MATEMATIKA, 2014, Volume 30, Number 1, 59–67

Prime Graph of the Commutative Ring $\mathbb{Z}_n$

[1]K. Patra and [2]S. Kalita
[1,2]Gauhati University
Gopinath Bardoloi Nagar, Guwahati - 781014, Assam, India
e-mail: [1]kuntalapatra@gmail.com, [2]sanjoykalita1@gmail.com

Abstract Prime graph is a graph associated to rings denoted by $PG(R)$. $PG(R)$ is defined as the graph whose vertices are the elements of the ring R and any two elements x and y of R are adjacent in $PG(R)$ if and only if $xRy = 0$ or $yRx = 0$. In this paper the chromatic number of prime graph of some rings namely $\mathbb{Z}_n$, where $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, are studied.

Keywords Prime Graph; Chromatic Numbers.

2010 Mathematics Subject Classification 05C25, 05C15, 13E15

1 Introduction

The study of rings with the help of graphs began when a graph of a commutative ring was defined by Beck [1]. It has got more attention when it was modified by Anderson and Livingston [2]. After them, many authors have introduced various kind of graphs associated to both commutative and non-commutative rings. Another graph structure associated to a ring called *prime graph* was introduced by Satyanarayana *et al.* [3]. Prime graph is defined as a graph whose vertices are all elements of the ring and any two distinct vertices $x, y \in R$ are adjacent if and only if $xRy = 0$ or $yRx = 0$. This graph is denoted by $PG(R)$.

Let $PG(R)$ be the prime graph of a ring R, then $PG(R)$ is a star graph if and only if R is a prime ring in which all the vertices are adjacent to the zero element of R [3]. If R is not a prime ring then there exists at least two non-zero elements of R which are adjacent in $PG(R)$. In this paper we investigate the chromatic number of prime graph $\chi PG(\mathbb{Z}_n)$ of ring $\mathbb{Z}_n$ for different values of n.

2 Preliminaries

Definition 1 A *graph* G consists of a set V called vertex set together with a set E of unordered pairs of distinct elements of V called edge set.

The cardinality of V and E are called *order* and *size* of the graph G respectively.

Definition 2 If in a simple graph every pair of vertices are adjacent then the graph is called a *complete* graph and is denoted by K_p.

Definition 3 A Graph H is said to be a *subgraph* of a graph G if all the vertices and edges of H are also the vertices and edges of G.

Definition 4 A *star graph* S_n is a graph with n vertices such that exactly one vertex has degree $n-1$ and the remaining $n-1$ vertices have degree 1.

Definition 5 A *clique* is a complete subgraph of a graph. A *maximal clique* is a clique which cannot be enlarged by adding additional vertices to it. A *maximum clique* is a clique of the largest possible size in a given graph. The order of the maximum clique in a graph G is called *clique number* of the graph and is denoted by $cl(G)$.

Definition 6 A *coloring* of a graph is an assignment of k-colors to the vertices of G such that no two adjacent vertices are assigned the same color.

The *chromatic number* $\chi(G)$ of a graph G is the minimum value of k for which G has a k-coloring.

Definition 7 A non-zero ring R is said to be a prime ring if for any two elements $a, b \in R$, $arb = 0$ for all $r \in R$ implies $a = 0$ or $b = 0$.

For example a *matrix ring* over an *integral domain* is a prime ring. Any *domain* is a prime ring

Definition 8 Let R be a ring. The *prime graph* of the ring R denoted by $PG(R)$ is defined as the graph whose vertex set $V = R$ and the edge set $E = \{(x, y) : \ xRy = 0 \ or \ yRx = 0, x \neq y\}$.

Some results that have been proved by Satyanarayana et al are mentioned below:

Theorem 1 [3] *If R is a semiprime ring, then the following conditions are equivalent:*

(i) *R is a prime ring.*

(ii) *$PG(R)$ is star graph.*

(iii) *$PG(R)$ is tree.*

Corollary 1 [3] If R is a ring with $|R| \geq 2$. Then R is a prime ring if and only if $Diam(PG(R)) = 2$ and $rad(PG(R)) = 1$.

Corollary 2 [3] The following conditions are equivalent:

(i) R is not a prime ring.

(ii) triangle is a subgraph of $PG(R)$.

(iii) there exists a chain of length greater than 2 in $PG(R)$.

