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On the Line Graph Associated to the Total Graph of a Module

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Abstract Let R be a commutative ring with unity and M be an R-module with $T\Gamma(M)$ be its total graph. The subject of this article is the investigation of the properties of the corresponding line graph $L(T\Gamma(M))$. In particular, we determine the girth and clique number of $L(T\Gamma(M))$. In addition to that we find a condition for $L(T\Gamma(M))$ to be Eulerian.

Keywords Total graph; line graph; module.

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1 Introduction

Let R be a commutative ring with identity and M be an R-module. Let $Z(R)$ be the set of zero divisors of R and $Z^*(R) = Z(R) - \{0\}$. Many works have been done in which the graphs associated to rings were introduced and their properties established. One of the most common is the zero-divisor graph. This idea first appears in [1], where for a ring R, the set of vertices is taken to be R and two vertices x and y are adjacent if and only if $xy = 0$. This work was mostly concerned with colorings of rings. Later in the paper [2], the authors define the zero-divisor graph $\Gamma(R)$ where the set of vertices is taken to be $Z^*(R)$.The zero-divisor graph of a commutative ring has also been studied by several other authors. The zero-divisor graph has also been introduced for semigroups and other algebraic structures.

In 2008, Anderson and Badawi [3] introduce the total graph $T(\Gamma(R))$, whose set of vertices is R and two vertices x and y are adjacent if and only if $x + y \in Z(R)$. After Anderson and Badawi, the concept of total graph has also been introduced for modules over commutative ring by several authors. Ebrahimi Atani and Habibi in [4] introduce the total torsion element graph of a module over a commutative ring.

Let M be a R-module,

$$R^* = R - \{0\} \text{ and} M^* = M - \{0\}.$$

Let $T(M) = \{m \in M | rm = 0 \text{ for some } r \in R^*\}$ be the set of its torsion elements and $Tof(M) = M - T(M)$ be the set of its non-torsion elements. Pucanovic [5] define the total graph of a module $T\Gamma(M)$ as the elements of M be the vertices and two distinct vertices x and y are adjacent if and only if $x + y \in T(M)$.

Given a graph G, its line graph $L(G)$ is a graph such that every vertex of $L(G)$ represents an edge of G, and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G. So, the set of vertices of $L(G)$ is exactly the set of edges of G, and $L(G)$ represents the adjacencies between edges of G. Thus the properties of a graph G that depend only on adjacency between edges may be translated into equivalent properties in $L(G)$ that depend on adjacency between vertices. This is very useful for various problems in graph theory. For example, a matching in G is a set of edges such that no two of them are adjacent. To a matching in G there corresponds an independent set in $L(G)$, that is a set of vertices in $L(G)$ no two of which are adjacent. If G is connected and if its line graph $L(G)$ is known,one may,according to [6], completely determine G except in the case when $L(G)$ is a triangle. Eric and Pucanovic [7] studied the structure and properties of the line graph associated to the total graph $L(T\Gamma(R))$. Knowing the structure of the total graph $T\Gamma(M)$, our goal is to investigate the structure of its line graph $L(T\Gamma(M))$ and to look into various relations between them. In this paper we determine some properties of this line graph.

For the algebraic part of this paper, notation and terminology is standard and one may find it in [8], [9], or in [10]. For the graph theoretical part, notation and terminology may be find in [11], or in [12]. In what follows, by a graph G we mean the simple undirected graph(without loops and parallel edges) with the set of vertices $V = V(G)$ and the set of edges $E = E(G)$. The degree of the vertex $v \in V$, denoted by $deg(v)$ is the number of vertices adjacent to the vertex v and

$$\delta(G)= \min \{deg(v)|v \in V(G)\}$$

is the minimal degree of the graph. A graph is regular of degree r if every vertex has the degree r. The vertices x and y are adjacent if they are connected by an edge. If for every two vertices x and y there exists a path connecting them , then we say that the graph is connected. A graph G is complete if any two vertices are adjacent. If the vertices of the graph G may be separated into two disjoint sets of cardinalities m and n, such that vertices are adjacent if and only if they do not belong to the same set, then the graph G is a complete bipartite graph.

