graph of a ring. *Turk J. Math.* to appear, doi:10.3906/mat-1112–35.