(iv) $PG(R)$ is not a tree.

(v) $PG(R)$ is not a star graph.

3 Prime graph of $\mathbb{Z}_n$

In this section we first investigate the chromatic number of prime graph of $\mathbb{Z}_n$ for some particular values of n.

Theorem 2 *For the ring $\mathbb{Z}_{p^2}$, $\chi PG(\mathbb{Z}_{p^2}) = p$ where p is a prime.*

Proof Let a and b be any two elements of $\mathbb{Z}_{p^2} - \{0\}$, then a and b are adjacent in $PG(\mathbb{Z}_{p^2})$ if and only if $p|a$ and $p|b$. There are $p-1$ elements which are divisible by p and all are adjacent to each other. So these vertices induce a complete subgraph K_{p-1}.

Also the vertex 0 is adjacent to all the other vertices of $PG(\mathbb{Z}_{p^2})$ and the remaining vertices are adjacent only to 0. So K_p is a clique of $PG(\mathbb{Z}_{p^2})$. Hence $\chi PG(\mathbb{Z}_{p^2}) \geq p$. All vertices of the clique can be coloured with p colors. The elements which are not divisible by p are not adjacent to each other. So these vertices can be colored with any one color assigned to the vertices of the clique except 0. Thus the graph $PG(\mathbb{Z}_{p^2})$ can be properly colored with p colors. Hence $\chi PG(\mathbb{Z}_{p^2}) \leq p$. Therefore, $\chi PG(\mathbb{Z}_{p^2}) = p$.

In case when $p = 2$, $\mathbb{Z}_4 - \{0\} = \{1, 2, 3\}$. Here 2 is the only element divisible by 2. As the prime graph we are considering is a simple graph so the prime graph of $\mathbb{Z}_4$ is a star graph and $\chi PG(\mathbb{Z}_4) = 2$. So in this case also the result holds.

Example 1 The prime graph of $\mathbb{Z}_9$ is given in Figure 1 and from figure it is it is clear that $\chi PG(\mathbb{Z}_9) = 3$.

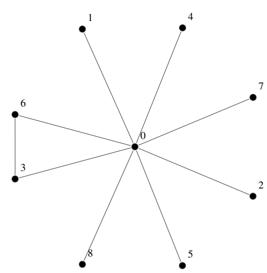

Figure 1: The Prime Graph of $\mathbb{Z}_9$

Theorem 3 *For the ring $\mathbb{Z}_{p^3}$, $\chi PG(\mathbb{Z}_{p^3}) = p + 1$ where p is a prime.*

Proof Let a and b be any two elements of $\mathbb{Z}_{p^3} - \{0\}$, then a and b are adjacent in $PG(\mathbb{Z}_{p^3})$ if and only if either one of a and b is divisible by p and other by p^2 or both divisible by p^2. There are $p-1$ elements divisible by p^2 and all are adjacent to each other. These vertices induce complete subgraph K_{p-1}.

Also there are $p(p-1)$ elements divisible by p but not by p^2. These elements are not adjacent to each other but adjacent to all elements divisible by p^2. 0 is adjacent to all the other vertices of $PG(\mathbb{Z}_{p^3})$. So any one element of those divisible by p together with all the elements divisible by p^2 and the vertex 0 induce a clique in $PG(\mathbb{Z}_{p^3})$ of order $p+1$. So $\chi PG(\mathbb{Z}_{p^3}) \geq p + 1$.

The elements of the clique can be colored with $p+1$ colors. The vertices divisible by p are not adjacent to each other so these vertices can be assigned with the same color that assigned to the vertex of this set considered in the clique. The vertices not divisible by p are not adjacent to each other and are adjacent to only 0. So these vertices can be assigned any one of the $p+1$ colors which is not assigned to 0. Thus $\chi PG(\mathbb{Z}_{p^3}) \leq p + 1$.

Therefore, $\chi PG(\mathbb{Z}_{p^3}) = p + 1$.

