Positive Integer Solutions of Some Pell Equations

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Abstract. Let $d$ be a positive integer which is not a perfect square. In this paper, by using continued fraction expansion of $\sqrt{d}$, we find fundamental solution of some Pell equation. Moreover, we get all positive integer solutions of some Pell equations in terms of generalized Fibonacci and Lucas sequences.

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1 Introduction

Let $d$ be a positive integer that is not a perfect square. It is well known that the Pell equation $x^2 - dy^2 = 1$ has always positive integer solutions. But, whether or not there exists a positive integer solution to the equation $x^2 - dy^2 = -1$ depends on the period length of the continued fraction expansion of $\sqrt{d}$. In this paper, if a solution exists, we will use continued fraction expansion of $\sqrt{d}$ in order to get all positive integer solutions of the equations $x^2 - dy^2 = \pm 1$ when $d \in \{k^2 \pm 2, k^2 \pm k\}$ for any natural number $k$. Moreover, we will find all positive integer solutions of the equations $x^2 - dy^2 = \pm 4$ in terms of generalized Fibonacci and Lucas sequences.

Now we briefly mention the generalized Fibonacci and Lucas sequences $(U_n(k, s))$ and $(V_n(k, s))$. Let $k$ be a natural number and $s$ be nonzero integer with $k^2 + 4s > 0$. Generalized Fibonacci sequence is defined by

$$U_0(k, s) = 0, U_1(k, s) = 1 \quad \text{and} \quad U_{n+1}(k, s) = kU_n(k, s) + sU_{n-1}(k, s)$$

for $n \geq 1$ and generalized Lucas sequence is defined by

$$V_0(k, s) = 2, V_1(k, s) = k \quad \text{and} \quad V_{n+1}(k, s) = kV_n(k, s) + sV_{n-1}(k, s)$$

for $n \geq 1$, respectively. It is well known that

$$U_n(k, s) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n(k, s) = \alpha^n + \beta^n$$

where $\alpha = (k + \sqrt{k^2 + 4s})/2$ and $\beta = (k - \sqrt{k^2 + 4s})/2$. The above identities are known as Binet’s formulae. Clearly, $\alpha + \beta = k$, $\alpha - \beta = \sqrt{k^2 + 4s}$, and $\alpha\beta = -s$.

Especially, if $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$, then we get

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad Q_n = \alpha^n + \beta^n.$$  

$P_n$ and $Q_n$ are called Pell and Pell-Lucas sequences, respectively. For more information about generalized Fibonacci and Lucas sequences, one can consult [1–7].
2 Preliminaries

Let $d$ be a positive integer which is not a perfect square and $N$ be any nonzero fixed integer. Then the equation $x^2 - dy^2 = N$ is known as Pell equation. For $N = \pm 1$, the equations $x^2 - dy^2 = 1$ and $x^2 - dy^2 = -1$ are known as classical Pell equation. If $a^2 - db^2 = N$, we say that $(a, b)$ is a solution to the Pell equation $x^2 - dy^2 = N$. We use the notations $(a, b)$ and $a + b\sqrt{d}$ interchangeably to denote solutions of the equation $x^2 - dy^2 = N$. Also, if $a$ and $b$ are both positive, we say that $a + b\sqrt{d}$ is positive solution to the equation $x^2 - dy^2 = N$.

Lemma 1 Let $d$ be a positive integer that is not a perfect square. Then, there is a continued fraction expansion of $\sqrt{d}$ such that $\sqrt{d} = [a_0, a_1, a_2, ..., a_{l-1}, 2a_0]$ where $l$ is the period length and the $a_j$’s are given by the recursion formulas:

$$a_0 = \sqrt{d}, \ a_k = \lfloor a_k \rfloor \text{ and } a_{k+1} = \frac{1}{a_k - a_k}, \ k = 0, 1, 2, 3, ...$$

Recall that $a_l = 2a_0$ and $a_{l+k} = a_k$ for $k \geq 1$. The $n^{th}$ convergent of $\sqrt{d}$ for $n \geq 0$ is given by

$$\frac{p_n}{q_n} = [a_0, a_1, ..., a_n] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}}.$$

By means of the $k^{th}$ convergent of $\sqrt{d}$, we can give the fundamental solution to the equations $x^2 - dy^2 = 1$ and $x^2 - dy^2 = -1$.

