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Positive Integer Solutions of Some Pell Equations

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Abstract Let d be a positive integer which is not a perfect square. In this paper, by using continued fraction expansion of $\sqrt{d}$, we find fundamental solution of some Pell equation. Moreover, we get all positive integer solutions of some Pell equations in terms of generalized Fibonacci and Lucas sequences.

Keywords Pell Equations, Continued Fraction, Generalized Fibonacci and Lucas numbers

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1 Introduction

Let d be a positive integer that is not a perfect square. It is well known that the Pell equation $x^2 - dy^2 = 1$ has always positive integer solutions. But, whether or not there exists a positive integer solution to the equation $x^2 - dy^2 = -1$ depends on the period length of the continued fraction expansion of $\sqrt{d}$. In this paper, if a solution exists, we will use continued fraction expansion of $\sqrt{d}$ in order to get all positive integer solutions of the equations $x^2 - dy^2 = \pm 1$ when $d \in \left\{k^2 \pm 2,\ k^2 \pm k\right\}$ for any natural number k. Moreover, we will find all positive integer solutions of the equations $x^2 - dy^2 = \pm 4$ in terms of generalized Fibonacci and Lucas sequences.

Now we briefly mention the generalized Fibonacci and Lucas sequences $(U_n(k,s))$ and $(V_n(k,s))$. Let k be a natural number and s be nonzero integer with $k^2+4s > 0$. Generalized Fibonacci sequence is defined by

$$U_0(k,s) = 0, U_1(k,s) = 1 \text{ and } U_{n+1}(k,s) = kU_n(k,s) + sU_{n-1}(k,s)$$

for $n \geqslant 1$ and generalized Lucas sequence is defined by

$$V_0(k,s) = 2, V_1(k,s) = k \text{ and } V_{n+1}(k,s) = kV_n(k,s) + sV_{n-1}(k,s)$$

for $n \geqslant 1$, respectively. It is well known that

$$U_n(k,s) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n(k,s) = \alpha^n + \beta^n \tag{1}$$

where $\alpha = \left(k + \sqrt{k^2+4s}\right)/2$ and $\beta = \left(k - \sqrt{k^2+4s}\right)/2$. The above identities are known as Binet's formulae. Clearly, $\alpha + \beta = k$, $\alpha - \beta = \sqrt{k^2+4s}$, and $\alpha\beta = -s$.

Especially, if $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$, then we get

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } Q_n = \alpha^n + \beta^n. \tag{2}$$

P_n and Q_n are called Pell and Pell-Lucas sequences, respectively. For more information about generalized Fibonacci and Lucas sequences, one can consult [1–7].

2 Preliminaries

Let d be a positive integer which is not a perfect square and N be any nonzero fixed integer. Then the equation $x^2 - dy^2 = N$ is known as Pell equation. For $N = \pm 1$, the equations $x^2 - dy^2 = 1$ and $x^2 - dy^2 = -1$ are known as classical Pell equation. If $a^2 - db^2 = N$, we say that (a, b) is a solution to the Pell equation $x^2 - dy^2 = N$. We use the notations (a, b) and $a+b\sqrt{d}$ interchangeably to denote solutions of the equation $x^2 - dy^2 = N$. Also, if a and b are both positive, we say that $a + b\sqrt{d}$ is positive solution to the equation $x^2 - dy^2 = N$.

Let $x_1 + y_1\sqrt{d}$ be a positive solution to the equation $x^2 - dy^2 = N$. We say that $x_1 + y_1\sqrt{d}$ is the fundamental solution to the equation $x^2 - dy^2 = N$, if $x_2 + y_2\sqrt{d}$ is a different solution to the equation $x^2 - dy^2 = N$, then $x_1 + y_1\sqrt{d} < x_2 + y_2\sqrt{d}$. Recall that if $a + b\sqrt{d}$ and $r + s\sqrt{d}$ are two solutions to the equation $x^2 - dy^2 = N$, then $a = r$ if and only if $b = s$, and $a + b\sqrt{d} < r + s\sqrt{d}$ if and only if $a < r$ and $b < s$.