Example 2 Let us take the ring $\mathbb{Z}_{125}$, the non-zero elements which are adjacent in $PG(\mathbb{Z}_{125})$ are $\{5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95, 100, 105, 110, 115, 120\}$. Out of these vertices $\{25, 50, 75, 100\}$ are adjacent to each other and form a complete graph of order 4. These vertices can be assigned 4 distinct colors. The vertices $\{5, 10, 15, 20, 30, 35, 40, 45, 55, 60, 65, 70, 80, 85, 90, 95, 105, 110, 115, 120\}$ are adjacent to all of the vertices $\{25, 50, 75, 100\}$ but these vertices are not adjacent to each other. So a single color can be assigned to these vertices which is different from the 4 colors assigned. Also one more color required to assign vertex 0. Rest of the vertices can be colored with any one of these 6 colors. Therefore, $\chi PG(\mathbb{Z}_{125}) = 6$. The graph $PG(\mathbb{Z}_{125})$ is given in Figure 2.

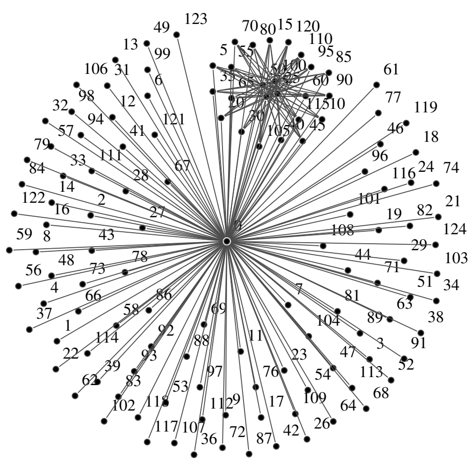

Figure 2: The Graph $PG(\mathbb{Z}_{125})$

Theorem 4 *For the ring $\mathbb{Z}_{p^4}$, $\chi PG(\mathbb{Z}_{p^4}) = p^2$ where p is a prime.*

Proof Any two non-zero elements of $\mathbb{Z}_{p^4}$ that are divisible by p^2 are always adjacent in $PG(\mathbb{Z}_{p^4})$. There are $p^2 - 1$ elements in $\mathbb{Z}_{p^4}$ divisible by p^2. So these elements induce a complete subgraph of oder $p^2 - 1$. Any elements divisible by p but not by p^2 is adjacent to only to the elements divisible by p^3. These elements are also not adjacent to each other. Since 0 is adjacent to all elements of the ring, so we have a clique of order p^2. Hence

$$\chi PG(\mathbb{Z}_{p^4}) \geq p^2.$$

The clique can be properly colored with p^2 colors. Now the vertices of $PG(\mathbb{Z}_{p^4})$ that are divisible by only p can be colored with any color that is not assigned to the vertices divisible by p^3. Rest of all elements can be colored with any one of the p^2 colors not assigned to 0. Thus all vertices of $PG(\mathbb{Z}_{p^4})$ is properly colored with p^2 colors i.e. $PG(\mathbb{Z}_{p^4})$ has a proper coloring of p^2 colors. Hence $\chi PG(\mathbb{Z}_{p^4}) \leq p^2$.

Therefore, $\chi PG(\mathbb{Z}_{p^4}) = p^2$.

Theorem 5 *For the ring $\mathbb{Z}_{p^5}$, $\chi PG(\mathbb{Z}_{p^5}) = p^2 + 1$ where p is a prime.*

Proof For $k > 2$, all elements $mp^k \in \mathbb{Z}_{p^5}$, $1 \leq m \leq p^{5-k} - 1$, $p \nmid m$ are adjacent to each other in $PG(\mathbb{Z}_{p^5})$. There are $p^2 - 1$ elements divisible by p^k, $k > 2$. These vertices induce the subgraph K_{p^2-1}.

For $k = 2$, the elements $mp^k \in \mathbb{Z}_{p^5}$, $p \nmid m$ are not adjacent to each other but adjacent to all elements mp^k, $k > 2$.

The elements which are divisible by p are adjacent only to elements mp^4, $1 \leq m \leq p-1$. Therefore all elements mp^k, $k > 2$, any one of the elements mp^2 and 0 together induce a clique of order $p^2 + 1$. Hence $\chi PG(\mathbb{Z}_{p^5}) \geq p^2 + 1$.

