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Second Hankel Determinant for a Subclass of Tilted Starlike Functions with Respect to Conjugate Points

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Abstract Let $S_c^*(\alpha, \delta, A, B)$ be the class of functions which are analytic and univalent in an open unit disc,$E = \{z : |z| < 1\}$ of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots + a_nz^n + \cdots = z + \sum_{n=2}^{\infty} a_nz^n$ and normalized with $f(0) = 0$and $f'(0) - 1 = 0$ and satisfy $\left(e^{i\alpha}\frac{zf'(z)}{g(z)} - \delta - i\sin\alpha\right)\frac{1}{t_{\alpha\delta}} \prec \frac{1+Az}{1+Bz}$, $\quad -1 \le B < A \le 1, \quad z \in E$ where $g(z) = \frac{f(z)+\overline{f(\bar{z})}}{2}$, $\quad t_{\alpha\delta} = \cos\alpha - \delta, \cos\alpha - \delta > 0, 0 \le \delta < 1$ and $|\alpha| < \frac{\pi}{2}$. In this paper, we determine the sharp upper bound of the functional $\left|a_2a_4 - a_3^2\right|$ for this class of functions. The results generalize some known existing results in the literature.

Keywords Univalent functions; starlike functions; conjugate points; Hankel determinant

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1 Introduction

Let H be the class of functions ω which are analytic and univalent in the unit disc, $E = \{z : |z| < 1\}$ given by

$$\omega(z) = \sum_{n=1}^{\infty} t_nz^n$$

and satisfies the conditions $\omega(0) = 0, |\omega(z)| < 1, z \in E$.

Let S be the class of functions f which are analytic and univalent in E and of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_nz^n \tag{1}$$

and normalized with $f(0) = 0$ and $f'(0) - 1 = 0$.

Also, let S_s^* be the subclass of S consisting of functions given by (1) satisfying the condition

$$Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0, \quad z \in E.$$

These functions are called starlike functions with respect to symmetric points and were introduced by Sakaguchi [1] in 1959.

As cited in [2], in 1987, El-Ashwah and Thomas defined the class of starlike functions with respect to conjugate points and the class of starlike functions with respect to symmetric conjugate points respectively as follows:

$$S_c^* = \left\{f \in S : Re\left(\frac{2zf'(z)}{f(z) + \overline{f(\overline{z})}}\right) > 0, z \in E\right\},$$

$$S_{sc}^{*}=\left\{f\in S:Re\left(\frac{2zf'(z)}{f(z)-\overline{f(\overline{-z})}}\right)>0,z\in E\right\}.$$

Moreover, we introduce $S_c^{*}(\alpha,\delta)$ as the class of functions f which are analytic and univalent in E and of the form (1) and normalized with $f(0)=0$ and $f'(0)-1=0$ and satisfy

$$Re\left(e^{i\alpha}\frac{zf'(z)}{g(z)}\right)>\delta \tag{2}$$

where $g(z)=\frac{f(z)+\overline{f(\bar{z})}}{2}$, $\cos\alpha-\delta>0, 0\leq\delta<1$ and $|\alpha|<\frac{\pi}{2}$. The functions of the class $S_c^{*}(\alpha,\delta)$ are called tilted starlike functions with respect to conjugate points of order δ.

Further, let two functions $F(z)$ and $G(z)$ be analytic in E. If there exists a functions $\omega\in H$ which is analytic in E with $\omega(0)=0$ and $|\omega(z)|<1$ such that $F(z)=G(\omega(z))$ for every $z\in E$, then we say that $F(z)$ is subordinate to $G(z)$ and it can be written as $F(z)\prec G(z)$. We also note that if $G(z)$ is univalent in E, then the subordination is equivalent to $F(0)=G(0)$ and $F(E)\subset G(E)$.

In term of subordination, Abdul Wahid *et al.* [3] introduced a subclass of $S_c^{*}(\alpha,\delta)$ denoted by $S_c^{*}(\alpha,\delta,A,B)$ as in the following definition.