Now we give the fundamental solution to the equations $x^2 - dy^2 = \pm 1$ by means of the period length of the continued fraction expansion of $\sqrt{d}$ (See [8]).

**Lemma 1** Let $l$ be the period length of continued fraction expansion of $\sqrt{d}$. If $l$ is even, then the fundamental solution to the equation $x^2 - dy^2 = 1$ is given by

$$x_1 + y_1 \sqrt{d} = p_{l-1} + q_{l-1} \sqrt{d}$$

and the equation $x^2 - dy^2 = -1$ has no integer solutions. If $l$ is odd, then the fundamental solution to the equation $x^2 - dy^2 = 1$ is given by

$$x_1 + y_1 \sqrt{d} = p_{2l-1} + q_{2l-1} \sqrt{d}$$

and the fundamental solution to the equation $x^2 - dy^2 = -1$ is given by

$$x_1 + y_1 \sqrt{d} = p_{l-1} + q_{l-1} \sqrt{d}.$$
Theorem 1 Let \( d \equiv 1, 2, 3 (\mod 4) \). Then the equation \( x^2 - dy^2 = -4 \) has a positive integer solution if and only if the equation \( x^2 - dy^2 = -1 \) has a positive integer solution.

Theorem 2 Let \( d \equiv 2 (\mod 4) \) or \( d \equiv 3 (\mod 4) \). If fundamental solution to the equations \( x^2 - dy^2 = \pm 1 \) is \( (x_1, y_1) \), then fundamental solution to the equation \( x^2 - dy^2 = \pm 4 \) is \( (2x_1, 2y_1) \).

Theorem 3 Let \( d \equiv 0 (\mod 4) \). If fundamental solution to the equation \( x^2 - (d/4)y^2 = 1 \) is \( x_1 + y_1\sqrt{d/4} \), then fundamental solution to the equation \( x^2 - dy^2 = 4 \) is \( (2x_1, y_1) \).

If we know fundamental solution to the equations \( x^2 - dy^2 = \pm 1 \) and \( x^2 - dy^2 = \pm 4 \), then we can give all positive integer solutions to these equations. For more information about Pell equation, one can consult [11–17].

Theorem 4 Let \( x_1 + y_1\sqrt{d} \) be the fundamental solution to the equation \( x^2 - dy^2 = 1 \). Then all positive integer solutions to the equation \( x^2 - dy^2 = 1 \) are given by

\[
x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n
\]

with \( n \geq 1 \).

Theorem 5 Let \( x_1 + y_1\sqrt{d} \) be the fundamental solution to the equation \( x^2 - dy^2 = -1 \). Then all positive integer solutions to the equation \( x^2 - dy^2 = -1 \) are given by

\[
x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^{2n-1}
\]

with \( n \geq 1 \).

Theorem 6 Let \( x_1 + y_1\sqrt{d} \) be the fundamental solution to the equation \( x^2 - dy^2 = 4 \). Then all positive integer solutions to the equation \( x^2 - dy^2 = 4 \) are given by

\[
x_n + y_n\sqrt{d} = \frac{(x_1 + y_1\sqrt{d})^n}{2^{n-1}}
\]

with \( n \geq 1 \).

Theorem 7 Let \( x_1 + y_1\sqrt{d} \) be the fundamental solution to the equation \( x^2 - dy^2 = -4 \). Then all positive integer solutions to the equation \( x^2 - dy^2 = -4 \) are given by

\[
x_n + y_n\sqrt{d} = \frac{(x_1 + y_1\sqrt{d})^{2n-1}}{4^{n-1}}
\]

with \( n \geq 1 \).
From now on, we will assume that \( k \) is a natural number. We give continued fraction expansion of \( \sqrt{d} \) for \( d = k^2 \pm 2 \) and \( d = k^2 \pm k \). The proofs of the following four theorems are easy and they can be found many text books on number theory as an exercise (see, for example [8]).