The vertices of the induce subgraph K_{p^2-1} can be colored with $p^2 - 1$ colors. Since the vertices of $PG(\mathbb{Z}_{p^5})$ that are only divisible by p^2, are adjacent to all vertices of the subgraph K_{p^2-1} but not adjacent to each other these can be colored with a single color distinct from the $p^2 - 1$ colors. The zero vertex can not be assigned any of these colors as it is adjacent to all the vertices of $PG(\mathbb{Z}_{p^5})$ so one more color is required to color this vertex. The vertices that are divisible by p are neither adjacent to each other nor to the vertices divisible only by p^2. So these vertices can be colored with any one of the colors assigned the vertices divisible by p^2. Rest of all vertices are adjacent only to vertex 0 so these can be colored with any of these colors not assigned to 0. Thus $PG(\mathbb{Z}_{p^5})$ has a proper coloring with $(p^2 - 1) + 1 + 1 = p^2 + 1$ colors. Hence $\chi PG(\mathbb{Z}_{p^5}) \leq p^2 + 1$.

Therefore, $\chi PG(\mathbb{Z}_{p^5}) = p^2 + 1$

In above results we have seen that chromatic number of $PG(\mathbb{Z}_{p^m})$ for $m = 2n$ and $m = 2n + 1$ differ by 1. Now we find the chromatic number of the prime graph of $\mathbb{Z}_n$ where n is a power of some prime.

Theorem 6 *Let the ring be $\mathbb{Z}_{p^{2n}}$, then $\chi PG(\mathbb{Z}_{p^{2n}}) = p^n$ where p is a prime and n is a positive integer.*

Proof For $1 \leq k \leq 2n - 1$, let us define sets

$$V_k = \{mp^k \mid 1 \leq m \leq p^{2n-k} - 1, (m, p) = 1\}.$$

For $k_1,\ k_2 \geq n$, let $a_1 = m_1 p^{k_1}$ and $a_2 = m_2 p^{k_2}$ are two elements of $\mathbb{Z}_{p^{2n}}$. Then $a_1 r a_2 = (m_1 p^{k_1}) r (m_2 p^{k_2}) = m_1 r m_2 p^{k_1+k_2} = 0$ for all $r \in \mathbb{Z}_{p^{2n}}$ ($\because k_1 + k_2 \geq 2n$), therefore $a_1(\mathbb{Z}_{p^{2n}})a_2 = 0$.

Therefore for $k \geq n$ the elements of each set V_k are adjacent to each other as well as adjacent to all elements of every sets $V_j,\ j \geq n,\ j \neq k$. Now $o(\bigcup_{k \geq n} V_k) = p^n - 1$. So elements of these sets induce a subgraph K_{p^n-1}.

For $j_1, j_2 < n$, let $b_1 = m_1 p^{j_1} \in V_{j_1}$ and $b_2 = m_2 p^{j_2} \in V_{j_2}$ are two elements of $\mathbb{Z}_{p^{2n}}$. Then $b_1 r b_2 = (m_1 p^{j_1}) r (m_2 p^{j_2}) = m_1 r m_2 p^{j_1+j_2} \neq 0$ for some $r \in \mathbb{Z}_{p^{2n}}$, so $b_1(\mathbb{Z}_{p^{2n}})b_2 \neq 0$. Therefore no two elements of each $V_k (k < n)$ are adjacent to each other. Also these elements are adjacent only to the element of the sets V_i $(i \geq 2n - k)$.

Thus all vertices of $\bigcup_{k \geq n} V_k$ and the vertex 0 together induces the maximal clique of $PG(\mathbb{Z}_{p^{2n}})$ of order p^n. Hence $\chi PG(\mathbb{Z}_{p^{2n}}) \geq p^n$.

The vertices of the clique can be colored with p^n colors. Now the vertices of the sets V_k for $k < n$ can be colored with the colors assigned to the vertices of the sets $V_i, n \leq i < 2n-k$. The vertices which are not divisible by $p^k, 1 \leq k \leq 2n - 1$ can be assigned any of the p^n colors not assigned to 0. So $PG(\mathbb{Z}_{p^{2n}})$ has a p^n -proper coloring. Hence $\chi PG(\mathbb{Z}_{p^{2n}}) \leq p^n$.

Therefore, $\chi PG(\mathbb{Z}_{p^{2n}}) = p^n$.