For complete and complete bipartite graphs, we use the notation K^n and $K^{m,n}$ respectively. In particular, $K^{1,n}$ is a star graph. The maximal positive integer r such that $K^r \subseteq G$ for some graph G is the clique number of that graph. For vertices $x, y \in G$ one defines the distance $d(x, y)$, as the length of the shortest path between x and y, if the vertices $x, y \in G$ are connected and $d(x, y) = \infty$, if they are not. Then, the diameter of the graph G is

$$diam(G) = sup\{d(x, y)|x, y \in G\}.$$

The cycle is a closed path which begins and ends in the same vertex. The cycle of n vertices is denoted by C_n. The girth of the graph G,denoted by $gr(G)$ is the length of the shortest cycle in G and $gr(G) = \infty$ if G has no cycles.

2 On the Structure and Properties of $L(T\Gamma(M))$

Let R be a commutative ring with unity and M be an R-module with $T\Gamma(M)$ be its total graph. We define the line graph $L(T\Gamma(M))$ of $T\Gamma(M)$ as the graph with all the edges of $T\Gamma(M)$ as vertices and any two distinct vertices in $L(T\Gamma(M))$ are adjacent if and only if their corresponding edges share a common vertex in $T\Gamma(M)$. If for elements $x, y \in M$ one has $x+y \in T(M)$, then we have a vertex in the graph $L(T\Gamma(M))$ and we denote that vertex by $[x, y]$. Let us consider an example.

Example 1: Every finite complete graph may be realized as a total graph of a module. In particular, if $R = \mathbb{Z}$ and $M = \mathbb{Z}_4$, an R-module with the usual multiplication, then the total graph $T\Gamma(M) \cong K^4$. This total graph $T\Gamma(M)$ and its associated line graph $L(T\Gamma(M))$ can be observed from Figure 1

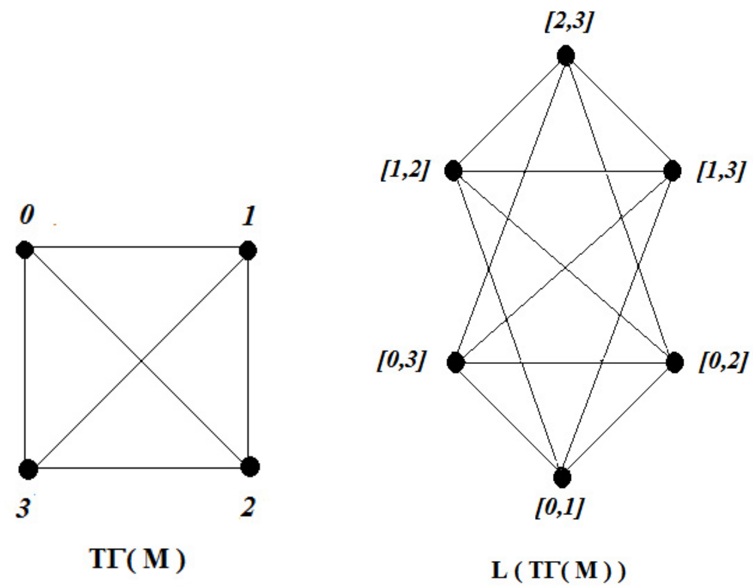

Figure 1: The Total Graph $T\Gamma(M)$ and Its Line Graph $L(T\Gamma(M))$

From the definition of the graph $T\Gamma(M)$ it follows that the degree of every vertex of this graph depends on the cardinality of $T(M)$ as well as on whether 2 is a zero divisor in R or not. Therefore, we begin with the following proposition.

Proposition 2.1 Let x be a vertex of the graph $T(\Gamma(M))$. Then

$$deg(x) = \begin{cases} |T(M)| - 1, & \text{if } 2 \in Z(R) \text{ or } x \in T(M), \\ |T(M)|, & \text{otherwise.} \end{cases}$$

Proof If $x_i \in T(M)$, the vertex $x \in M$ is adjacent to vertices $x_i - x$. Then $deg(x) = |T(M)| - 1$ if and only if $x = x_i - x$ for some $x_i \in T(M)$ i.e. if and only if $2x \in T(M)$. If $2x \notin T(M)$, then $deg(x) = |T(M)|$. If $2 \in Z(R)$, then $2x \in T(M)$ for all $x \in M$, thus $deg(x) = |T(M)| - 1$ i.e. all vertices of the graph $T(\Gamma(M))$ are of degree $|T(M)| - 1$. Again, if $2 \notin Z(R)$, then two cases arise.
Case-1. If $x \in T(M)$, then $deg(x) = |T(M)| - 1$.
Case-2. If $x \notin T(M)$, then $deg(x) = |T(M)|$.