Continued fraction plays a significant role in solutions of the Pell equations $x^2 - dy^2 = 1$ and $x^2 - dy^2 = -1$. Let d be a positive integer that is not a perfect square. Then, there is a continued fraction expansion of $\sqrt{d}$ such that $\sqrt{d} = \left[a_0, \overline{a_1, a_2, ..., a_{l-1}, 2a_0}\right]$ where l is the period length and the a_j's are given by the recussion formulas;

$$\alpha_0 = \sqrt{d},\ a_k = \lfloor \alpha_k \rfloor \text{ and } \alpha_{k+1} = \frac{1}{\alpha_k - a_k},\ k = 0, 1, 2, 3, ...$$

Recall that $a_l = 2a_0$ and $a_{l+k} = a_k$ for $k \geq 1$. The n^{th} convergent of $\sqrt{d}$ for $n \geq 0$ is given by

$$\frac{p_n}{q_n} = [a_0, a_1, ..., a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \cfrac{1}{a_{n-1} + \frac{1}{a_n}}}}.$$

By means of the k^{th} convergent of $\sqrt{d}$, we can give the fundamental solution to the equations $x^2 - dy^2 = 1$ and $x^2 - dy^2 = -1$.

Now we give the fundamental solution to the equations $x^2 - dy^2 = \pm 1$ by means of the period length of the continued fraction expansion of $\sqrt{d}$ (See [8]).

Lemma 1 Let l be the period length of continued fraction expansion of $\sqrt{d}$. If l is even, then the fundamental solution to the equation $x^2 - dy^2 = 1$ is given by

$$x_1 + y_1\sqrt{d} = p_{l-1} + q_{l-1}\sqrt{d}$$

and the equation $x^2 - dy^2 = -1$ has no integer solutions. If l is odd, then the fundamental solution to the equation $x^2 - dy^2 = 1$ is given by

$$x_1 + y_1\sqrt{d} = p_{2l-1} + q_{2l-1}\sqrt{d}$$

and the fundamental solution to the equation $x^2 - dy^2 = -1$ is given by

$$x_1 + y_1\sqrt{d} = p_{l-1} + q_{l-1}\sqrt{d}.$$

Now we give the following three theorems from [9]. See also [10].

Theorem 1 *Let $d \equiv 1, 2, 3(mod4)$. Then the equation $x^2 - dy^2 = -4$ has a positive integer solution if and only if the equation $x^2 - dy^2 = -1$ has a positive integer solution.*

Theorem 2 *Let $d \equiv 2(mod4)$ or $d \equiv 3(mod4)$. If fundamental solution to the equations $x^2 - dy^2 = \pm 1$ is (x_1, y_1), then fundamental solution to the equation $x^2 - dy^2 = \pm 4$ is $(2x_1, 2y_1)$.*

Theorem 3 *Let $d \equiv 0 \ (mod4)$. If fundamental solution to the equation $x^2 - (d/4)y^2 = 1$ is $x_1 + y_1\sqrt{d/4}$, then fundamental solution to the equation $x^2 - dy^2 = 4$ is $(2x_1, y_1)$.*

If we know fundamental solution to the equations $x^2 - dy^2 = \pm 1$ and $x^2 - dy^2 = \pm 4$, then we can give all positive integer solutions to these equations. For more information about Pell equation, one can consult [11–17].

Theorem 4 *Let $x_1 + y_1\sqrt{d}$ be the fundamental solution to the equation $x^2 - dy^2 = 1$. Then all positive integer solutions to the equation $x^2 - dy^2 = 1$ are given by*

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$$

with $n \geq 1$.

Theorem 5 *Let $x_1 + y_1\sqrt{d}$ be the fundamental solution to the equation $x^2 - dy^2 = -1$. Then all positive integer solutions to the equation $x^2 - dy^2 = -1$ are given by*

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^{2n-1}$$

with $n \geq 1$.

