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Some Results on Common Fixed Point Theorems in ϵ-Chainable Fuzzy Metric Spaces

Rajinder Sharma
College of Applied Sciences, Sohar
Department of General Requirements (Mathematics Section), Sultanate of Oman.
e-mail: rajind.math@gmail.com

Abstract In this paper, we prove common fixed point theorems for four weakly compatible mappings in ϵ-chainable fuzzy metric space without taking under consideration the continuity of any mappings. The given results extend and generalize several known results of fixed point theory in different spaces. The present study point out that the continuity of any mapping is not required for existence of fixed point in a complete ϵ-chainable fuzzy metric space.

Keywords Common Fixed Point; ϵ-Chainable Fuzzy metric space; Weakly Compatible Mappings

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1 Introduction

Banach contraction principle plays an important role in the development of various interesting results in fixed point theory as well as in approximation theory but suffers from one drawback that the mappings taken under consideration requires continuity throughout the space. Fuzzy set being a brain child of Zadeh [1] acts as a roadmap for other researchers involved in the field of nonlinear analysis for the development of fuzzy metric spaces in fixed point theory and its applications. Working in the same line Chang [2] established fuzzy topological properties. Kramosil and Michalek [3] , Erceg [4], Fisher [5] and Deng [6] have introduced the concept of fuzzy metric spaces in different ways. Sessa [7] generalized the concept of commutativity to weak commutativity. Kaleva and Seikkala [8] and Mishra *et al.* [9] explores further by proving some common fixed point theorems on fuzzy metric spaces. Furthermore, Jungck [10,11] introduced more generalized commutativity, which is called compatibility. This concept was frequently used to prove existence theorems in common fixed point theory. Grabiec [12] praiseworthy work on fixed points in fuzzy metric spaces by extending fixed point theorems of Banach and Edelstein to fuzzy metric spaces motivated other researcher to explore new horizons in the field of fixed point theory and its applications.

It is self evident that strict contractive condition doesn't guarantees the existence of common fixed point in the setting of metric space unless and until the space is assumed to be compact or the strict conditions are replaced by stronger conditions as in taken by George and Veermani [13].

Jachymski [14] considered some families of maps and proved common fixed point theorems for it. Cho [15], George and Veeramani [16] came out with some results on analysis for fuzzy metric spaces. Chang *et al.* [17] proved coincidence point and minimization theorems in fuzzy metric spaces. Jungck and Rhoades [18] introduced weak compatibility and proved common fixed point theorem for set valued functions without continuity of mappings. Pant [19] came out with common fixed points of non-compatible maps and R-weak commutativity in fuzzy metric spaces. Sharma [20], Song [21] and Cho and Jung [22] proved some common fixed point theorems in fuzzy metric spaces in different ways. Sharma and Deshpande [23] proved some common fixed point theorems for finite number of discontinuous, non-compatible mappings on non-complete fuzzy metric spaces. Furthermore, Sharma and Deshpande [24] extended their own work by proving some common fixed point theorems for finite number of discontinuous, non-compatible mappings on non-complete intuitionistic fuzzy metric spaces. The main objective of this paper is to prove some common fixed point theorems in a complete ϵ-chainable fuzzy metric space by striking off the condition of continuity.

2 Preliminaries

Definition 1 [3] The 3-tuple $(X, M, *)$ is called a fuzzy metric space (shortly FM-space) if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$,

(FM-1) $M(x, y, 0) = 0$,
(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, \cdot) : [0, 1] \to [0, 1]$ is left continuous.

In what follows $(X, M, *)$ will denote a fuzzy metric space, note that $M(x, y, t)$ can be thought as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$ and $M(x, y, t) = 0$ with ∞ and we can find some topological properties and examples of fuzzy metric spaces in [9,10].

In the following example, we know that every metric induces a fuzzy metric.

Example 1 [14] Let (X, d) be a metric space. Define $a * b = ab$ or $a * b = min\{a, b\}$ and for all $x, y \in X$ and $t > 0$,

$$M(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric M induced by the metric d, the standard fuzzy metric.

Lemma 1 [12] For all $x, y \in X, M(x, y, \cdot)$ is non-decreasing.

Definition 2 [12] Let $(X, M, *)$ be a fuzzy metric space :

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$), if $\lim_{n\to\infty} M(x_n, x, t) = 1$, for all $t > 0$.