Theorem 7 *For the ring $\mathbb{Z}_{p^{2n+1}}$, $\chi PG(\mathbb{Z}_{p^{2n+1}}) = p^n + 1$ where p is a prime n is a positive integer.*

Proof For $1 \leq k \leq 2n$ let us define the set

$$V_k = \{mp^k \mid 1 \leq m \leq p^{(2n+1)-k} - 1, (m,p) = 1\}.$$

For $k_1,\ k_2 > n$, let $a_1 = m_1 p^{k_1}$ and $a_2 = m_2 p^{k_2}$ are two elements of $\mathbb{Z}_{p^{2n+1}}$. Then $a_1 r a_2 = m_1 p^{k_1} r m_2 p^{k_2} = m_1 r m_2 p^{k_1+k_2} = 0$ for all $r \in \mathbb{Z}_{p^{2n+1}}$, ($\because k_1 + k_2 > 2n+1$) and so $a_1(\mathbb{Z}_{p^{2n+1}})a_2 = 0$. Therefore for $k > n$ the elements of each set V_k are adjacent to each other and also adjacent to all elements of the sets $V_j,\ j > n,\ j \neq k$. Now $o(\bigcup_{k>n} V_k) = p^n - 1$. So these sets induce a subgraph K_{p^n-1}.

For $j_1, j_2 \leq n$, let $b_1 = m_1 p^{j_1} \in V_{j_1}$ and $b_2 = m_2 p^{j_2} \in V_{j_2}$ are two elements of $\mathbb{Z}_{p^{2n+1}}$. Then $b_1 r b_2 = (m_1 p^k) r (m_2 p^k) = m_1 r m_2 p^{2k} \neq 0$ for some $r \in \mathbb{Z}_{p^{2n+1}}$, so $b_1(\mathbb{Z}_{p^{2n+1}})b_2 \neq 0$.

Therefore any two elements of each $V_k (k \leq n)$ are not adjacent to each other. But they are adjacent to every element of all sets V_j with $j \geq 2n+1-k$. All elements of the set V_n are adjacent to every elements of V_j with $j \geq n+1$ *i.e.* all elements of $\bigcup_{k \geq n+1} V_k$.

Therefore all vertices of $\bigcup_{k \geq n+1} V_k$, any one vertex of V_n and the vertex 0 together induces a complete graph K_{p^n+1} which is also the maximal clique in $PG(\mathbb{Z}_{p^{2n+1}})$.

Hence $\chi PG(\mathbb{Z}_{p^{2n+1}}) \geq p^n + 1$.

The vertices of the induced subgraph K_{p^n-1} and the vertex 0 can be colored with p^n colors. Now the vertices of the sets V_n can be colored with one color but different from those p^n colors. The vertices of the sets V_k for $k < n$ can be colored with the colors assigned to the vertices of the sets $V_i, n \leq i < 2n+1-k$. The vertices which are not divisible by $p^k, 1 \leq k \leq 2n-1$ can be assigned any of the p^n colors not assigned to 0. So $PG(\mathbb{Z}_{p^{2n+1}})$ has a $p^n + 1$ -proper coloring. Hence $\chi PG(\mathbb{Z}_{p^{2n+1}}) \leq p^n + 1$.

Therefore, $\chi PG(\mathbb{Z}_{p^{2n+1}}) = p^n + 1$.

In the following theorems we find the chromatic number of prime graph of $\mathbb{Z}_n$, where n has more than one prime factor.

Theorem 8 *If $R = \mathbb{Z}_{p^\alpha q}$, then $\chi PG(R) = p^{\frac{\alpha}{2}} + 1$ where p and q are distinct primes and α is an even positive integer.*

Proof Let us define for $1 \leq s \leq \alpha - 1$ the set $V_s = \{mp^s q \mid 1 \leq m \leq p^{\alpha-s} - 1, (m,p) = 1\}$ and for $1 \leq t \leq \alpha$, $V'_t = \{m'p^t \mid 1 \leq m' \leq p^{\alpha-t} - 1, (m',p) = 1\}$.

For $s \geq \frac{\alpha}{2}$, all elements of each V_s are adjacent to each other and also adjacent to the elements of the sets $V_r,\ r \geq \frac{\alpha}{2},\ r \neq s$ and $o(\bigcup_{s \geq \frac{\alpha}{2}} V_s) = p^{\frac{\alpha}{2}} - 1$.

For $s < \frac{\alpha}{2}$, elements of each V_s are not adjacent to each other. But they are adjacent to the elements of the sets V_r, for all $r \geq \alpha - s$.

