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Numerical Solution of Polynomial Equations using Ostrowski Homotopy Continuation Method

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Abstract It is known that homotopy continuation methods (HCM) used in conjunction with a classical numerical method to compute roots of nonlinear algebraic equations can improve the performance of the classical numerical method. In this paper, we develop an Ostrowski-HCM method and apply it to solve several polynomial equations. The results obtained indicate that Ostrowski-HCM performs better than Ostrowski's method.

Keywords Numerical method; Polynomial equations; Ostrowski's method; Homotopy continuation method

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1 Introduction

The problem of solving or finding the roots of a polynomial equation, represented by $f(x) = 0$, is an important problem in the field of numerical analysis which requires efficient and accurate solution procedures. The computation of roots of polynomial equations has many real world applications in physics, chemistry, economics and so on. There are also problems which necessitate the solution of a system of polynomial equations

$$\mathbf{F}(\mathbf{x}) = \mathbf{0}, \tag{1}$$

where $\mathbf{F}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) & f_2(\mathbf{x}) & \cdots & f_n(\mathbf{x}) \end{pmatrix}^{\mathrm{T}}$ and $\mathbf{x} = \{x_1, x_2, ..., x_n\}$ and this also requires efficient and accurate solution procedures.

When searching for the roots of $f(x) = 0$ or $\mathbf{F}(\mathbf{x}) = \mathbf{0}$, it is not uncommon to run into difficulties such as overflow and divergence usually caused by using a bad initial guess or bad structure of equations. A recent study by Wu [1] explored using the homotopy continuation method (HCM) to overcome these difficulties.

According to Gritton *et al.* [2], there are four classifications of numerical methods to find the roots of $f(x) = 0$ or $\mathbf{F}(\mathbf{x}) = \mathbf{0}$: local methods, global methods, interval methods and graphical methods. We focus on the first two methods. The authors in [2] stated that local methods require an initial guess of a root which is close to the intended root for the particular application while global methods can use an arbitrary initial guess. As an example, the Newton method is a local method while HCM itself, on its own, is a global method. In this paper, we combine a local method called Ostrowski's method (OM) with HCM to obtain the Ostrowski-HCM (OHCM).

Section 2 discusses Ostrowski's method and its variants. Section 3 discusses Ostrowski's method and its problems with divergence. Section 4 contains the details and inner workings of OHCM. Section 5 contains the numerical experiments and results, while Section 6 gives the conclusion.

2 Ostrowski's Method and Its Variants

Ostrowski's method was introduced by Alexander Markowich Ostrowski in 1960 [3] to find the roots of a single-variable nonlinear function. The method employs two-step iterations using the following equations

$$y_i = x_i - \frac{f(x_i)}{f'(x_i)}, \qquad i = 0, 1, 2, ..., k-1, \tag{2a}$$

$$x_{i+1} = y_i - \frac{x_i - y_i}{f(x_i) - 2f(y_i)} f(y_i), \qquad i = 0, 1, 2, ..., k-1, \tag{2b}$$

where (2a) is the classical Newton's method while (2b) is the Ostrowski's formula. Ostrowski [4] then expanded (2b) and reformulated as follows

$$\begin{aligned} y_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ x_{i+1} &= y_i - \frac{f(x_i)}{f(x_i) - 2f(y_i)} \frac{f(y_i)}{f'(x_i)}, \qquad i = 0, 1, 2, ..., k-1. \end{aligned} \tag{3}$$

Ostrowski's method which has fourth order convergence is an extension of Newton's method which has second order convergence. More recent studies by Grau and Diaz-Barrero [5], Sharma and Guha [6] , Chun and Ham [7] and Kou *et al.* [8] use Osrowski's method as a basis for the methods they developed.