(ii) A sequence $\{x_n\}$ in X called a Cauchy sequence if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1, \text{ for all } t > 0 \text{ and } p > 0.$$

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Remark 1 Since $*$ is continuous, if follows from (FM-4) that the limit of the sequence in fuzzy metric space is uniquely determined.
Let $(X, M, *)$ be a fuzzy metric space with the following condition:
(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$.

Lemma 2 [25] If for all $x, y \in X$, $t > 0$ and for a number $k \in (0, 1)$

$$M(x, y, kt) \geq M(x, y, t) \text{ then } x = y.$$

Lemma 3 [15] Let $\{y_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$ with the condition (FM-6). If $\exists$ a number $k \in (0, 1)$ such that

$$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$$

for all $t > 0$ and $n = 1, 2, \ldots$ then $\{y_n\}$ is a Cauchy sequence in X.

Definition 3 [22] Let $(X, M, *)$ be a fuzzy metric space and $\epsilon > 0$. A finite sequence $x = x_0, x_1, x_2, \ldots, x_n = y$ is called ϵ-chain from x to y if

$$M(x_i, x_{i-1}, t) > 1 - \epsilon \text{ for all } t > 0 \text{ and } i = 1, 2, \ldots, n.$$

A fuzzy metric space $(X, M, *)$ is called ϵ-chainable if for any $xy \in X$, $\exists$ an ϵ-chain from x to y.

Definition 4 [9] Let S and T be mappings from a fuzzy metric space $(X, M, *)$ into itself. The mappings S and T are said to be compatible if

$$\lim_{n\to\infty} M(STx_n, TSx_n, t) = 1,$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \text{ for some } z \in X.$$

Definition 5 [18] Two self mappings S and T are said to be weakly compatible if they commute at their coincidence points; i.e., if $Tu = Su$ for some $u \in X$, then $TSu = STu$

3 Main Results

Theorem 1 *Let $(X, M, *)$ be a complete ϵ-chainable fuzzy metric space and let A, B, S and T be self mappings of X satisfying the following conditions:*

$$AX \subset TX \text{ and } BX \subset SX, \tag{1}$$

$\exists k \in (0, 1)$ such that

$$M(Ax, By, kt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) * M(Ax, Ty, t),$$
$$\text{for every } x, y \in X \text{and} > 0, \tag{2}$$

the pairs AS and BT are weakly compatible. (3)

Then A, B, S and T have a unique common fixed point in X.

Proof Working on the same pattern as in [7], we can find a Cauchy sequence $\{y_n\}$ in X such that $y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n = 1, 2, \ldots$ from completeness, $y_n \to z$ for some $z \in X$, and so $\{Ax_{2n-2}\}$, $\{Sx_{2n}\}$, $\{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ also converges to z.
Similarly as in [22] we can show that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, hence $\exists z \in X$ such that $\{x_n\}$ converges to a point z. Hence $\exists u, v \in X$ such that $Su = z$ and $Tv = z$ respectively. By (2), we have

$$\begin{aligned} M(Au, y_{2n}, kt) =& M(Au, Bx_{2n-1}, kt) \\ \geq& M(Su, Tx_{2n-1}, t) * M(Au, Su, t) * M(Bx_{2n-1}, Tx_{2n-1}, t) * \\ & M(Au, Tx_{2n-1}, t). \end{aligned}$$

Taking the limit as $n \to \infty$,

$$M(Au, z, kt) \geq M(z, z, t) * M(Au, z, t) * M(z, z, t) * M(Au, z, t),$$
$$M(Au, zkt) \geq 1 * M(Au, z, t) * 1 * M(Au, z, t).$$

which gives $M(Au, z, kt) \geq M(Au, z, t)$.
Therefore by Lemma 2, we have $Au = z$. Since $Su = z$, thus $Au = z = Su$, i.e. u is a coincidence point of A and S.
Similar to (2), we have

$$\begin{aligned} & M(y_{2n-1}, Bv, kt) = M(Ax_{2n-2}, Bv, kt) \\ & \geq M(Sx_{2n-2}, Tv, t) * M(Ax_{2n-2}, Sx_{2n-2}, t) * M(Bv, Tv, t) * M(Ax_{2n-2}, Tv, t). \end{aligned}$$