Theorem 6 *Let $x_1 + y_1\sqrt{d}$ be the fundamental solution to the equation $x^2 - dy^2 = 4$. Then all positive integer solutions to the equation $x^2 - dy^2 = 4$ are given by*

$$x_n + y_n\sqrt{d} = \frac{(x_1 + y_1\sqrt{d})^n}{2^{n-1}}$$

with $n \geq 1$.

Theorem 7 *Let $x_1 + y_1\sqrt{d}$ be the fundamental solution to the equation $x^2 - dy^2 = -4$. Then all positive integer solutions to the equation $x^2 - dy^2 = -4$ are given by*

$$x_n + y_n\sqrt{d} = \frac{(x_1 + y_1\sqrt{d})^{2n-1}}{4^{n-1}}$$

with $n \geq 1$.

From now on, we will assume that k is a natural number. We give continued fraction expansion of $\sqrt{d}$ for $d = k^2 \pm 2$ and $d = k^2 \pm k$. The proofs of the following four theorems are easy and they can be found many text books on number theory as an exercise (see, for example [8]).

Theorem 8 *Let $k > 0$. Then*

$$\sqrt{k^2+2} = \left[k, \overline{k, 2k}\right].$$

Theorem 9 *Let $k > 2$ Then*

$$\sqrt{k^2-2} = \left[k-1, \overline{1, k-2, 1, 2(k-1)}\right].$$

Theorem 10 *Let $k > 1$. Then*

$$\sqrt{k^2+k} = \left[k, \overline{2, 2k}\right].$$

Theorem 11 *Let $k > 2$. Then*

$$\sqrt{k^2-k} = \left[k-1, \overline{2, 2(k-1)}\right].$$

Corollary 1 Let $k > 0$ and $d = k^2 + 2$. Then the fundamental solution to the equation $x^2 - dy^2 = 1$ is

$$x_1 + y_1\sqrt{d} = k^2 + 1 + k\sqrt{d}$$

and the equation $x^2 - dy^2 = -1$ has no positive integer solutions.

Proof The period length of the continued fraction expansion of $\sqrt{k^2+2}$ is 2 by Theorem 8. Therefore the fundamental solution to the equation $x^2-dy^2 = 1$ is $p_1+q_1\sqrt{d}$ by Lemma 1. Since

$$\frac{p_1}{q_1} = k + \frac{1}{k} = \frac{k^2+1}{k},$$

the proof follows. Since the period of length of the continued fraction expansion of $\sqrt{k^2+2}$ is even, the equation $x^2 - dy^2 = -1$ has no positive integer solutions by Lemma 1.

Corollary 2 Let $k > 2$ and $d = k^2 - 2$. Then the fundamental solution to the equation $x^2 - dy^2 = 1$ is

$$x_1 + y_1\sqrt{d} = k^2 - 1 + k\sqrt{d}$$

and the equation $x^2 - dy^2 = -1$ has no positive integer solutions.

Proof The period length of the continued fraction expansion of $\sqrt{k^2-2}$ is 4 by Theorem 9. Therefore the fundamental solution to the equation $x^2-dy^2=1$ is $p_3+q_3\sqrt{d}$ by Lemma 1. Since

$$\frac{p_3}{q_3}=k-1+\frac{1}{1+\frac{1}{k-2+\frac{1}{1}}}=\frac{k^2-1}{k},$$

the proof follows. Since the period of length of the continued fraction expansion of $\sqrt{k^2-2}$ is even, the equation $x^2-dy^2=-1$ has no positive integer solutions by Lemma 1.

Corollary 3 Let $k>1$ and $d=k^2+k$. Then the fundamental solution to the equation $x^2-dy^2=1$ is

$$x_1+y_1\sqrt{d}=2k+1+2\sqrt{d}$$

and the equation $x^2-dy^2=-1$ has no positive integer solutions.