For $1 \leq t \leq \alpha$ the elements of each V'_t are non-adjacent in $PG(R)$ and also elements of any two sets V'_{t_1} and V'_{t_2}, $1 \leq t_1,\ t_2 \leq \alpha$ are non adjacent.

Now elements of V'_t, $t \geq \frac{\alpha}{2}$ are adjacent to all elements of V_s, $s \geq \frac{\alpha}{2}$. So they are adjacent to all elements of $\bigcup_{s \geq \frac{\alpha}{2}} V_s$. Since all elements are adjacent to 0, we obtain a maximal clique of order $(p^{\frac{\alpha}{2}} - 1) + 1 + 1 = p^{\frac{\alpha}{2}} + 1$. Therefore, $\chi PG(R) \geq p^{\frac{\alpha}{2}} + 1$.

All the vertices of the set $\bigcup_{s \geq \frac{\alpha}{2}} V_s$ and the vertex 0 can be colored with $p^{\frac{\alpha}{2}}$ colors. The vertices of the set $\bigcup_{t \geq \frac{\alpha}{2}} V'_t$ are not adjacent to each other but adjacent to all vertices belongs to the set $\bigcup_{s\geq\frac{\alpha}{2}} V_s$. So these vertices can be colored with one color distinct from the $p^{\frac{\alpha}{2}}$ colors. The vertices belongs to the sets $\bigcup_{s<\frac{\alpha}{2}} V_s$ and $\bigcup_{t<\frac{\alpha}{2}} V'_t$ are not adjacent to the vertices of the set $V_{\frac{\alpha}{2}}$, so the vertices of these two sets can be colored with the colors assigned to the vertices of the set $V_{\frac{\alpha}{2}}$. The vertices divisible by q are neither adjacent to each other nor to the vertices belongs to the set $\bigcup_{s\geq 1} V_s$ so these vertices can be assigned any colors from the $p^{\frac{\alpha}{2}}$ colors. Thus $PG(R)$ has a proper coloring with $p^{\frac{\alpha}{2}}+1$ colors. Hence $\chi PG(R) \leq p^{\frac{\alpha}{2}}+1$.

Therefore, $\chi PG(R) = p^{\frac{\alpha}{2}}+1$.

Theorem 9 *If $R = \mathbb{Z}_{p^\alpha q}$, then $\chi PG(R) = p^{\frac{(\alpha-1)}{2}}+2$ where p and q are distinct primes and α is an odd positive integer.*

Proof Let us define for $1 \leq s \leq \alpha-1$ the set $V_s = \{mp^s q \mid 1 \leq m \leq p^{\alpha-s}-1, (m,p) = 1\}$ and for $1 \leq t \leq \alpha$, $V'_t = \{m'p^t \mid 1 \leq m' \leq p^{\alpha-t}-1, (m',p) = 1\}$.

For $s \geq \frac{\alpha+1}{2}$, all elements of each V_s are adjacent to each other and also adjacent to the elements of the sets $V_r, r \geq \frac{\alpha+1}{2}, r \neq s$ and $o(\bigcup_{s\geq\frac{\alpha+1}{2}} V_s) = p^{\frac{\alpha-1}{2}}-1$.

For $s \leq \frac{\alpha-1}{2}$, elements of each V_s are not adjacent to each other. But they are adjacent to the elements of the sets V_r, for all $r \geq \alpha - s$. In particular the elements of $V_{\frac{\alpha-1}{2}}$ are adjacent to all elements of the sets $V_r, r \geq \frac{\alpha+1}{2}$.

For $1 \leq t \leq \alpha$ the elements of each V'_t are non-adjacent in $PG(R)$ and also elements of any two sets V'_{t_1} and V'_{t_2}, $1 \leq t_1,\ t_2 \leq \alpha$ are non adjacent.

Now elements of $V'_t,\ t \geq \frac{\alpha+1}{2}$ are adjacent to all elements of $V_s,\ s \geq \frac{\alpha-1}{2}$. In particular the elements of the set $V_{\frac{\alpha-1}{2}}$ and $V'_t,\ t \geq \frac{\alpha+1}{2}$ induce a complete bipartite subgraph. They are also adjacent to all elements of $\bigcup_{s\geq\frac{\alpha+1}{2}} V_s$. Since all elements are adjacent to 0, we obtain a maximal clique of order $(p^{\frac{\alpha-1}{2}}-1)+2+1 = p^{\frac{\alpha-1}{2}}+2$. The remaining elements cannot be adjacent to all of these elements, so $\chi PG(R) = p^{\frac{\alpha-1}{2}}+2$.