Definition 1 *$f\in S_c^{*}(\alpha,\delta,A,B)$ if and only if*

$$\left(e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha\right)\frac{1}{t_{\alpha\delta}}\prec\frac{1+Az}{1+Bz},z\in E.$$

By definition of subordination, it follows that $f\in S_c^{*}(\alpha,\delta,A,B)$ if and only if

$$\left(e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha\right)\frac{1}{t_{\alpha\delta}}=\frac{1+A\omega(z)}{1+B\omega(z)}=p(z),\omega\in H$$

where $p(z)=1+\sum\limits_{n=1}^{\infty}p_nz^n$.

In 1976, Noonan and Thomas [4] defined the q^{th} Hankel determinant of f for $q\geq 1$ and $n\geq 1$ given by

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

This determinant has been investigated by several researchers. For instance, as stated in [5], Noor determined the rate of growth of $H_q(n)$ as $n\to\infty$ for functions in (1) with bounded boundary and Ehrenborg [6] studied the Hankel determinant of order $(n+1)$ of the exponential polynomials. Also, as cited in [5], in 2006, Janteng *et al.* studied the Hankel determinant for the class S_s^{*}.

Recently, Singh [5] obtained the Second Hankel determinant for the classes S_c^{*} and S_{sc}^{*}. For our discussion in this paper, we consider the Hankel determinant in the case $q=2$ and $n=2$,

$$H_2(2)=\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

In this paper, we established the sharp upper bound of the functional $\left|a_2a_4-a_3^2\right|$ for functions in the class $S_c^*(\alpha,\delta,A,B)$.

2 Preliminary Results

Let P be the class of all functions p of the form

$$p(z)=1+p_1z+p_2z^2+\cdots+p_nz^n+\cdots=1+\sum_{n=1}^{\infty}p_nz^n$$

that is analytic in $E$and satisfying the condition $Re\,(p(z))>0$ for $z\in E$.

We need the following lemmas for proving our result.

Lemma 1 [7]
If $p\in P$,*then* $|p_n|\leq 2\,(k=1,2,3,\ldots)$.

Lemma 2 [8,9]
If $p\in P$,*then*

$$2p_2=p_1^2+\left(4-p_1^2\right)x,$$

$$4p_3=p_1^3+2p_1\left(4-p_1^2\right)x-p_1\left(4-p_1^2\right)x^2+2\left(4-p_1^2\right)\left(1-|x|^2\right)z,$$

for some values of x *and* z *satisfying* $|x|\leq 1,|z|\leq 1$ *and* $p_1\in[0,2]$.

3 Main Result

Theorem 1 *If* $f\in S_c^*(\alpha,\delta,A,B)$, *then*

$$\left|a_2a_4-a_3^2\right|\leq\frac{T^2}{4}$$

where $T=(A-B)\,t_{\alpha\delta}$ *and* $t_{\alpha\delta}=\cos\alpha-\delta$.

The result obtained is sharp.

Proof: From Definition 1, we have

$$\left(e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha\right)\frac{1}{t_{\alpha\delta}}=\frac{1+A\omega(z)}{1+B\omega(z)},\omega\in H \tag{3}$$

where $g(z)=\frac{f(z)+\overline{f(\bar{z})}}{2}$ and $t_{\alpha\delta}=\cos\alpha-\delta$.

Now, let

$$h(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+k_1z+k_2z^2+\cdots+k_nz^n+\cdots;n\geq 1. \tag{4}$$

From (4) we get

$$\omega(z)=\frac{h(z)-1}{h(z)+1}. \tag{5}$$

By using (5), thus (3) can also be written as

$$\left(e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha\right)\frac{1}{t_{\alpha\delta}}=\frac{1-A+h(z)(1+A)}{1-B+h(z)(1+B)}. \tag{6}$$