Taking the limit $n \to \infty$, we have

$$M(z, Bv, kt) \geq M(z, z, t) * M(z, z, t) * M(Bv, z, t) * M(z, z, t).$$

This gives $M(z, Bv, kt) \geq M(Bv, z, t)$.
Therefore by Lemma 2, we have $Bv = z$. Since $Tv = z$, thus $Bv = Tv = z$, i.e. v is a coincidence point of B and T.
Since the pair $\{A, S\}$ is weakly compatible therefore A and S commute at their coincidence point i.e. $ASu = SAu$ or $Az = Sz$
Similarly the pair $\{B, T\}$ is weakly compatible therefore B and T commute at their coincidence point i.e. $BTv = TBv$ or $Bz = Tz$.
Now we prove that $Az = z$, by (2), we have

$$\begin{aligned} M(Az, Bx_{2n-1}, kt) \geq& M(Sz, Tx_{2n-1}, t) * M(Az, Sz, t) * M(Bx_{2n-1}, Tx_{2n-1}, t) * \\ & M(Az, Tx_{2n-1}, t). \end{aligned}$$

Taking the limit $n \to \infty$, we have

$$\begin{aligned} M(Az, z, kt) &\geq M(Sz, z, t) * M(Az, Sz, t) * M(z, z, t) * M(Az, z, t), \\ M(Az, z, kt) &\geq M(z, z, t) * M(Az, z, t) * M(z, z, t) * M(Az, z, t), \\ M(Az, z, kt) &\geq 1 * M(Az, z, t) * 1 * M(Az, z, t), \\ M(Az, z, kt) &\geq M(Az, z, t). \end{aligned}$$

Therefore by Lemma 2, we have $Az = z$. Since $Az = Sz$, thus $Az = Sz = z$.
Similar to (2), we have

$$M(Ax_{2n-2}, Bz, kt) \geq M(Sx_{2n-2}, Tz, t) * M(Ax_{2n-2}, Sx_{2n-2}, t) * M(Bz, Tz, t) * M(Ax_{2n-2}, Tz, t).$$

Taking the limit $n \to \infty$, we have

$$M(z, Bz, kt) \geq M(z, z, t) * M(z, z, t) * M(Bz, z, t) * M(z, z, t).$$

This gives

$$M(z, Bz, kt) \geq M(Bz, z, t).$$

Therefore by Lemma 2, we have $Bz = z$. Since $Tz = Bz$, thus $Bz = Tz = z$.
For uniqueness let $w(z \neq w)$ be another common fixed point of A, B, S and T. By (2), we have

$$M(z, w, kt) = M(Az, Bw, t) \geq M(Sz, Tw, t) * M(Az, Sz, t) * M(Bw, Tw, t) * M(Az, Tw, t) \geq M(z, w, t).$$

From Lemma 2, $z = w$.
This completes the proof of the theorem.

Corollary 1 Let $(X, M, *)$ be a complete ϵ-chainable fuzzy metric space and let A, B, S and T be self mappings of X satisfying (1) and (2) of Theorem 1 and there exists $k \in (0, 1)$ such that

$$M(Ax, By, kt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(Sx, By, 2t) * M(By, Ty, t) * M(Ax, Ty, t),$$

for all $x, y \in X$ and $t > 0$.
Then A, B, S and T have a unique common fixed point in X.

Corollary 2 Let $(X, M, *)$ be a complete ϵ-chainable fuzzy metric space and let A and B be self mappings of X satisfying the following condition ; there exists $k \in (0, 1)$ such that

$$M(Ax, By, kt) \geq M(x, y, t),$$

for all $x, y \in X$ and $t > 0$.
Then A and B have a unique common fixed point in X.

Theorem 2 *Let $(X, M, *)$ be a complete ϵ-chainable fuzzy metric space and let A, S and T be mappings from X into itself and*

$$A(X) \subset S(X) \cap T(X), \tag{4}$$

the pairs $\{A, T\}$ and $\{A, S\}$ are weakly compatible, (5)

there exists a number $q \in (0, 1)$ such that

$$M(Ax, Ay, qt) \geq min\{M(Ty, Ay, t), M(Sx, Ax, t), M(Sx, Ty, t), \frac{1}{2}[M(Sx, Ay, t) + M(Ty, Ax, t)]\} \tag{6}$$

for all $x, y \in X, t > 0$. Then S, T and A have a unique common fixed point.