It follows that $deg(x) = \begin{cases} |T(M)| - 1, & \text{if } 2 \in Z(R) \text{ or } x \in T(M), \\ |T(M)|, & \text{otherwise.} \ \square \end{cases}$

In what follows we will denote by $\omega(G)$ the clique number of the graph G.

Proposition 2.2

$$\omega(L(T\Gamma(M))) = \begin{cases} |T(M)|-1, & \text{if } 2 \in Z(R), \\ |T(M)|, & \text{otherwise.} \end{cases}$$

Proof

Case-1: Assume that $2 \in Z(R)$, By the Proposition 2.1, $deg(x) = |T(M)|-1$ for every $x \in V(T\Gamma(M))$. For some vertex a and the edge ay we have an vertex $[a, y]$ in the line graph. There are $|T(M)|-1$ such vertices in $L(T\Gamma(M))$. These vertices form subgraph K^{α} where $\alpha = |T(M)|-1$. For any set of $\beta > \alpha$ edges in $T\Gamma(M)$, corresponding vertices in the line graph are not adjacent. Thus $\omega(L(T\Gamma(M))) = |T(M)|-1$.

Case-2: Assume that $2 \notin Z(R)$, By the Proposition 2.1 we have $deg(x) = |T(M)|$ for some $x \in V(T\Gamma(M))$. For some vertex a and the edge ay we have an vertex $[a, y]$ in the line graph. There are $|T(M)|$ such vertices in $L(T\Gamma(M))$. These vertices form subgraph K^{α} where $\alpha = |T(M)|$. For any set of $\beta > \alpha$ edges in $T\Gamma(M)$, corresponding vertices in the line graph are not adjacent. Thus $\omega(L(T\Gamma(M))) = |T(M)|$. □

Lemma 2.3 ([5], Theorem 2.4)
Let R be a commutative ring and M an R-module. Then $T\Gamma(M)$ is totally disconnected if and only if R has characteristic 2 and M is torsion-free.

Proposition 2.4

$$gr(L(T\Gamma(M))) = \begin{cases} 3, & \text{if } |T(M)| \geq 3 \text{ and } M \neq \mathbb{Z}_2 \times \mathbb{Z}_2, \\ 4, & M \cong \mathbb{Z}_2 \times \mathbb{Z}_2, \\ \infty, & \text{if } |T(M)| \leq 2. \end{cases}$$

Proof

Case-1. $|T(M)| = 1$:

Then $T(M) = \{0\}$, and therefore M is torsion-free. We have two possible cases.

1. If $Char(R) = 2$, then by lemma 2.3 above $T\Gamma(M)$ is totally disconnected. Therefore, $L(T\Gamma(M))$ is an empty graph.
2. If $Char(R) \neq 2$, then $T\Gamma(M)$ is the disjoint union of $|M|/2$ graphs K^2 and an isolated vertex 0. In that case , $L(T\Gamma(M))$ is a totally disconnected graph with $|M|/2$ vertices. Therefore, $gr(L(T\Gamma(M))) = \infty$.

Case-2. $|T(M)| = 2$:

In this case we have only two non-isomorphic modules namely $\mathbb{Z}_4$ and $\mathbb{Z}[x]/(x^2)$. They have isomorphic total graphs-the disjoint union of two complete graphs K^2. It follows that $L(T\Gamma(M))$ is the union of two isolated vertices, therefore $gr(L(T\Gamma(M))) = \infty$.

Case-3. $|T(M)| \geq 3$:

1. If $T(M)$ is a submodule and if x, y are different elements from $T^*(M)$, then $T\Gamma(M)$ contains the triangle $0 - x - y - 0$; so, $L(T\Gamma(M))$ contains the triangle $[0, x] - [x, y] - [y, 0] - [0, x]$. Therefore, $gr(L(T\Gamma(M))) = 3$.