Studies [5-7] extended Ostrowski's method to one that has sixth order convergence. The method put forward in [5] is

$$\begin{aligned} y_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ z_i &= y_i - \frac{f(x_i)}{f(x_i) - 2f(y_i)} \frac{f(y_i)}{f'(x_i)}, \qquad i = 0, 1, 2, ..., k-1. \\ x_{i+1} &= z_i - \frac{f(x_i)}{f(x_i) - 2f(y_i)} \frac{f(z_i)}{f'(x_i)}, \end{aligned} \tag{4}$$

The study in [6] also explores a modification of Ostrowski's method with accelerated sixth order convergence. The method put forward is

$$\begin{aligned} y_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ z_i &= y_i - \frac{f(x_i)}{f(x_i) - 2f(y_i)} \frac{f(y_i)}{f'(x_i)}, \qquad i = 0, 1, 2, ..., k-1, \\ x_{i+1} &= z_i - \frac{f(x_i) + (\beta+2) f(y_i)}{f(x_i) + \beta f(y_i)} \frac{f(z_i)}{f'(x_i)}, \end{aligned} \tag{5}$$

where $\beta \in R$. When $\beta = -2$ scheme (5) reduces to scheme (4). The study in [6] compared equation (5) with two other methods i.e. equation (3) and equation (4) as introduced in studies [4] and [5] respectively. The numerical results overwhelmingly support the new method introduced by Sharma and Guha [6] rather than the schemes introduced in [4] and [6].

Chun and Ham [7] put forward the following scheme

$$\begin{aligned} y_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ z_i &= y_i - \frac{f(x_i)}{f(x_i)-2f(y_i)}\frac{f(y_i)}{f'(x_i)}, \qquad i = 0,1,2,...,k-1, \\ x_{i+1} &= z_i - R(u_i)\frac{f(z_i)}{f'(x_i)}, \end{aligned} \tag{6}$$

where $R(\lambda)$ is a real-valued function and $u_i = \frac{f(y_i)}{f(x_i)}$. The authors observed that if $R(\lambda) = \frac{1}{1-2\lambda}$ and $R(\lambda) = \frac{1+(\beta+2)\lambda}{1+\beta\lambda}$, then Eq. (6) can be reduced to (4) and (5) respectively. In [7], three methods were developed by using three different real-valued functions $R(\lambda)$ and a comparison was made with the existing schemes such as (2a), (3), (4) and (5). The numerical results showed that the schemes derived from (6) improve the computational efficiencies.

The work in [8] presented some variants of Ostrowski's method with seventh order convergence. Their method is given by

$$\begin{aligned} y_i &= x_i - \frac{f(x_i)}{f'(x_i)}, \\ z_i &= y_i - \frac{f(y_i)}{f(x_i)-2f(y_i)}\left(x_i - y_i\right), \qquad i = 0,1,2,...,k-1. \\ x_{i+1} &= z_i - \left[\left(1+\frac{f(y_i)}{f(x_i)-2f(y_i)}\right)^2 + \frac{f(z_i)}{f(y_i)}\right]\frac{f(z_i)}{f'(x_i)}, \end{aligned} \tag{7}$$

The authors in [8] claim that (7) has a higher efficiency index than (3) and (4). The efficiency index can be calculated as follows [9]

$$\lim_{k\to\infty}\left[\frac{\eta_{k+1}}{\eta_k}\right]^{\frac{1}{m}} = q^{\frac{1}{m}}, \tag{8}$$

where q is the convergence order of the method and m refer to the number of function evaluations per iteration required by the method. Hence the scheme (8) is very efficient and performs better than the schemes of (3) and (4).

The solution methods presented in [5-8] all tackle the problem of finding roots of single variable of polynomial equations. For systems of equations, one of the most commonly used solution methods is Newton's method, given by

$$\mathbf{x}_{i+1} = \mathbf{x}_i - \left[\mathbf{F}'(\mathbf{x}_i)\right]^{-1}\mathbf{F}(\mathbf{x}_i), \quad i = 0,1,2,...,k-1. \tag{9}$$

Burden and Faires [10] defined (9) in two ways. Firstly as

$$\begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix} - \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{pmatrix}^{-1} \begin{pmatrix} f_1(x_0,y_0) \\ f_2(x_0,y_0) \end{pmatrix}, \tag{10}$$

and secondly as

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = -\begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{pmatrix}^{-1} \begin{pmatrix} f_1(x_0, y_0) \\ f_2(x_0, y_0) \end{pmatrix}, \tag{11}$$

where $t = [0, 1]$. Standard numerical methods can then be used to solve the differential equations in (11).