Proof The period length of the continued fraction expansion of $\sqrt{k^2+k}$ is 2 by Theorem 10. Therefore the fundamental solution to the equation $x^2-dy^2=1$ is $p_1+q_1\sqrt{d}$ by Lemma 1. Since

$$\frac{p_1}{q_1}=k+\frac{1}{2}=\frac{2k+1}{2},$$

the proof follows. Since the period of length of the continued fraction expansion of $\sqrt{k^2+k}$ is even, the equation $x^2-dy^2=-1$ has no positive integer solutions by Lemma 1.

Corollary 4 Let $k>2$ and $d=k^2-k$. Then the fundamental solution to the equation $x^2-dy^2=1$ is

$$x_1+y_1\sqrt{d}=2k-1+2\sqrt{d}$$

and the equation $x^2-dy^2=-1$ has no positive integer solutions.

Proof The period length of the continued fraction expansion of $\sqrt{k^2-k}$ is 2 by Theorem 11. Therefore the fundamental solution to the equation $x^2-dy^2=1$ is $p_1+q_1\sqrt{d}$ by Lemma 1. Since

$$\frac{p_1}{q_1}=k-1+\frac{1}{2}=\frac{2k-1}{2},$$

the proof follows. Since the period of length of the continued fraction expansion of $\sqrt{k^2-k}$ is even, the equation $x^2-dy^2=-1$ has no positive integer solutions by Lemma 1.

3 Main Theorems

In this section, we give all positive integer solutions to some Pell equations.

Theorem 12 *Let $k>0$ and $d=k^2+2$. Then all positive integer solutions of the equation $x^2-dy^2=1$ are given by*

$$(x,y)=\left(\frac{V_n(2k^2+2,-1)}{2},kU_n(2k^2+2,-1)\right)$$

with $n\geq 1$.

Proof By Corollary 1 and Theorem 4, all positive integer solutions of the equation $x^2 - dy^2 = 1$ are given by

$$x_n + y_n\sqrt{d} = \left(k^2 + 1 + k\sqrt{d}\right)^n$$

with $n \geq 1$. Let $\alpha = k^2+1+k\sqrt{d}$ and $\beta = k^2+1-k\sqrt{d}$. Then $\alpha+\beta = 2k^2+2, \alpha-\beta = 2k\sqrt{d}$ and $\alpha\beta = 1$. Thus,

$$x_n + y_n\sqrt{d} = \alpha^n \text{ and } x_n - y_n\sqrt{d} = \beta^n.$$

Then it follows that

$$x_n = \frac{\alpha^n + \beta^n}{2} = \frac{V_n(2k^2+2,-1)}{2}$$

and

$$y_n = \frac{\alpha^n - \beta^n}{2\sqrt{d}} = k\frac{\alpha^n - \beta^n}{2k\sqrt{d}} = k\frac{\alpha^n - \beta^n}{\alpha - \beta} = kU_n(2k^2+2,-1)$$

by (1).

Since the proof of the following theorem is similar, we omit it.

Theorem 13 *Let $k > 2$ and $d = k^2 - 2$. Then all positive integer solutions of the equation $x^2 - dy^2 = 1$ are given by*

$$(x, y) = \left(\frac{V_n(2k^2-2,-1)}{2}, kU_n(2k^2-2,-1)\right)$$

with $n \geq 1$.

Theorem 14 *Let $k > 1$ and $d = k^2 + k$. Then all positive integer solutions of the equation $x^2 - dy^2 = 1$ are given by*

$$(x, y) = \left(\frac{V_n(4k+2,-1)}{2}, 2U_n(4k+2,-1)\right)$$

with $n \geq 1$.