**Theorem 8** Let \( k > 0 \). Then
\[
\sqrt{k^2 + 2} = \left[ k, k, 2k \right].
\]

**Theorem 9** Let \( k > 2 \). Then
\[
\sqrt{k^2 - 2} = \left[ k - 1, 1, k - 2, 1, 2(k - 1) \right].
\]

**Theorem 10** Let \( k > 1 \). Then
\[
\sqrt{k^2 + k} = \left[ k, 2, 2k \right].
\]

**Theorem 11** Let \( k > 2 \). Then
\[
\sqrt{k^2 - k} = \left[ k - 1, 2, 2(k - 1) \right].
\]

**Corollary 1** Let \( k > 0 \) and \( d = k^2 + 2 \). Then the fundamental solution to the equation \( x^2 - dy^2 = 1 \) is
\[
x_1 + y_1 \sqrt{d} = k^2 + 1 + k \sqrt{d}
\]
and the equation \( x^2 - dy^2 = -1 \) has no positive integer solutions.

**Proof** The period length of the continued fraction expansion of \( \sqrt{k^2 + 2} \) is 2 by Theorem 8. Therefore the fundamental solution to the equation \( x^2 - dy^2 = 1 \) is \( p_1 + q_1 \sqrt{d} \) by Lemma 1. Since
\[
\frac{p_1}{q_1} = k + \frac{1}{k} = \frac{k^2 + 1}{k},
\]
the proof follows. Since the period of length of the continued fraction expansion of \( \sqrt{k^2 + 2} \) is even, the equation \( x^2 - dy^2 = -1 \) has no positive integer solutions by Lemma 1.

**Corollary 2** Let \( k > 2 \) and \( d = k^2 - 2 \). Then the fundamental solution to the equation \( x^2 - dy^2 = 1 \) is
\[
x_1 + y_1 \sqrt{d} = k^2 - 1 + k \sqrt{d}
\]
and the equation \( x^2 - dy^2 = -1 \) has no positive integer solutions.
Proof The period length of the continued fraction expansion of $\sqrt{k^2 - 2}$ is 4 by Theorem 9. Therefore the fundamental solution to the equation $x^2 - dy^2 = 1$ is $p_3 + q_3 \sqrt{d}$ by Lemma 1. Since
\[
\frac{p_3}{q_3} = k - 1 + \frac{1}{1 + \frac{1}{k - 2 + \frac{1}{k}}},
\]
the proof follows. Since the period of length of the continued fraction expansion of $\sqrt{k^2 - 2}$ is even, the equation $x^2 - dy^2 = -1$ has no positive integer solutions by Lemma 1.

Corollary 3 Let $k > 1$ and $d = k^2 + k$. Then the fundamental solution to the equation $x^2 - dy^2 = 1$ is
\[
x_1 + y_1 \sqrt{d} = 2k + 1 + 2\sqrt{d}
\]
and the equation $x^2 - dy^2 = -1$ has no positive integer solutions.

Proof The period length of the continued fraction expansion of $\sqrt{k^2 + k}$ is 2 by Theorem 10. Therefore the fundamental solution to the equation $x^2 - dy^2 = 1$ is $p_1 + q_1 \sqrt{d}$ by Lemma 1. Since
\[
\frac{p_1}{q_1} = k + \frac{1}{2} = \frac{2k + 1}{2},
\]
the proof follows. Since the period of length of the continued fraction expansion of $\sqrt{k^2 + k}$ is even, the equation $x^2 - dy^2 = -1$ has no positive integer solutions by Lemma 1.

Corollary 4 Let $k > 2$ and $d = k^2 - k$. Then the fundamental solution to the equation $x^2 - dy^2 = 1$ is
\[
x_1 + y_1 \sqrt{d} = 2k - 1 + 2\sqrt{d}
\]
and the equation $x^2 - dy^2 = -1$ has no positive integer solutions.