Rearranging (6), we get

$$\begin{aligned}e^{i\alpha}\frac{zf'(z)}{g(z)}&=\frac{[1-A+h(z)(1+A)]\,t_{\alpha\delta}}{1-B+h(z)(1+B)}+i\sin\alpha+\delta\\&=\frac{[1-A+h(z)(1+A)]\,t_{\alpha\delta}+[1-B+h(z)(1+B)](i\sin\alpha+\delta)}{1-B+h(z)(1+B)}\\&=\frac{t_{\alpha\delta}-At_{\alpha\delta}+h(z)t_{\alpha\delta}+Ah(z)t_{\alpha\delta}+(i\sin\alpha+\delta)-B(i\sin\alpha+\delta)}{1-B+h(z)(1+B)}\\&\quad+\frac{h(z)(i\sin\alpha+\delta)+Bh(z)(i\sin\alpha+\delta)}{1-B+h(z)(1+B)}\\&=\frac{e^{i\alpha}+e^{i\alpha}h(z)-At_{\alpha\delta}-B(i\sin\alpha+\delta)+Bt_{\alpha\delta}-Bt_{\alpha\delta}+Ah(z)t_{\alpha\delta}}{1-B+h(z)(1+B)}\\&\quad+\frac{Bh(z)(i\sin\alpha+\delta)+Bh(z)t_{\alpha\delta}-Bh(z)t_{\alpha\delta}}{1-B+h(z)(1+B)}.\end{aligned}$$

Thus, we get

$$e^{i\alpha}\frac{zf'(z)}{g(z)}=\frac{\left\{e^{i\alpha}(1-B)-T\right\}+h(z)\left\{e^{i\alpha}(1+B)+T\right\}}{1-B+h(z)(1+B)}$$

Using the series expansion then we have

$$\begin{aligned}&e^{i\alpha}(1-B)\left(z+2a_2z^2+3a_3z^3+\cdots\right)\\&\qquad+e^{i\alpha}(1+B)\left\{\left(z+2a_2z^2+3a_3z^3+\cdots\right)\left(1+k_1z+k_2z^2+k_3z^3+\cdots\right)\right\}\\&=\left\{e^{i\alpha}(1-B)-T\right\}\left(z+a_2z^2+a_3z^3+\cdots\right)\\&\qquad+\left\{e^{i\alpha}(1+B)+T\right\}\left\{\left(z+a_2z^2+a_3z^3+\cdots\right)\left(1+k_1z+k_2z^2+k_3z^3+\cdots\right)\right\}.\end{aligned} \tag{7}$$

Equating the coefficients of z^2, z^3 and z^4 in (7) gives us

$$2a_2e^{i\alpha}=k_1T,$$

$$a_2=\frac{k_1Te^{-i\alpha}}{2}. \tag{8}$$

$$4a_3e^{i\alpha}=k_2T+\left[T-e^{i\alpha}(1+B)\right]a_2k_1, \tag{9}$$

and

$$6a_4e^{i\alpha}=k_3T+\left[T-e^{i\alpha}(1+B)\right]a_2k_2+\left[T-2e^{i\alpha}(1+B)\right]a_3k_1. \tag{10}$$

Using (8) into (9) gives us

$$\begin{aligned}4a_3e^{i\alpha}&=k_2T+\frac{\left[T-e^{i\alpha}(1+B)\right]k_1^2Te^{-i\alpha}}{2},\\&=\frac{2k_2T+k_1^2T^2e^{-i\alpha}-(1+B)k_1^2T}{2}\end{aligned}$$

and thus

$$a_3=\frac{2k_2Te^{-i\alpha}+k_1^2T^2e^{-2i\alpha}-(1+B)\,k_1^2Te^{-i\alpha}}{8}. \tag{11}$$