Proof Define a sequence $\{x_n\}$ such that

$$Ax_{2n} = Sx_{2n-1} \text{ and } Ax_{2n-1} = Tx_{2n}. \qquad n = 1, 2, \ldots$$

In the similar manner as in Theorem 1, we can prove that $\{Ax_n\}$ is a Cauchy sequence. Since the space X is complete so there exists

$$z = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_{2n-1} \text{ and } = \lim_{n\to\infty} Ax_{2n-1} = \lim_{n\to\infty} Tx_{2n}.$$

Since $A(X) \subset S(X)$, there exists a point $u \in X$ such that $Su = z$. Then using (6) we write

$$M(Au, Ax_{2n+1}, qt) \geq min \left\{ \begin{array}{l} M(Tx_{2n+1}, Ax_{2n+1}, t), M(Su, Au, t), M(Su, Tx_{2n+1}, t), \\ 1/2[M(Su, Ax_{2n+1}, t) + M(Tx_{2n+1}, Au, t)] \end{array} \right\}.$$

Now taking the limit $n \to \infty$, we write

$$M(Au, z, qt) \geq \min\{1, M(z, Au, t), 1, \frac{1}{2}[1 + M(z, Au, t)]\}.$$

This gives

$$M(Au, z, t) \geq M(z, Au, t).$$

Therefore by Lemma 2, we have

$$Au = z.$$

Thus

$$Au = z = Su.$$

Similarly, since $A(X) \subset T(X)$, there exists a point $v \in X$ such that $Tv = z$. Again using (6) we write

$$M(Au, Av, qt) \geq \min\{M(Tv, Av, t), M(Su, Au, t), M(Su, Tv, t), \frac{1}{2}[M(Su, Av, t) + M(Tv, Au, t)]\},$$

$$M(z, Av, qt) \geq \min\{M(z, Av, t), 1, 1, \frac{1}{2}[M(z, Av, t) + 1]\}.$$

This gives

$$M(z, Av, qt) \geq M(z, Av, t).$$

Therefore by Lemma 2, we have

$$Av = z.$$

Thus

$$Av = z = Tv.$$

Since $Au = Su = z$ by definition of weak compatibility it follows that

$$ASu = SAu \quad \text{and so } Az = ASu = SAu = Sz.$$

Thus from (6), we have

$$M(Az, Ax_{2n+1}, qt) \geq min \left\{ \begin{array}{l} M\left(Tx_{2n+1}, Ax_{2n+1}, t\right), M\left(Sz, Az, t\right), M\left(Sz, Tx_{2n+1}, t\right), \\ 1/2[M\left(Sz, Ax_{2n+1}, t\right) + M\left(Tx_{2n+1}, Az, t\right)] \end{array} \right\}.$$

Now taking the limit $n \to \infty$, we write

$$M(Az, z, qt) \geq \min\{1, 1, M(Az, z, t), \frac{1}{2}[M(Az, z, t) + M(z, Az, t)]\}.$$

This gives

$$M(Az, z, qt) \geq M(Az, z, t)).$$

Therefore by Lemma 2, we have $Az = z$. Thus $Az = z = Sz$.

Similarly by definition of weak compatibility it follows that $ATv = TAv$ and so $Az = ATv = TAv = Tz$. Since we have already proved that $Az = z$. Thus $Az = Tz = z$. Therefore $Az = Tz = Sz = z$. Hence z is a common fixed point of A, S and T.

Now, we'll show that z is a unique common fixed point. Let $w (w \neq z)$ be another common fixed point of S, T and A. By (6) we write

$$\begin{aligned} M(Az, Aw, qt) &\geq \min\{M(Tw, Aw, t), M(Sz, Az, t), M(Sz, Tw, t), \frac{1}{2}[M(Sz, Aw, t) \\ &\quad + M(Tw, Az, t)]\}, \\ M(z, w, qt) &\geq \min\{1, 1, M(z, w, t), M(z, w, t)\} \end{aligned}$$

which implies that

$$M(z, w, qt) \geq M(z, w, t).$$

Therefore by Lemma 2, we write $z = w$. This completes the proof of Theorem 2.