Proof The period length of the continued fraction expansion of $\sqrt{k^2 - k}$ is 2 by Theorem 11. Therefore the fundamental solution to the equation $x^2 - dy^2 = 1$ is $p_1 + q_1 \sqrt{d}$ by Lemma 1. Since
\[
\frac{p_1}{q_1} = k - 1 + \frac{1}{2} = \frac{2k - 1}{2},
\]
the proof follows. Since the period of length of the continued fraction expansion of $\sqrt{k^2 - k}$ is even, the equation $x^2 - dy^2 = -1$ has no positive integer solutions by Lemma 1.

3 Main Theorems

In this section, we give all positive integer solutions to some Pell equations.

Theorem 12 Let $k > 0$ and $d = k^2 + 2$. Then all positive integer solutions of the equation $x^2 - dy^2 = 1$ are given by
\[
(x, y) = \left(\frac{V_n(2k^2 + 2, -1)}{2}, kU_n(2k^2 + 2, -1)\right)
\]
with $n \geq 1$. 
Proof By Corollary 1 and Theorem 4, all positive integer solutions of the equation \(x^2 - dy^2 = 1\) are given by
\[x_n + y_n \sqrt{d} = \left(k^2 + 1 + k \sqrt{d}\right)^n\]
with \(n \geq 1\). Let \(\alpha = k^2 + 1 + k \sqrt{d}\) and \(\beta = k^2 + 1 - k \sqrt{d}\). Then \(\alpha + \beta = 2k^2 + 2\), \(\alpha - \beta = 2k \sqrt{d}\) and \(\alpha \beta = 1\). Thus,
\[x_n + y_n \sqrt{d} = \alpha^n\text{ and } x_n - y_n \sqrt{d} = \beta^n.\]
Then it follows that
\[x_n = \frac{\alpha^n + \beta^n}{2} = \frac{V_n(2k^2 + 2, -1)}{2}\]
and
\[y_n = \frac{\alpha^n - \beta^n}{2 \sqrt{d}} = k \frac{\alpha^n - \beta^n}{2k \sqrt{d}} = k \frac{\alpha^n - \beta^n}{\alpha - \beta} = k U_n(2k^2 + 2, -1)\]
by (1).

Since the proof of the following theorem is similar, we omit it.

**Theorem 13** Let \(k > 2\) and \(d = k^2 - 2\). Then all positive integer solutions of the equation \(x^2 - dy^2 = 1\) are given by
\[(x, y) = \left(\frac{V_n(2k^2 - 2, -1)}{2}, k U_n(2k^2 - 2, -1)\right)\]
with \(n \geq 1\).

**Theorem 14** Let \(k > 1\) and \(d = k^2 + k\). Then all positive integer solutions of the equation \(x^2 - dy^2 = 1\) are given by
\[(x, y) = \left(\frac{V_n(4k + 2, -1)}{2}, 2 U_n(4k + 2, -1)\right)\]
with \(n \geq 1\).

Proof By Corollary 3 and Theorem 4, all positive integer solutions of the equation \(x^2 - dy^2 = 1\) are given by
\[x_n + y_n \sqrt{d} = \left(2k^2 + 1 + 2 \sqrt{d}\right)^n\]
with \(n \geq 1\). Let \(\alpha = 2k^2 + 1 + 2 \sqrt{d}\) and \(\beta = 2k^2 + 1 - 2 \sqrt{d}\). Then \(\alpha + \beta = 4k + 2\), \(\alpha - \beta = 4 \sqrt{d}\) and \(\alpha \beta = 1\). Thus, we have
\[x_n + y_n \sqrt{d} = \alpha^n\text{ and } x_n - y_n \sqrt{d} = \beta^n.\]
Then it follows that
\[x_n = \frac{\alpha^n + \beta^n}{2} = \frac{V_n(4k + 2, -1)}{2}\]
and
\[y_n = \frac{\alpha^n - \beta^n}{2 \sqrt{d}} = 2 \frac{\alpha^n - \beta^n}{4 \sqrt{d}} = 2 \frac{\alpha^n - \beta^n}{\alpha - \beta} = 2 U_n(4k + 2, -1)\]
by (1).