All the vertices of the set $\bigcup_{s\geq\frac{\alpha+1}{2}} V_s$ and the vertex 0 can be colored with $p^{\frac{\alpha-1}{2}}$ colors. The vertices of the sets $V_{\frac{\alpha-1}{2}}$ and $\bigcup_{t\geq\frac{\alpha-1}{2}} V'_t$ induced a bipartite subgraph and are adjacent to all vertices belongs to the set $\bigcup_{s\geq\frac{\alpha+1}{2}} V_s$. So color these vertices two more colors are required. The vertices belongs to the sets $\bigcup_{s<\frac{\alpha-1}{2}} V_s$ and $\bigcup_{t<\frac{\alpha-1}{2}} V'_t$ not adjacent to the vertices of the set $V_{\frac{\alpha-1}{2}}$, so the vertices of these two sets can be colored with the colors assigned to the vertices of the set $V_{\frac{\alpha-1}{2}}$. The vertices divisible by q are neither adjacent to each other nor to the vertices belongs to the set $\bigcup_{s\geq 1} V_s$ so these vertices can be assigned any colors from the $p^{\frac{\alpha-1}{2}}$ colors. Thus $PG(R)$ has a proper coloring with $p^{\frac{\alpha-1}{2}}+2$ colors. Hence $\chi PG(R) \leq p^{\frac{\alpha-1}{2}}+2$.

Therefore, $\chi PG(R) = p^{\frac{\alpha-1}{2}}+2$.

Theorem 10 *Let $R = \mathbb{Z}_n$ where $n = \prod_{i=1}^r p_i^{\alpha_i}$. Let for $1 \leq i \leq m$ each α_i is an odd positive integer. Then $\chi PG(\mathbb{Z}_n) = \prod_{i=1}^r p^{\left[\frac{\alpha_i}{2}\right]}+m$, where $[x]$ is the greatest integer function.*

Proof Let $v, v' \in \mathbb{Z}_n$ such that $v = a\prod_{i=1}^r p_i^{k_i}$ and $v' = a'\prod_{i=1}^r p_i^{k'_i}$, $(k_i, k'_i \leq \alpha_i)$ for all i. Then $vxv' = axa'\prod_{i=1}^r p_i^{k_i+k'_i}$ for all $x \in \mathbb{Z}_n$. So $vxv' = 0$, for all $x \in \mathbb{Z}_n$ i.e. v and v' are

adjacent if and only if $k_i + k'_i \geq \alpha_i$, for all i, $1 \leq i \leq r$.
Case I. Let for $1 \leq i \leq m$, $k_i, k'_i > \left[\frac{\alpha_i}{2}\right]$ and for $m+1 \leq i \leq r$, $k_i, k'_i \geq \left[\frac{\alpha_i}{2}\right]$ then for all $1 \leq i \leq r$, $k_i + k'_i \geq \alpha_i$. This implies that v and v' are adjacent. Therefore all $v = a\prod_{i=1}^{r} p_i^{k_i}$ where for $1 \leq i \leq m$, $k_i, ki' > \left[\frac{\alpha_i}{2}\right]$ and for $m+1 \leq i \leq r$, $k_i, k'_i \geq \left[\frac{\alpha_i}{2}\right]$ are adjacent to each other. Total number of such elements is $\prod_{i=1}^{r} p^{\left[\frac{\alpha_i}{2}\right]} - 1$ and they induce a complete graph.
Case II. Let for fixed j, $1 \leq j \leq m$, $v_j = a_j \prod_{i=1}^{r} p_i^{k_i}$ where $k_j = \left[\frac{\alpha_j}{2}\right]$, for $1 \leq i \leq m$, $i \neq j$, $k_i > \left[\frac{\alpha_i}{2}\right]$ and for $m+1 \leq i \leq r$, $k_i = \left[\frac{\alpha_i}{2}\right]$ then $v_j x v'_j \neq 0$, for some $x \in \mathbb{Z}_n$. Therefore the vertices with above conditions are not adjacent to each other. But for $1 \leq j_1, j_2 \leq m, j_1 \neq j_2$, $v_{j_1} x v_{j_2} = 0$, for all $x \in \mathbb{Z}_n$ as $k_{j_1} + k'_{j_1} \geq \alpha_{j_1}$ and $k_{j_2} + k'_{j_2} \geq \alpha_{j_2}$. Therefore these elements induce an $m-partite$ graph. Also all these elements are adjacent to all those elements we have considered in *Case I.*
Case III. Let for $1 \leq i \leq r$, $k_i, k'_i = \left[\frac{\alpha_i}{2}\right]$ then $vxv' \neq 0$, for some $x \in \mathbb{Z}_n (\because k_i + k'_i < \alpha_i)$. So these elements are not adjacent to each other. These elements are adjacent to all those elements considered in Case I but not adjacent to all elements considered in Case II.
Case IV. For $1 \leq i \leq m$, $k_i, k'_i > \left[\frac{\alpha_i}{2}\right]$ and for $m+1 \leq i \leq r$, $k_i, k'_i < \left[\frac{\alpha_i}{2}\right]$ then the elements are not adjacent to each other as for $m+1 \leq i \leq r$, $k_i + k'_i < \alpha_i$. But these elements are adjacent with only those elements with $k'_i > \alpha_i - k_i$, $m+1 \leq i \leq r$. Rest of the elements cannot be adjacent to all those elements considered in Case I and Case II. Therefore total number of non-zero elements that induce a complete graph is