Substituting (8) and (11) into (10) we obtain

$$\begin{aligned}6a_4e^{i\alpha} &= k_3T+\frac{k_1k_2Te^{-i\alpha}\left[T-e^{i\alpha}(1+B)\right]}{2}\\ &\quad+\frac{\left(T-2e^{i\alpha}(1+B)\right)\left[2k_1k_2Te^{-i\alpha}+k_1^3T^2e^{-2i\alpha}-(1+B)\,k_1^3Te^{-i\alpha}\right]}{8},\\ &= \frac{8k_3T+4k_1k_2T^2e^{-i\alpha}-4(1+B)\,k_1k_2T+2k_1k_2T^2e^{-i\alpha}+k_1^3T^3e^{-2i\alpha}}{8}\\ &\quad-\frac{(1+B)\,k_1^3T^2e^{-i\alpha}-4(1+B)\,k_1k_2T-2(1+B)\,k_1^3T^2e^{-i\alpha}+2(1+B)^2\,k_1^3T}{8}\end{aligned}$$

and thus

$$\begin{aligned}a_4 &= \frac{8k_3Te^{-i\alpha}+6k_1k_2T^2e^{-2i\alpha}-8(1+B)\,k_1k_2Te^{-i\alpha}+k_1^3T^3e^{-3i\alpha}}{48}\\ &\quad-\frac{3(1+B)\,k_1^3T^2e^{-2i\alpha}+2(1+B)^2\,k_1^3Te^{-i\alpha}}{48}.\end{aligned} \tag{12}$$

Then, by squaring (11) we have

$$\begin{aligned}a_3^2 &= \left(\frac{2k_2Te^{-i\alpha}+k_1^2T^2e^{-2i\alpha}-(1+B)\,k_1^2Te^{-i\alpha}}{8}\right)^2\\ &= \frac{4k_2^2T^2e^{-2i\alpha}+4k_1^2k_2T^3e^{-3i\alpha}-4(1+B)\,k_1^2k_2T^2e^{-2i\alpha}+k_1^4T^4e^{-4i\alpha}}{64}\\ &\quad-\frac{2(1+B)\,k_1^4T^3e^{-3i\alpha}+(1+B)^2\,k_1^4T^2e^{-2i\alpha}}{64}\end{aligned} \tag{13}$$

and using (8) and (12) gives us

$$\begin{aligned}a_2a_4 &= \frac{k_1Te^{-i\alpha}}{2}\left\{\frac{8k_3Te^{-i\alpha}+6k_1k_2T^2e^{-2i\alpha}-8(1+B)\,k_1k_2Te^{-i\alpha}+k_1^3T^3e^{-3i\alpha}}{48}\right.\\ &\quad\left.-\frac{3(1+B)\,k_1^3T^2e^{-2i\alpha}+2(1+B)^2\,k_1^3Te^{-i\alpha}}{48}\right\}\\ &= \frac{8k_1k_3T^2e^{-2i\alpha}+6k_1^2k_2T^3e^{-3i\alpha}-8(1+B)\,k_1^2k_2T^2e^{-2i\alpha}+k_1^4T^4e^{-4i\alpha}}{96}\\ &\quad-\frac{3(1+B)\,k_1^4T^3e^{-3i\alpha}+2(1+B)^2\,k_1^4T^2e^{-2i\alpha}}{96}.\end{aligned} \tag{14}$$

Equations (13) and (14) together yield

$$\begin{aligned}
&a_2a_4 - a_3^2 \\
&= \frac{8k_1k_3T^2e^{-2i\alpha} + 6k_1^2k_2T^3e^{-3i\alpha} - 8(1+B)k_1^2k_2T^2e^{-2i\alpha} + k_1^4T^4e^{-4i\alpha}}{96} \\
&\quad - \frac{3(1+B)k_1^4T^3e^{-3i\alpha} + 2(1+B)^2k_1^4T^2e^{-2i\alpha}}{96} - \left\{\frac{4k_2^2T^2e^{-2i\alpha} + 4k_1^2k_2T^3e^{-3i\alpha}}{64}\right. \\
&\quad \left. - \frac{4(1+B)k_1^2k_2T^2e^{-2i\alpha} + k_1^4T^4e^{-4i\alpha} - 2(1+B)k_1^4T^3e^{-3i\alpha} + (1+B)^2k_1^4T^2e^{-2i\alpha}}{64}\right\} \\
&= \frac{k_1k_3T^2e^{-2i\alpha}}{12} - \frac{(1+B)k_1^2k_2T^2e^{-2i\alpha}}{48} - \frac{k_1^4T^4e^{-4i\alpha}}{192} + \frac{(1+B)^2k_1^4T^2e^{-2i\alpha}}{192} \\
&\quad - \frac{k_2^2T^2e^{-2i\alpha}}{16} \\
&= T^2e^{-2i\alpha}\left\{\frac{k_1k_3}{12} - \frac{(1+B)k_1^2k_2}{48} + \frac{(1+B)^2k_1^4}{192} - \frac{k_2^2}{16}\right\} - \frac{k_1^4T^4e^{-4i\alpha}}{192}.
\end{aligned}$$