Since the proof of the following theorem is similar, we omit it.
**Theorem 15** Let \( k > 2 \) and \( d = k^2 - k \). Then all positive integer solutions of the equation \( x^2 - dy^2 = 1 \) are given by

\[
(x, y) = \left( \frac{V_n(4k^2 - 2)}{2}, 2U_n(4k^2 - 2) \right)
\]

with \( n \geq 1 \).

Now we give all positive integer solutions of the equations \( x^2 - (k^2 \pm 2)y^2 = \pm 4 \) and \( x^2 - (k^2 \pm k)y^2 = \pm 4 \).

**Theorem 16** Let \( k > 0 \). Then all positive integer solutions of the equation \( x^2 - (k^2+2)y^2 = 4 \) are given by

\[
(x, y) = (V_n(2k^2 + 2), 2kU_n(2k^2 + 2))
\]

with \( n \geq 1 \).

**Proof** If \( k \) is odd, then \( d = k^2 + 2 \equiv 3 \pmod{4} \) and if \( k \) is even, then \( d = k^2 + 2 \equiv 2 \pmod{4} \). Thus, by Corollary 1 and Theorem 2, it follows that \( 2k^2 + 2 + 2k\sqrt{k^2 + 2} \) is the fundamental solution to the equation \( x^2 - (k^2 + 2)y^2 = 4 \). Therefore, by Theorem 6, all positive integer solutions of the equation \( x^2 - dy^2 = 4 \) are given by

\[
x_n + y_n\sqrt{d} = \frac{(2k^2 + 2 + 2k\sqrt{k^2 + 2})^n}{2^{n-1}} = 2 \left( \frac{2k^2 + 2 + 2k\sqrt{k^2 + 2}}{2} \right)^n.
\]

Let \( \alpha = \frac{2k^2 + 2 + 2k\sqrt{k^2 + 2}}{2} \) and \( \beta = \frac{2k^2 + 2 - 2k\sqrt{k^2 + 2}}{2} \). Then \( \alpha + \beta = 2k^2 + 2, \alpha - \beta = 2k\sqrt{d} \) and \( \alpha\beta = 1 \). Thus it is seen that

\[x_n + y_n\sqrt{d} = 2\alpha^n \quad \text{and} \quad x_n - y_n\sqrt{d} = 2\beta^n.\]

Therefore we get

\[x_n = \alpha^n + \beta^n = V_n(2k^2 + 2, -1)\]

and

\[y_n = \frac{\alpha^n - \beta^n}{\sqrt{d}} = \frac{2k\alpha^n - \beta^n}{2k\sqrt{d}} = \frac{2k\alpha^{2n} - \beta^{2n}}{\alpha - \beta} = 2kU_n(2k^2 + 2, -1)\]

by (1). Then the proof follows.

**Theorem 17** Let \( k > 2 \). Then all positive integer solutions of the equation \( x^2 - (k^2 - 2)y^2 = 4 \) are given by

\[
(x, y) = (V_n(2k^2 - 2), 2kU_n(2k^2 - 2))
\]

with \( n \geq 1 \).
Proof If \( k \) is odd, then \( d = k^2 - 2 \equiv 3(\text{mod}4) \) and if \( k \) is even, then \( d = k^2 - 2 \equiv 2(\text{mod}4) \). Thus, by Corollary 2 and Theorem 2, it follows that \( 2k^2 - 2 + 2k\sqrt{k^2 - 2} \) is the fundamental solution to the equation \( x^2 - (k^2 - 2)y^2 = 4 \). Therefore, by Theorem 6, all positive integer solutions of the equation \( x^2 - dy^2 = 4 \) are given by
\[
x_n + y_n\sqrt{d} = \left(\frac{2k^2 - 2 + 2k\sqrt{k^2 - 2}}{2}\right)^n.
\]

Let \( \alpha = \frac{2k^2 - 2 + 2k\sqrt{k^2 - 2}}{2} \) and \( \beta = \frac{2k^2 - 2 - 2k\sqrt{k^2 - 2}}{2} \). Then \( \alpha + \beta = 2k^2 - 2, \alpha - \beta = 2k\sqrt{d} \) and \( \alpha\beta = 1 \). Thus it is seen that \( x_n + y_n\sqrt{d} = 2\alpha^n \) and \( x_n - y_n\sqrt{d} = 2\beta^n \).