$$\left(\prod_{i=1}^{r} p^{\left[\frac{\alpha_i}{2}\right]} - 1\right) + m.$$

Since all elements of the ring are adjacent to 0, we obtain a maximal clique of order $\prod_{i=1}^{r} p^{\left[\frac{\alpha_i}{2}\right]} + m$. Hence $\chi PG(\mathbb{Z}_n) \geq \prod_{i=1}^{r} p^{\left[\frac{\alpha_i}{2}\right]} + m$.

All the vertices that satisfy the condition of *Case I* induce a complete subgraph of order $\prod_{i=1}^{r} p^{\left[\frac{\alpha_i}{2}\right]} - 1$, so these vertices can be properly colored with $\prod_{i=1}^{r} p^{\left[\frac{\alpha_i}{2}\right]} - 1$ colors. The vertices satisfying condition of *Case II* induce a complete m-partite subgraph,so these vertices can be colored with m colors. But all the vertices of this m-partite subgraph are adjacent to all the vertices of the complete graph of order $\prod_{i=1}^{r} p^{\left[\frac{\alpha_i}{2}\right]} - 1$ so that these m colors are distinct from those $p^{\left[\frac{\alpha_i}{2}\right]} - 1$ colors. The vertices that satisfy the condition of *Case III* and *Case IV* are neither adjacent to each other nor to any vertices of the m-partite subgraph. So these vertices can be colored with the m colors. The vertex 0 required one more color to color it. Thus the graph $PG(\mathbb{Z}_n)$ can be properly colored with

$$(\prod_{i=1}^{r} p^{\left[\frac{\alpha_i}{2}\right]} - 1) + m + 1 = \prod_{i=1}^{r} p^{\left[\frac{\alpha_i}{2}\right]} + m$$

Hence $\chi PG(\mathbb{Z}_n) \leq \prod_{i=1}^{r} p^{\left[\frac{\alpha_i}{2}\right]} + m$. Therefore, $\chi PG(\mathbb{Z}_n) = \prod_{i=1}^{r} p^{\left[\frac{\alpha_i}{2}\right]} + m$.

4 Conclusion

From the above results it is observed that the chromatic number of the prime graph of the ring depends on the prime factorization of n. If n is prime then the prime graph of $\mathbb{Z}_n$ is a star graph, but the converse is not true.

Acknowledgments

The authors would like to express their thanks to UGC for their financial support. The authors also express their thanks for the valuable successions made by the referee in improving the paper.

References

[1] Beck, I. Coloring of commutative rings, *J. Algebra*. 1988. 116(1): 208-226.

[2] Anderson, D. F. and Livingston P. S., The zero-divisor graph of a commutative ring. *J. Algebra*. 1999. 217(2): 434–447.

[3] Satyanarayana, B., Shyam Prasad, K. and Nagaraju, D.. Prime Graph of a Ring, *J. of Combinatorics, Information and System Sources*. 2010. 35(1): 27–42.