Taking modulus for both sides then we have

$$|a_2a_4 - a_3^2| = \left|T^2e^{-2i\alpha}\left\{\frac{k_1k_3}{12} - \frac{(1+B)k_1^2k_2}{48} + \frac{(1+B)^2k_1^4}{192} - \frac{k_2^2}{16}\right\} - \frac{k_1^4T^4e^{-4i\alpha}}{192}\right|.$$

Using Lemma 2, we obtain

$$\begin{aligned}
&|a_2a_4 - a_3^2| \\
&= \left|T^2e^{-2i\alpha}\left\{\frac{k_1\left[k_1^3 + 2(4-k_1^2)k_1x - k_1(4-k_1^2)x^2 + 2(4-k_1^2)\left(1-|x|^2\right)z\right]}{48}\right.\right. \\
&\quad \left.\left. - \frac{(1+B)k_1^2\left[k_1^2 + x(4-k_1^2)\right]}{96} + \frac{(1+B)^2k_1^4}{192} - \frac{\left[k_1^2 + x(4-k_1^2)\right]^2}{64}\right\} - \frac{k_1^4T^4e^{-4i\alpha}}{192}\right| \\
&= \left|T^2e^{-2i\alpha}\left\{\frac{\left[4k_1^4 + 8k_1^2x(4-k_1^2) - 4k_1^2x^2(4-k_1^2) + 8k_1(4-k_1^2)\left(1-|x|^2\right)z\right]}{192}\right.\right. \\
&\quad - \frac{2(1+B)\left[k_1^4 + k_1^2x(4-k_1^2)\right] + (1+B)^2k_1^4 - 3k_1^4 - 6k_1^2x(4-k_1^2)}{192} \\
&\quad \left.\left. - \frac{3x^2(4-k_1^2)^2}{192}\right\} - \frac{k_1^4T^4e^{-4i\alpha}}{192}\right| \\
&= \left|T^2e^{-2i\alpha}\left\{\frac{k_1^4\left[1 - 2(1+B) + (1+B)^2\right] + k_1^2x(4-k_1^2)\left[8 - 2(1+B) - 6\right]}{192}\right.\right. \\
&\quad \left.\left. - \frac{x^2(4-k_1^2)\left[4k_1^2 + 3(4-k_1^2)\right] + 8k_1(4-k_1^2)\left(1-|x|^2\right)z}{192}\right\} - \frac{k_1^4T^4e^{-4i\alpha}}{192}\right|
\end{aligned}$$

$$= \left| T^2 e^{-2i\alpha} \left\{ \frac{k_1^4 B^2 - 2Bk_1^2 x \left(4-k_1^2\right) - x^2 \left(4-k_1^2\right)\left[k_1^2+12\right]}{192} + \frac{8k_1\left(4-k_1^2\right)\left(1-|x|^2\right)z}{192} \right\} - \frac{k_1^4 T^4 e^{-4i\alpha}}{192} \right|.$$

By Lemma 1, $|k_n| \le 2$. Then, let $k_1 = k$, we may assume without restriction that$k \in [0,2]$ which gives

$$\left|a_2 a_4 - a_3^2\right| = \left| T^2 e^{-2i\alpha} \left\{ \frac{k^4 B^2 - 2Bk^2 x \left(4-k^2\right) - x^2 \left(4-k^2\right)\left[k^2+12\right]}{192} + \frac{8k\left(4-k^2\right)\left(1-|x|^2\right)z}{192} \right\} - \frac{k^4 T^4 e^{-4i\alpha}}{192} \right|.$$