Therefore we get
\[
x_n = \alpha^n + \beta^n = V_n(2k^2 - 2, -1)
\]
and
\[
y_n = \frac{\alpha^n - \beta^n}{\sqrt{d}} = 2k\frac{\alpha^n - \beta^n}{2k\sqrt{d}} = \frac{2k\alpha^{2n} - \beta^{2n}}{\alpha - \beta} = 2kU_n(2k^2 - 2, -1)
\]
by (1). Then the proof follows.

**Theorem 18** Let \( k > 1 \) and \( k \neq 4 \). Then all positive integer solutions of the equation \( x^2 - (k^2 - k)y^2 = 4 \) are given by
\[
(x, y) = (V_n(4k - 2, -1), 4U_n(4k - 2, -1))
\]
with \( n \geq 1 \).

**Proof** When \( k \neq 4, b = 1, 2, 3 \) and \( a^2 - (k^2 - k)b^2 = 4 \), it can be seen that \( k = -3, 0 \) or 1. Since \( (4k - 2)^2 - (k^2 - k)4^2 = 4 \), it follows that the fundamental solution to the equation \( x^2 - (k^2 - k)y^2 = 4 \) is \( 4k - 2 + 4\sqrt{k^2 - k} \). Thus by Theorem 6, all positive integer solutions of the equation \( x^2 - dy^2 = 4 \) are given by
\[
x_n + y_n\sqrt{d} = \left(\frac{4k - 2 + 4\sqrt{k^2 - k}}{2}\right)^n.
\]

Let \( \alpha = \frac{4k - 2 + 4\sqrt{k^2 - k}}{2} \) and \( \beta = \frac{4k - 2 - 4\sqrt{k^2 - k}}{2} \). Then \( \alpha + \beta = 4k - 2, \alpha - \beta = 4\sqrt{d} \) and \( \alpha\beta = 1 \). Thus it follows that \( x_n + y_n\sqrt{d} = 2\alpha^n \) and \( x_n - y_n\sqrt{d} = 2\beta^n \). Then, we get
\[
x_n = \alpha^n + \beta^n = V_n(4k - 2, -1)
\]
and
\[
y_n = \frac{\alpha^n - \beta^n}{\sqrt{d}} = 4\frac{\alpha^n - \beta^n}{4\sqrt{d}} = \frac{4\alpha^n - \beta^n}{\alpha - \beta} = 4U_n(4k - 2, -1)
\]
by (1).

**Theorem 19** Let \( k > 1 \) and \( k \neq 3 \). Then all positive integer solutions of the equation \( x^2 - (k^2 + k)y^2 = 4 \) are given by
\[
(x, y) = (V_n(4k + 2, -1), 4U_n(4k + 2, -1))
\]
with \( n \geq 1 \).
Proof When \( k \neq 3 \), \( b = 1, 2, 3 \) and \( a^2 - (k^2 + k)b^2 = 4 \), it can be seen that \( k = -1 \) or 0. Since \( (4k + 2)^2 - (k^2 + k)4^2 = 4 \), it follows that the fundamental solution to the equation \( x^2 - (k^2 + k)y^2 = 4 \) is \( 4k + 2 + 4\sqrt{k^2 + k} \). Thus, by Theorem 6, all positive integer solutions of the equation \( x^2 - dy^2 = 4 \) are given by

\[
x_n + yn\sqrt{d} = \left(\frac{4k + 2 + 4\sqrt{k^2 + k}}{2}\right)^n.
\]