Letting the number of independent variables $n = 2$, we can write the Taylor series expansion of $\mathbf{F}(\mathbf{x})$ as

$$u_{i+1} = u_i + (x_{i+1} - x_i)\frac{\partial u_i}{\partial x} + (y_{i+1} - y_i)\frac{\partial u_i}{\partial y}, \tag{12}$$

and

$$v_{i+1} = v_i + (x_{i+1} - x_i)\frac{\partial v_i}{\partial x} + (y_{i+1} - y_i)\frac{\partial v_i}{\partial y}, \tag{13}$$

where $u(x, y) = f_1(x, y)$and $v(x, y) = f_2(x, y)$. Chapra and Canale [11] simplified (12) and (13) by using Cramer's rule to get

$$x_{i+1} = x_i - \frac{u_i \dfrac{\partial v_i}{\partial y} - v_i \dfrac{\partial u_i}{\partial y}}{\dfrac{\partial u_i}{\partial x}\dfrac{\partial v_i}{\partial y} - \dfrac{\partial u_i}{\partial y}\dfrac{\partial v_i}{\partial x}}, \tag{14}$$

and

$$y_{i+1} = y_i - \frac{v_i \dfrac{\partial u_i}{\partial x} - u_i \dfrac{\partial v_i}{\partial x}}{\dfrac{\partial u_i}{\partial x}\dfrac{\partial v_i}{\partial y} - \dfrac{\partial u_i}{\partial y}\dfrac{\partial v_i}{\partial x}}. \tag{15}$$

Studies [5-11] have a similarity in the sense that they all present methods to solve polynomial equations and systems of polynomial equations with a good initial guess. However, there are fewer studies of solving an equation with a bad initial guess. A collection of studies by Wu [1,12-15] deals with this problem with the use of HCM. In addition, there are also not many studies on Ostrowski's method for solving a system of polynomial equations $\mathbf{F}(\mathbf{x}) = \mathbf{0}$. For example, Ostrowski [3] and Grau-Sanchez *et al.* [16] focused on this problem. Motivated by this, we develop a new hybrid method for solving $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ by combining Ostrowski's method with HCM to obtain Ostrowski-HCM (OHCM).

3 Divergence and Ostrowski's Method

Burden and Faires [10] stated that Newton's method requires an accurate initial approximation to the solution of $f(x) = 0$ to ensure convergence. Further, Chapra and Canale [11] stated that the convergence of Newton's method depends on the nature of the function and on the accuracy of the initial guess. In other words, divergence occurs when we choose inappropriate initial values or there exists a bad structure of equations to track the approximate solutions of single polynomial equation as well as a system of polynomial equations. We consider the following two examples.

Example 1 Consider the following polynomial equation as in Matinfar and Aminzadeh [17]

$$f(x) = x^5 + x^4 + 4x^2 - 15 = 0, \tag{16}$$

where $x = 1.347428098968305$ is an exact solution. Table 1 shows the performance of Ostrowski's method for solving single polynomial equations (16).

Table 1: Divergence and Ostrowski's Method for Equation (16)

Initial Value	Ostrowski's Method
$x_0 = 0$	*Indeterminate at first iteration*
$x_0 = -1$	*Converge after 13 iterations*
$x_0 = 0.5$	*Converge after 4 iterations*

Example 2 Consider the following polynomial equation as in Chapra and Canale [11]

$$f(x) = x^{10} - 1 = 0, \tag{17}$$

where $x = 1$ is an exact solution. Table 2 shows the performance of Ostrowski's method for solving single polynomial equations (17).

Table 2: Divergence and Ostrowski's Method for Equation (17)

Initial Value	Ostrowski's Method
$x_0 = 0$	*Indeterminate at first iteration*
$x_0 = 0.1$	*Converge after 76 iterations*
$x_0 = 0.5$	*Converge after 16 iterations*

The stopping criterion used is $\eta = 10^{-5}$. There are some points for which that Ostrowski's method do not work at initial iterations, i.e. $x_0 = 0$ in Table 1 and Table 2, because it causes division by zero in the Ostrowski's formula as $f'(x_0) = 0$. There are also some situations that causes Ostrowski's method to diverge when we use an initial value x_0 that is not sufficiently close to the true roots. For example, the initial value considered in Table 1, $x_0 = -1$, is not close to the exact solution $x = 1.347428098968305$. Apart from divergence due to an inappropriate initial guess, another type of divergence comes from the use of bad structure of equations. In Table 2, it is observed that there are still divergence problems even when we use the good initial guess of $x_0 = 0.5$, which is close to the true root $x = 1$.