Proof By Corollary 3 and Theorem 4, all positive integer solutions of the equation $x^2 - dy^2 = 1$ are given by

$$x_n + y_n\sqrt{d} = \left(2k + 1 + 2\sqrt{d}\right)^n$$

with $n \geq 1$. Let $\alpha = 2k+1+2\sqrt{d}$ and $\beta = 2k+1-2\sqrt{d}$. Then $\alpha+\beta = 4k+2, \alpha-\beta = 4\sqrt{d}$ and $\alpha\beta = 1$. Thus, we have

$$x_n + y_n\sqrt{d} = \alpha^n \text{ and } x_n - y_n\sqrt{d} = \beta^n.$$

Then it follows that

$$x_n = \frac{\alpha^n + \beta^n}{2} = \frac{V_n(4k+2,-1)}{2}$$

and

$$y_n = \frac{\alpha^n - \beta^n}{2\sqrt{d}} = 2\frac{\alpha^n - \beta^n}{4\sqrt{d}} = 2\frac{\alpha^n - \beta^n}{\alpha - \beta} = 2U_n(4k+2,-1)$$

by (1).

Since the proof of the following theorem is similar, we omit it.

Theorem 15 *Let $k > 2$ and $d = k^2 - k$. Then all positive integer solutions of the equation $x^2 - dy^2 = 1$ are given by*

$$(x, y) = \left(\frac{V_n(4k-2,-1)}{2}, 2U_n(4k-2,-1)\right)$$

with $n \geq 1$.

Now we give all positive integer solutions of the equations $x^2 - (k^2 \pm 2)y^2 = \pm 4$ and $x^2 - (k^2 \pm k)y^2 = \pm 4$.

Theorem 16 *Let $k > 0$. Then all positive integer solutions of the equation $x^2 - (k^2+2)y^2 = 4$ are given by*

$$(x, y) = (V_n(2k^2+2,-1), 2kU_n(2k^2+2,-1))$$

with $n \geq 1$.

Proof If k is odd, then $d = k^2+2 \equiv 3(mod 4)$ and if k is even, then $d = k^2+2 \equiv 2(mod 4)$. Thus, by Corollary 1 and Theorem 2, it follows that $2k^2+2+2k\sqrt{k^2+2}$ is the fundamental solution to the equation $x^2 - (k^2+2)y^2 = 4$. Therefore, by Theorem 6, all positive integer solutions of the equation $x^2 - dy^2 = 4$ are given by

$$x_n + y_n\sqrt{d} = \frac{(2k^2+2+2k\sqrt{k^2+2})^n}{2^{n-1}} = 2\left(\frac{2k^2+2+2k\sqrt{k^2+2}}{2}\right)^n.$$

Let $\alpha = \frac{2k^2+2+2k\sqrt{k^2+2}}{2}$ and $\beta = \frac{2k^2+2-2k\sqrt{k^2+2}}{2}$. Then $\alpha+\beta = 2k^2+2, \alpha-\beta = 2k\sqrt{d}$ and $\alpha\beta = 1$. Thus it is seen that

$$x_n + y_n\sqrt{d} = 2\alpha^n \text{ and } x_n - y_n\sqrt{d} = 2\beta^n.$$

Therefore we get

$$x_n = \alpha^n + \beta^n = V_n(2k^2+2,-1)$$

and

$$y_n = \frac{\alpha^n - \beta^n}{\sqrt{d}} = 2k\frac{\alpha^n-\beta^n}{2k\sqrt{d}} = 2k\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta} = 2kU_n(2k^2+2,-1)$$

by (1). Then the proof follows.

Theorem 17 *Let $k > 2$. Then all positive integer solutions of the equation $x^2 - (k^2-2)y^2 = 4$ are given by*

$$(x, y) = (V_n(2k^2-2,-1), 2kU_n(2k^2-2,-1))$$

with $n \geq 1$.