Let \( \alpha = \frac{4k + 2 + 4\sqrt{k^2 + k}}{2} \) and \( \beta = \frac{4k + 2 - 4\sqrt{k^2 + k}}{2} \). Then \( \alpha + \beta = 4k + 2, \alpha - \beta = 4\sqrt{d} \) and \( \alpha\beta = 1 \). Thus it follows that \( x_n + y_n\sqrt{d} = 2\alpha^n \) and \( x_n - y_n\sqrt{d} = 2\beta^n \). Therefore we get

\[
x_n = \alpha^n + \beta^n = V_n(4k + 2, -1)
\]

and

\[
y_n = \frac{\alpha^n - \beta^n}{\sqrt{d}} = \frac{4\alpha^n - \beta^n}{4\sqrt{d}} = 4\frac{\alpha^n - \beta^n}{\alpha - \beta} = 4U_n(4k + 2, -1)
\]

by (1).

Theorem 20 All positive integer solutions of the equation \( x^2 - 12y^2 = 4 \) are given by

\[
(x, y) = (V_n(4, -1), U_n(4, -1))
\]

with \( n \geq 1 \).

Proof The fundamental solution to the equation \( x^2 - 12y^2 = 4 \) is \( 4 + \sqrt{12} \) since \( 4^2 - 12 \cdot 1^2 = 4 \). Thus, by Theorem 6, all positive integer solutions of the equation \( x^2 - 12y^2 = 4 \) are given by

\[
x_n + y_n\sqrt{12} = \left(\frac{4 + \sqrt{12}}{2}\right)^n = 2\left(\frac{4 + 2\sqrt{3}}{2}\right)^n = 2(2 + \sqrt{3})^n.
\]

Let \( \alpha = 2 + \sqrt{3} \) and \( \beta = 2 - \sqrt{3} \). Then \( \alpha + \beta = 4, \alpha - \beta = 2\sqrt{3} \) and \( \alpha\beta = 1 \). Thus it follows that \( x_n + y_n\sqrt{12} = 2\alpha^n \) and \( x_n - y_n\sqrt{12} = 2\beta^n \). Therefore we get

\[
x_n = \alpha^n + \beta^n = V_n(4, -1)
\]

and

\[
y_n = \frac{\alpha^n - \beta^n}{\sqrt{12}} = \frac{\alpha^n - \beta^n}{2\sqrt{3}} = \frac{\alpha^n - \beta^n}{\alpha - \beta} = U_n(4, -1)
\]

by (1).

Theorem 21 Let \( k > 0 \). Then the equation \( x^2 - (k^2 + 2)y^2 = -4 \) has no positive integer solutions.
Proof If \( k \) is odd, then \( k^2 + 2 \equiv 3 (\text{mod}4) \) and if \( k \) is even, then \( k^2 + 2 \equiv 2 (\text{mod}4) \). Thus by Corollary 1 and Theorem 1, the proof follows.

Since the proof of the following theorem is similar, we omit it.

**Theorem 22** Let \( k > 2 \). Then the equation \( x^2 - (k^2 - 2)y^2 = -4 \) has no positive integer solutions.

**Theorem 23** Let \( k > 1 \). Then the equation \( x^2 - (k^2 + k)y^2 = -4 \) has no positive integer solutions.

**Proof** Let \( d = k^2 + k \) and \( a^2 - db^2 = -4 \) for some positive integers \( a \) and \( b \). Assume that \( k = 4t + 1 \) or \( k = 4t + 2 \). Then \( d \equiv 2 (\text{mod}4) \). Thus, by Theorem 1 and Corollary 3, the equation \( x^2 - dy^2 = -4 \) has no positive integer solutions. Assume that \( k = 4t \). Then \( d = 16t^2 + 4t \) and \( d \equiv 0 (\text{mod}4) \). Thus \( a \) is even and \( a^2 - (16t^2 + 4t)b^2 = -4 \). This implies that

\[
(a/2)^2 - (4t^2 + t)b^2 = -1. \tag{3}
\]