In [10] it was stated that the HCM can be used as a stand-alone method and does not require a particularly good choice of initial guess. In this case, one can choose any arbitrary initial guess (even bad initial values) when using HCM. This fulfills the definition of a global method as discussed in Gritton *et al.* [2] and is the motivation for the development of Ostrowski-HCM so as to overcome the deficiencies of Ostrowski's method.

4 Ostrowski Homotopy Continuation Method

Recently, there have been several studies that combine local methods with HCM such as Newton-HCM, Adomian-HCM and Secant-HCM. All of the aforementioned hybrid methods use functional transformation to combine their chosen local method with HCM.

For example, Wu [1,13] combined Newton's method with HCM to obtain Newton-HCM:

$$x_{i+1} = x_i - \frac{H(x_i,t)}{D_x H(x_i,t)}, \quad i = 0,1,2,...,k-1, \tag{18}$$

where $H(x,t)$ refer to the homotopy function and $D_x H(x,t)$ refer to the Jacobian matrix. Note that equation (18) is only suitable for solving polynomial equations. To solve a system of polynomial equations, Wu [1] altered the above method to obtain

$$\mathbf{x}_{i+1} = \mathbf{x}_i - \left[D_{\mathbf{x}}\mathbf{H}(\mathbf{x}_i,t)\right]^{-1}\mathbf{H}(\mathbf{x}_i,t), \quad i = 0,1,2,...,k-1, \tag{19}$$

where $\mathbf{x}_{i+1}, \mathbf{x}_i, \mathbf{H}(\mathbf{x}_i,t)$ are supposed to be vectors with dimensions $n \times 1$. The homotopy function is defined as

$$\mathbf{H}(\mathbf{x},t) = (1-t)\mathbf{G}(\mathbf{x}) + t\,\mathbf{F}(\mathbf{x}), \quad i = 0,1,2,...,k-1. \tag{20}$$

The Jacobian matrix for the homotopy function is defined as

$$D_{\mathbf{x}}\mathbf{H}(\mathbf{x},t) = \begin{pmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x_2} & \cdots & \frac{\partial H_1}{\partial x_n} \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial x_2} & \cdots & \frac{\partial H_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial H_n}{\partial x_1} & \frac{\partial H_n}{\partial x_2} & \cdots & \frac{\partial H_n}{\partial x_n} \end{pmatrix}. \tag{21}$$

Notice that in creating Newton-HCM, $f(x)$ or $\mathbf{F}(\mathbf{x})$ is converted to the homotopy function $H(x,t)$. We will now use the same technique to construct our new method to be known as Ostrowski-HCM:

$$\begin{aligned} y_i &= x_i - \frac{H(x_i,t)}{H'(x_i,t)}, \\ x_{i+1} &= y_i - \frac{H(x_i,t)}{H(x_i,t) - 2H(y_i,t)}\frac{H(y_i,t)}{H'(x_i,t)}, \qquad i = 0,1,2,...,k-1, \end{aligned} \tag{22}$$

where $i = 0, 1, 2, ..., k-1$ and $t \in [0, 1]$. To solve a system of polynomial equations, we need to modify (22) to

$$\begin{aligned} \mathbf{y}_i &= \mathbf{x}_i - \left[D_{\mathbf{x}}\mathbf{H}(\mathbf{x}, t)\right]^{-1} \mathbf{H}(\mathbf{x}, t), \\ \mathbf{x}_{i+1} &= \mathbf{y}_i - \left[D_{\mathbf{x}}\mathbf{H}(\mathbf{x}, t)\right]^{-1} \frac{\mathbf{H}(\mathbf{x}_i, t)\mathbf{H}(\mathbf{y}_i, t)}{\mathbf{H}(\mathbf{x}_i, t) - 2\mathbf{H}(\mathbf{y}_i, t)}, \qquad i = 0, 1, 2, ..., k-1, \end{aligned} \tag{23}$$

where $\mathbf{y}_i = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}^T$, $\mathbf{x}_i = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T$, and $D_{\mathbf{x}}\mathbf{H}(\mathbf{x}, t)$ is the Jacobian matrix of size $n \times n$.