Proof If k is odd, then $d = k^2-2 \equiv 3(mod4)$ and if k is even, then $d = k^2-2 \equiv 2(mod4)$. Thus, by Corollary 2 and Theorem 2, it follows that $2k^2-2+2k\sqrt{k^2-2}$ is the fundamental solution to the equation $x^2-(k^2-2)y^2 = 4$. Therefore, by Theorem 6, all positive integer solutions of the equation $x^2-dy^2 = 4$ are given by

$$x_n + y_n\sqrt{d} = \frac{(2k^2-2+2k\sqrt{k^2-2})^n}{2^{n-1}} = 2\left(\frac{2k^2-2+2k\sqrt{k^2-2}}{2}\right)^n.$$

Let $\alpha = \frac{2k^2-2+2k\sqrt{k^2-2}}{2}$ and $\beta = \frac{2k^2-2-2k\sqrt{k^2-2}}{2}$. Then $\alpha+\beta = 2k^2-2, \alpha-\beta = 2k\sqrt{d}$ and $\alpha\beta = 1$. Thus it is seen that

$$x_n + y_n\sqrt{d} = 2\alpha^n \text{ and } x_n - y_n\sqrt{d} = 2\beta^n.$$

Therefore we get

$$x_n = \alpha^n + \beta^n = V_n(2k^2-2, -1)$$

and

$$y_n = \frac{\alpha^n-\beta^n}{\sqrt{d}} = 2k\frac{\alpha^n-\beta^n}{2k\sqrt{d}} = 2k\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta} = 2kU_n(2k^2-2, -1)$$

by (1). Then the proof follows.

Theorem 18 *Let $k > 1$ and $k \neq 4$. Then all positive integer solutions of the equation $x^2-(k^2-k)y^2 = 4$ are given by*

$$(x, y) = (V_n(4k-2, -1), 4U_n(4k-2, -1))$$

with $n \geq 1$.

Proof When $k \neq 4$, $b = 1, 2, 3$ and $a^2-(k^2-k)b^2 = 4$, it can be seen that $k = -3, 0$ or 1. Since $(4k-2)^2-(k^2-k)4^2 = 4$, it follows that the fundamental solution to the equation $x^2-(k^2-k)y^2 = 4$ is $4k-2+4\sqrt{k^2-k}$. Thus by Theorem 6, all positive integer solutions of the equation $x^2-dy^2 = 4$ are given by

$$x_n + y_n\sqrt{d} = \frac{(4k-2+4\sqrt{k^2-k})^n}{2^{n-1}} = 2\left(\frac{4k-2+4\sqrt{k^2-k}}{2}\right)^n.$$

Let $\alpha = \frac{4k-2+4\sqrt{k^2-k}}{2}$ and $\beta = \frac{4k-2-4\sqrt{k^2-k}}{2}$. Then $\alpha+\beta = 4k-2, \alpha-\beta = 4\sqrt{d}$ and $\alpha\beta = 1$. Thus it follows that $x_n + y_n\sqrt{d} = 2\alpha^n$ and $x_n - y_n\sqrt{d} = 2\beta^n$. Then, we get

$$x_n = \alpha^n + \beta^n = V_n(4k-2, -1)$$

and

$$y_n = \frac{\alpha^n-\beta^n}{\sqrt{d}} = 4\frac{\alpha^n-\beta^n}{4\sqrt{d}} = 4\frac{\alpha^n-\beta^n}{\alpha-\beta} = 4U_n(4k-2, -1)$$

by (1).

Theorem 19 *Let $k > 1$ and $k \neq 3$. Then all positive integer solutions of the equation $x^2-(k^2+k)y^2 = 4$ are given by*

$$(x, y) = (V_n(4k+2, -1), 4U_n(4k+2, -1))$$

with $n \geq 1$.