4 Conclusion

In this paper, the condition of continuity of any mapping had been waived off to prove some common fixed point theorems in ϵ-chainable fuzzy metric space in presence of weak compatible mappings. The proven results in ϵ-chainable fuzzy metric space for weakly compatible mappings without taking any mappings continuous shows that for existence of fixed point, continuity of any mappings is not needed. We believe that dropping of continuity with weak compatibility being a weaker condition to prove some common fixed point theorems makes our result more interesting and useful for other researcher involved in the said field.

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References

[1] Zadeh, L. A. Fuzzy Sets. *Inform Contr.* 1965. 8: 338-353.

[2] Chang, C. L. Fuzzy topological space. *J. Math. Anal. Appl.* 1968. 24: 182-190.

[3] Kramosil, I. and Michalek, J. Fuzzy metric and statistical metric spaces. *Kybernetika.* 1975. 11: 336-344.

[4] Erceg, M. A. Metric spaces in fuzzy set theory. *J. Math. Anal. Appl.* 1979. 69: 205-230.

[5] Fisher, B. Mappings with a common fixed point, *Math. Seminar Notes Kobe Univ.* 1979 7, 81-84.

[6] Deng, Z. K. Fuzzy pseudo metric spaces. *J. Math. Anal. Appl.* 1982. 86: 74-95.

[7] Sessa, S. On a weak commutativity condition of mappings in a fixed point considerations. *Publ. Inst. Math.* 1982. 32(46): 149-153.

[8] Kaleva, O. and Seikkala, S. On fuzzy metric spaces. *Fuzzy Sets and Systems.* 1984. 12: 215-229.

[9] Mishra, S. N., Sharma, N. and Singh, S. L. Common fixed points of maps in fuzzy metric spaces. *Internat. J. Math. Math. Sci.* 1994. 17: 253-258.

[10] Jungck, G. Compatible mappings and common fixed points. *Internat. J. Math. Math. Sci.* 1986. 9: 771-779.

[11] Jungck, G. Compatible mappings and common fixed points (2). *Internat. J. Math. and Math. Sci.* 1988. 11: 285-288.

[12] Grabiec, M. Fixed point in fuzzy metric spaces. *Fuzzy Sets and Systems.* 1988. 27: 385-389.

[13] George, A. and Veeramani, P. On some results in fuzzy metric spaces. *Fuzzy Sets and Systems.* 1994. 64: 395-399.

[14] Jachymski, J. Common fixed point theorems for some families of maps. *Ind. J. Pure Appl. Math.* 1994. 25: 925-937.

[15] Cho, Y. J. Fixed points in fuzzy metric spaces. *J. Fuzzy Math.* 1997. 5(4): 949-962.

[16] George, A. and Veeramani, P. On some results of analysis for fuzzy metric spaces. *Fuzzy Sets and Systems.* 1997. 90: 365-368.

[17] Chang, S. S., Cho, Y.J., Lee, B.S., Jung, J. S. and Kang, S.M. Coincidence point and minimization theorems in fuzzy metric spaces. *Fuzzy Sets and Systems.* 997. 88(1): 119-128.

[18] Jungck, G. and Rhoades, B. E. Fixed point for set valued functions without continuity. *Ind. J. Pure Appl. Maths.* 1998. 29(3): 227-238.

[19] Pant, R. P. R-weak commutativity and common fixed points of non compatible maps. *Ganita.* 1998. 49: 19-27.

[20] Sharma, S. On fuzzy metric space. *Southeast Asian Bull. Math.* 2002. 6(1): 145-157.

[21] Song, G. A common fixed point theorem in a fuzzy metric space space, *Fuzzy Set and Systems.* 2003. 135: 409-413.

[22] Cho, S. H. and Jung, J. H. On common fixed point theorems in fuzzy metric spaces. *Int. Math. Forum.* 2006. 1(29): 1441-1451.

[23] Sharma , S. and Deshpande, B. Common fixed point theorems for finite Number of mappings without continuity and compatibility on intuitionistic fuzzy metric spaces, *Chaos, Solitons & Fractals.* 2009. 40: 2242-2256.

[24] Sharma , S. and Deshpande, B. Common fixed point theorems for finite number of mappings without continuity and compatibility on fuzzy metric spaces, *Fuzzy Systems and Math.* 2010. 24 (2):73-83.