It can be easily seen that \( \sqrt{4t^2 + t} = [2t, 1, 4t] \). Since the period of length of the continued fraction expansion of \( \sqrt{4t^2 + t} \) is even, the equation \( x^2 - (4t^2 + t)y^2 = -1 \) has no positive integer solutions by Lemma 1. This contradicts with (3). Therefore the equation \( x^2 - dy^2 = -4 \) has no positive integer solutions. Assume that \( k = 4t + 3 \). Then \( d = 16t^2 + 28t + 12 \) and \( d \equiv 0 (\text{mod}4) \). Thus \( a \) is even and \( a^2 - (16t^2 + 28t + 12)b^2 = -4 \). This implies that

\[
(a/2)^2 - (4t^2 + 7t + 3)b^2 = -1. \tag{4}
\]

It can be easily seen that \( \sqrt{4t^2 + 7t + 3} = [2t + 1, 1, 2, 1, 2(2t + 1)] \). Since the period of length of the continued fraction expansion of \( \sqrt{4t^2 + 7t + 3} \) is even, the equation \( x^2 - (4t^2 + 7t + 3)y^2 = -1 \) has no positive integer solutions by Lemma 1. This contradicts with (4). Therefore the equation \( x^2 - dy^2 = -4 \) has no positive integer solutions.

Since the proofs of the following theorem is similar, we omit it.

**Theorem 24** Let \( k > 2 \). Then the equation \( x^2 - (k^2 - k)y^2 = -4 \) has no positive integer solutions.

Continued fraction expansion of \( \sqrt{2} \) is \([1, \overline{2}]\). Thus the period length of the continued fraction of \( \sqrt{2} \) is 1. Moreover, the fundamental solution to the equation \( x^2 - 2y^2 = 1 \) is \( 3 + 2\sqrt{2} \) and the fundamental solution to the equation \( x^2 - 2y^2 = -1 \) is \( 1 + \sqrt{2} \) by Lemma 1. Therefore, by using (2), we can give the following corollaries easily.

**Corollary 5** All positive integer solutions of the equation \( x^2 - 2y^2 = 1 \) are given by

\[
(x, y) = \left( \frac{Q_{2n}}{2}, P_{2n} \right)
\]

with \( n \geq 1 \).
Corollary 6 All positive integer solutions of the equation $x^2 - 2y^2 = -1$ are given by

$$(x, y) = \left(\frac{Q_{2n-1}}{2}, P_{2n-1}\right)$$

with $n \geq 1$.

It can be seen that fundamental solutions to the equations $x^2 - 2y^2 = 4$ and $x^2 - 2y^2 = -4$ are $6 + 4\sqrt{2}$ and $2 + 2\sqrt{2}$, respectively. Thus by Theorem 2 and identity (2), we can give following corollaries easily.

Corollary 7 All positive integer solutions of the equation $x^2 - 2y^2 = 4$ are given by

$$(x, y) = (Q_{2n}, 2P_{2n})$$

with $n \geq 1$.

Corollary 8 All positive integer solutions of the equation $x^2 - 2y^2 = -4$ are given by

$$(x, y) = (Q_{2n-1}, 2P_{2n-1})$$

with $n \geq 1$.

4 Conclusion

In this paper, by using continued fraction expansion of $\sqrt{d}$, we find fundamental solution of the $x^2 - dy^2 = \pm 1$, where $k$ is a natural number and $d = k^2 \pm 2$, or $k^2 \pm k$. Moreover, we investigate Pell equations of the form $x^2 - dy^2 = N$ when $N = \pm 1, \pm 4$ and we are looking for positive integer solutions in $x$ and $y$. We get all positive integer solutions of the Pell equations $x^2 - dy^2 = N$ in terms of generalized Fibonacci and Lucas sequences when $N = \pm 1, \pm 4$ and $d = k^2 \pm 2, k^2 \pm k$. Finally, all positive integer solutions of the equations $x^2 - 2y^2 = \pm 1$ and $x^2 - 2y^2 = \pm 4$ are given in terms of Pell and Pell-Lucas sequences.

References


[16] Duman, M. G. and Keskin, R. *Positive Integer Solutions of the Pell Equation* $x^2 - dy^2 = N, d \in \{k^2 \pm 4, k^2 \pm 1\}$ and $N \in \{\pm 1, \pm 4\}$. (submitted).