5 Numerical Experiments and Discussion

To show the effectiveness and efficiency of our proposed method, we compare Ostrowski-HCM with the classical Ostrowski's method to solve two examples of scalar polynomial equations and two examples of systems of polynomial equations.

5.1 Solving polynomial equations

We reconsider Example 3.1 and 3.2 and illustrate that Ostrowski-HCM can solve the divergence problem as well as accelerate convergence. The stopping criterion used was $|f(x)| < 10^{-5}$.

Table 3: Comparison between Ostrowski's Method and Ostrowski-HCM for Equation (16)

Initial Value	Ostrowski's Method	Ostrowski-HCM
$x_0 = 0$	*Indeterminate at first iteration*	*Converge after 4 iterations*
$x_0 = -1$	*Converge after 13 iterations*	*Converge after 9 iterations*
$x_0 = 0.5$	*Converge after 4 iterations*	*Converge after 3 iterations*

Table 4: Comparison between Ostrowski's Method and Ostrowski-HCM for Equation (17)

Initial Value	Ostrowski's Method	Ostrowski-HCM
$x_0 = 0$	*Indeterminate at first iteration*	*Converge after 10 iterations*
$x_0 = 0.1$	*Converge after 76 iterations*	*Converge after 4 iterations*
$x_0 = 0.5$	*Converge after 16 iterations*	*Converge after 2 iterations*

Table 3 and Table 4 show the behavior of Ostrowski's method and how Ostrowski-HCM is able to converge faster and solve the divergence problems. Note that divergence problems that arise when using $x_0 = 0$ (which causes $f'(x_0) = 0$) in both tables have been resolved in Ostrowski-HCM. The number of iterations it takes to converge when using Ostrowski-HCM with the other initial values in both tables have been vastly reduced, when compared to using Ostrowski-HCM with the same initial values. We next compare the behavior of Ostrowski's method with Ostrowski-HCM at the same number of iterations. This is shown in Figure 1 and Figure 2 for equations (16) and (17).

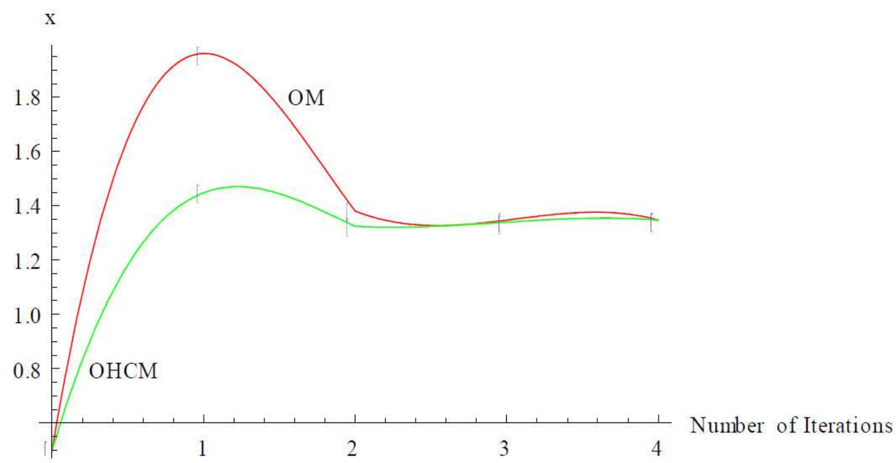

Figure 1: Comparison of Performance Ostrowski's Method and Ostrowski-HCM for Equation (17) with $x_0 = 0.5$.