2. If $T(M)$ is not a submodule,there exist $x, y \in T^*(M)$ such that $x+y \in Tof(M)$. If one also has $|T(M)| > 3$,i.e.,if there exists $z \in T^*(M)$ different from x and y , then in $T\Gamma(M)$ there exists a star subgraph $K^{1,3}$ (the vertex 0 is then adjacent to the vertices x,y and z). Since $L(K^{1,3}) = C_3$, then $[0, x] - [0, y] - [0, z] - [0, x]$ is a triangle in $L(T\Gamma(M))$ and $gr(L(T\Gamma(M))) = 3$.
   It remains to discuss the case $|T(M)| = 3$.
   In this case we have only three non-isomorphic modules namely $\mathbb{Z}_9$,$\mathbb{Z}_3[x]/(x^2)$ and $\mathbb{Z}_2\times\mathbb{Z}_2$. In the first two cases $T(M)$ is a submodule, while $T\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2) = C_4$. Since $L(C_4) = C_4$ we have that $gr(L(T\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2))) = 4$. □

It is not very difficult to see that the following proposition holds.

Proposition 2.5 Let M be an R-module such that $T\Gamma(M)$ is connected with finite diameter d. Then the corresponding line graph $L(T\Gamma(M))$ is also connected and $d - 1 \leq d_L \leq d + 1$ where d_L is the diameter of $L(T\Gamma(M))$.

Proposition 2.6 Let M be an R-module such that $T(M) \neq \{0\}$ and $Char(R) \neq 2$. Then $L(T\Gamma(M))$ is regular if and only if $2 \in Z(R)$. The degree of regularity is then $2|T(M)| - 4$. In particular , if $|V|$ and $|E|$ denote the number of vertices and edges of the line graph respectively, then we have that

$$|V| = \frac{|M|(|T(M)| - 1)}{2}, \quad |E| = \frac{|M|(|T(M)| - 1)(T(M) - 2)}{2}.$$

Proof

Case-1. Suppose first that $2 \in Z(R)$. Then , according to proposition 2.1 , $T\Gamma(M)$ is a regular graph of degree $|T(M)| - 1$ with $|M|(|T(M)| - 1)/2$ edges. Let $[x, y]$ be an arbitrary vertex of the line graph $L(T\Gamma(M))$. Then $deg([x, y]) = deg(x)+deg(y)-2 = 2|T(M)| - 4$, and the claim follows.

Case-2. Suppose that $2 \notin Z(R)$. Since $T(M) \neq \{0\}$, there exists $x \in T^*(M)$. Also, holds $1 \neq -1$. So , $deg([0, x]) = 2|T(M)| - 4 \neq deg([1, -1]) = 2|T(M)| - 2$. Therefore , $L(T\Gamma(M))$ is not regular. □

Lemma 2.7 ([5], Theorem 2.6)
Let M be an R-module such that $T(M)$ is a proper submodule of M. Then $T\Gamma(M)$ is disconnected.

In graph theory, an Eulerian path is a trail in a graph which visits every edge exactly once. Similarly, an Eulerian cycle of the graph G is an Eulerian path which starts and ends on the same vertex. The graph which has an Eulerian cycle is an Eulerian graph. It is known that a connected undirected graph with atleast one edge is Eulerian if and only if all of its vertices have even numbers for degrees.
Thus, we can concentrate on the case when M is a finite module such that $T(M)$ is not a proper submodule of M. Because if M is infinite, then it does not make any sense to seek Eulerian cycle and if M is finite and $T(M)$ is a proper submodule of M, then by lemma 2.7 we have $T\Gamma(M)$ is not connected.

Proposition 2.8 Let M be a finite R-module such that $T(M)$ is not a proper submodule of M. Then $L(T\Gamma(M))$ is Eulerian if $2 \in Z(R)$.

Proof Let us assume that M is a finite R-module and $2 \in Z(R)$. Then according to the proposition 2.1, every vertex of $T\Gamma(M)$ has degree $|T(M)| - 1$, that is, $T\Gamma(M)$ is a regular graph of degree $|T(M)| - 1$.

Let $[x, y]$ be an arbitrary vertex of $L(T\Gamma(M))$. Then $deg([x, y]) = deg(x) + deg(y) - 2 = 2|T(M)| - 4$, which is an even number. Consequently, $L(T\Gamma(M))$ is Eulerian. □

3 Conclusions

In this paper we found some properties of the line graph associated to the total graph of a module. In particular,we determine the girth and clique number of $L(T\Gamma(M))$. In addition to that, we find a condition for $L(T\Gamma(M))$ to be Eulerian. For future research, it would be interesting to find similar properties for the line graphs associated to the other graphs attached to the commutative rings as well as modules over rings. For example, one can consider the line graph of the intersection graphs of ideals of commutative rings and also same of submodules of modules.

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