Proof When $k \neq 3$, $b = 1, 2, 3$ and $a^2 - (k^2 + k)b^2 = 4$, it can be seen that $k = -1$ or 0. Since $(4k + 2)^2 - (k^2 + k)4^2 = 4$, it follows that the fundamental solution to the equation $x^2 - (k^2 + k)y^2 = 4$ is $4k + 2 + 4\sqrt{k^2 + k}$. Thus, by Theorem 6, all positive integer solutions of the equation $x^2 - dy^2 = 4$ are given by

$$x_n + y_n\sqrt{d} = \frac{(4k + 2 + 4\sqrt{k^2 + k})^n}{2^{n-1}} = 2\left(\frac{4k + 2 + 4\sqrt{k^2 + k}}{2}\right)^n.$$

Let $\alpha = \frac{4k+2+4\sqrt{k^2+k}}{2}$ and $\beta = \frac{4k+2-4\sqrt{k^2+k}}{2}$. Then $\alpha + \beta = 4k + 2, \alpha - \beta = 4\sqrt{d}$ and $\alpha\beta = 1$. Thus it follows that $x_n + y_n\sqrt{d} = 2\alpha^n$ and $x_n - y_n\sqrt{d} = 2\beta^n$. Therefore we get

$$x_n = \alpha^n + \beta^n = V_n(4k + 2, -1)$$

and

$$y_n = \frac{\alpha^n - \beta^n}{\sqrt{d}} = 4\frac{\alpha^n - \beta^n}{4\sqrt{d}} = 4\frac{\alpha^n - \beta^n}{\alpha - \beta} = 4U_n(4k + 2, -1)$$

by (1).

Theorem 20 *All positive integer solutions of the equation $x^2 - 12y^2 = 4$ are given by*

$$(x, y) = (V_n(4, -1), U_n(4, -1))$$

with $n \geq 1$.

Proof The fundamental solution to the equation $x^2-12y^2 = 4$ is $4+\sqrt{12}$ since $4^2-12.1^2 = 4$. Thus, by Theorem 6, all positive integer solutions of the equation $x^2 - 12y^2 = 4$ are given by

$$x_n + y_n\sqrt{12} = \frac{(4 + \sqrt{12})^n}{2^{n-1}} = 2\left(\frac{4 + 2\sqrt{3}}{2}\right)^n = 2(2 + \sqrt{3})^n.$$

Let $\alpha = 2 + \sqrt{3}$ and $\beta = 2 - \sqrt{3}$. Then $\alpha + \beta = 4, \alpha - \beta = 2\sqrt{3}$ and $\alpha\beta = 1$. Thus it follows that $x_n + y_n\sqrt{12} = 2\alpha^n$ and $x_n - y_n\sqrt{12} = 2\beta^n$. Therefore we get

$$x_n = \alpha^n + \beta^n = V_n(4, -1)$$

and

$$y_n = \frac{\alpha^n - \beta^n}{\sqrt{12}} = \frac{\alpha^n - \beta^n}{2\sqrt{3}} = \frac{\alpha^n - \beta^n}{\alpha - \beta} = U_n(4, -1)$$

by (1).

Theorem 21 *Let $k > 0$. Then the equation $x^2 - (k^2 + 2)y^2 = -4$ has no positive integer solutions.*

Proof If k is odd, then $k^2+2 \equiv 3(mod4)$ and if k is even, then $k^2+2 \equiv 2\ (mod4)$. Thus by Corollary 1 and Theorem 1, the proof follows.

Since the proof of the following theorem is similar, we omit it.

Theorem 22 *Let $k > 2$. Then the equation $x^2-(k^2-2)y^2=-4$ has no positive integer solutions.*

Theorem 23 *Let $k > 1$. Then the equation $x^2-(k^2+k)y^2=-4$ has no positive integer solutions.*