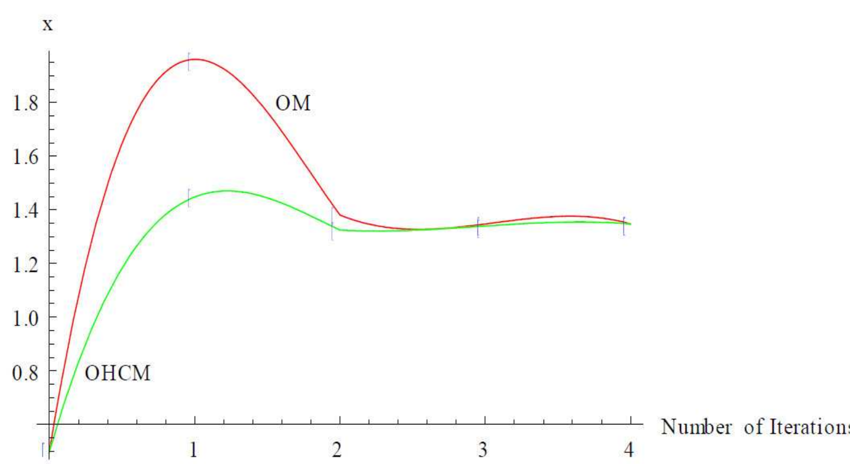

Figure 2: Comparison of Performance Ostrowski's Method and Ostrowski-HCM for Equation (18) with $x_0 = 0.5$.

The graphs show that Ostrowski-HCM converges more quickly to the approximate solutions while Ostrowski's method becomes inconsistent and only slowly converges.

5.2 Solving System of Polynomial Equations

We choose two systems of polynomial equations that has two and three independent variables. We test both equations by comparing the proposed method with the classical Ostrowski's method and with the stopping criterion $\|\mathbf{F}(\mathbf{x})\|_\infty < 10^{-3}$.

Example 3 Consider the following system of polynomial equations in [18] :

$$\begin{aligned} f_1(x,y) &= x^2 - 2x - y + 0.5 = 0, \\ f_2(x,y) &= x^2 + 4y^2 - 4 = 0, \end{aligned} \tag{24}$$

where the exact solutions are $(x_1, y_1) = (1.900676726367066, 0.3112185654192943)$ and $(x_2, y_2) = (-0.2222145550597218, 0.993808418599834)$. The auxiliary homotopy functions $g_1(x) = x - x_0$ and $g_2(y) = y - y_0$ are used for the Ostrowski-HCM. The results are shown in Table 5.

Table 5: Comparison between Ostrowski's Method and Ostrowski-HCM for Equation (24)

Initial values	Ostrowski's method	Ostrowski-HCM
$(x_0, y_0) = (0.001, 0.001)$	*Converge after 7 iterations*	*Converge after 5 iterations*
$(x_0, y_0) = (0, 0)$	*Indeterminate at first iteration*	*Converge after 5 iterations*
$(x_0, y_0) = (-0.001, -0.001)$	*Converge after 19 iterations*	*Converge after 5 iterations*

Example 4 Consider the following example in [19]:

$$\begin{aligned} f_1(x,y,z) &= x^2 + y^2 + z^2 - 1 = 0, \\ f_2(x,y,z) &= 2x^2 + y^2 - 4z = 0, \\ f_3(x,y,z) &= 3x^2 - 4y^2 + z^2 = 0, \end{aligned} \tag{25}$$

where$(x, y, z) = (0.69828860997151, -0.62852429796021, 0.342564189689569)$ is an exact solution of Eq. (25). The auxiliary homotopy functions $g_1(x) = x - x_0$, $g_2(y) = y - y_0$ and $g_3(z) = z - z_0$ are used. The results are shown in Table 6.

The results obtained clearly show Ostrowski-HCM has more advantages over Ostrowski's method in that Ostrowski-HCM converges more quickly than Ostrowski's method and has ability to converge even at a bad initial guess which resulted in the failure of Ostrowski's method.

Table 6: Comparison between Newton-HCM and Ostrowski-HCM for Equation (25)

Initial values	Ostrowski's method	Ostrowski-HCM
$(x_0, y_0, z_0) = (0, 0, 0)$	*Indeterminate at first iteration*	*Converge after 6 iterations*
$(x_0, y_0, z_0) = (0.001, 0.001, 0.001)$	*Converge after 7 iterations*	*Converge after 6 iterations*
$(x_0, y_0, z_0) = (0.0001, 0.0001, 0.0001)$	*Converge after 8 iterations*	*Converge after 6iterations*

6 Conclusion

The results show the improved performance of Ostrowski homotopy continuation method when compared with the Ostrowski's method. This improved performance has been demonstrated by applications to solve two scalar polynomial equations as well as two systems of polynomial equations. We have tested on several other examples and obtained the same conclusions. However to generalize the results will require an analytical investigation which we propose to do in the near future.

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