Since $\left|e^{-2i\alpha}\right| = 1, \quad \left|e^{-4i\alpha}\right| = 1$ and application of a triangle inequality with $|z| \le 1$ gives

$$\begin{aligned}&\left|a_2 a_4 - a_3^2\right| \\ &= T^2 \left\{ \frac{k^4 B^2 + 2k^2 |x| |B| \left(4-k^2\right) + |x|^2 \left(4-k^2\right)\left[k^2+12-8k\right] + 8k\left(4-k^2\right)}{192} \right\} + \frac{k^4 T^4}{192}.\end{aligned}$$

Replacing $|x|$ by ρ gives

$$\begin{aligned}&\left|a_2 a_4 - a_3^2\right| \\ &\le T^2 \left\{ \frac{k^4 B^2 + 2k^2 \rho |B| \left(4-k^2\right) + \rho^2 \left(4-k^2\right)\left[k^2+12-8k\right] + 8k\left(4-k^2\right)}{192} \right\} + \frac{k^4 T^4}{192} \\ &= F(k,\rho). \qquad (15)\end{aligned}$$

Next we assume that the upper bound for (15) occurs at an interior point of rectangle $k \times \rho = [0,2] \times [0,1]$.

First, differentiating (15) with respect to ρ we obtain

$$F'(k,\rho) = T^2 \left\{ \frac{2k^2 |B| \left(4-k^2\right) + 2\rho\left(4-k^2\right)\left[k^2+12-8k\right]}{192} \right\}. \qquad (16)$$

For $0 < \rho < 1$ and for any fixed k with $0 < k < 2$, from (16) we observe that $F'(k,\rho) > 0$. Therefore, $F(k,\rho)$ is an increasing function of ρ implying $\max(F(k,\rho)) = F(k,1) = G(k)$. Morever, for fixed $k \in [0,2]$, let

$$G(k) = T^2 \left\{ \frac{k^4\left(B^2 + 2|B| - 1\right) + 8k^2(|B|-1) + 48}{192} \right\} + \frac{k^4 T^4}{192}$$

then we have

$$G'(k) = T^2 \left\{ \frac{k^3 \left(B^2 + 2|B| - 1\right) + 4k(|B| - 1)}{48} \right\} + \frac{k^3 T^4}{48}$$

and

$$G''(k) = T^2 \left\{ \frac{3k^2 \left(B^2 + 2|B| - 1\right) + 4(|B| - 1)}{48} \right\} + \frac{3k^2 T^4}{48}.$$

By setting $G'(k) = 0$, the solutions for k are

$$k = 0, k = \pm 2\sqrt{\frac{1 - |B|}{B^2 + 2|B| - 1 + T^2}}.$$

Since $k \in [0, 2]$ by our assumption, we find that the maximum of $G(k)$ occurs at $k = 0$. Thus, from (15) the upper bound of $F(k, \rho)$ corresponds to $\rho = 1$ and $k = 0$ we obtain

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{T^2}{4}.$$

Remark 1: Setting $A = 1$ and $B = -1$ in Theorem 1, we obtain $\left|a_2 a_4 - a_3^2\right| \leq t_{\alpha\delta}^2$. This is the upper bound for the Second Hankel determinant for the class $S_c^*(\alpha, \delta)$ which is introduced earlier as in (2).

Remark 2: From Theorem 1, we observe that the result obtained is the same as the result for functions in the class S_{sc}^*.

4 Conclusion

In conclusion, we have obtained the sharp upper bound for the Second Hankel determinant for the class $S_c^*(\alpha, \delta)$ and $S_c^*(\alpha, \delta, A, B)$. By considering some specific values for the parameters α, β, A and B involved in $S_c^*(\alpha, \delta, A, B)$, we can reduce our result to some subclasses studied by previous researchers such as El-Ashwah and Thomas (as cited in [2]), Abdul Halim [2] and Mad Dahhar and Janteng [10].

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