Proof Let $d = k^2 + k$ and $a^2-db^2=-4$ for some positive integers a and b. Assume that $k=4t+1$ or $k=4t+2$. Then $d \equiv 2(mod4)$. Thus, by Theorem 1 and Corollary 3, the equation $x^2-dy^2=-4$ has no positive integer solutions. Assume that $k=4t$. Then $d=16t^2+4t$ and $d \equiv 0(mod4)$. Thus a is even and $a^2-(16t^2+4t)b^2=-4$. This implies that

$$(a/2)^2-(4t^2+t)b^2=-1. \tag{3}$$

It can be easily seen that $\sqrt{4t^2+t}=[2t,\overline{4,4t}]$. Since the period of length of the continued fraction expansion of $\sqrt{4t^2+t}$ is even, the equation $x^2-(4t^2+t)y^2=-1$ has no positive integer solutions by Lemma 1. This contradicts with (3). Therefore the equation $x^2-dy^2=-4$ has no positive integer solutions. Assume that $k=4t+3$. Then $d=16t^2+28t+12$ and $d \equiv 0(mod\ 4)$. Thus a is even and $a^2-(16t^2+28t+12)b^2=-4$. This implies that

$$(a/2)^2-(4t^2+7t+3)b^2=-1. \tag{4}$$

It can be easily seen that $\sqrt{4t^2+7t+3}=[2t+1,\overline{1,2,1,2(2t+1)}]$. Since the period of length of the continued fraction expansion of $\sqrt{4t^2+7t+3}$ is even, the equation $x^2-(4t^2+7t+3)y^2=-1$ has no positive integer solutions by Lemma 1. This contradicts with (4). Therefore the equation $x^2-dy^2=-4$ has no positive integer solutions.

Since the proofs of the following theorem is similar, we omit it.

Theorem 24 *Let $k > 2$. Then the equation $x^2-(k^2-k)y^2=-4$ has no positive integer solutions.*

Continued fraction expansion of $\sqrt{2}$ is $\left[1,\overline{2}\right]$. Thus the period length of the continued fraction of $\sqrt{2}$ is 1. Moreover, the fundamental solution to the equation $x^2-2y^2=1$ is $3+2\sqrt{2}$ and the fundamental solution to the equation $x^2-2y^2=-1$ is $1+\sqrt{2}$ by Lemma 1. Therefore, by using (2), we can give the following corollaries easily.

Corollary 5 All positive integer solutions of the equation $x^2-2y^2=1$ are given by

$$(x,y)=\left(\frac{Q_{2n}}{2},P_{2n}\right)$$

with $n \geq 1$.

Corollary 6 All positive integer solutions of the equation $x^2 - 2y^2 = -1$ are given by

$$(x, y) = \left(\frac{Q_{2n-1}}{2}, P_{2n-1}\right)$$

with $n \geq 1$.

It can be seen that fundamental solutions to the equations $x^2 - 2y^2 = 4$ and $x^2 - 2y^2 = -4$ are $6 + 4\sqrt{2}$ and $2 + 2\sqrt{2}$, respectively. Thus by Theorem 2 and identity (2), we can give following corollaries easily.

Corollary 7 All positive integer solutions of the equation $x^2 - 2y^2 = 4$ are given by

$$(x, y) = (Q_{2n}, 2P_{2n})$$

with $n \geq 1$.

Corollary 8 All positive integer solutions of the equation $x^2 - 2y^2 = -4$ are given by

$$(x, y) = (Q_{2n-1}, 2P_{2n-1})$$

with $n \geq 1$.

4 Conclusion

In this paper, by using continued fraction expansion of $\sqrt{d}$, we find fundamental solution of the $x^2 - dy^2 = \pm 1$, where k is a natural number and $d = k^2 \pm 2$, or $k^2 \pm k$. Moreover, we investigate Pell equations of the form $x^2 - dy^2 = N$ when $N = \pm 1, \pm 4$ and we are looking for positive integer solutions in x and y. We get all positive integer solutions of the Pell equations $x^2 - dy^2 = N$ in terms of generalized Fibonacci and Lucas sequences when $N = \pm 1, \pm 4$ and $d = k^2 \pm 2, k^2 \pm k$. Finally, all positive integer solutions of the equations $x^2 - 2y^2 = \pm 1$ and $x^2 - 2y^2 = \pm 4$ are given in terms of Pell and Pell-Lucas sequences.

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